Of all particular integrals of Laplace’s equation, these are of the greatest importance in respect of their applications, and were the only ones considered by the earlier investigators; the solu- tions of potential problems in which the bounding surfaces are exactly or approximately spherical are usually expressed as series in which the terms are these spherical harmonics. In the wider sense of the term, a spherical harmonic is any homogeneous function of the variables which satisfies Laplace’s equation, the degree of the function being not necessarily integral or real, and the functions are not necessarily rational in *x, y,* z, or single-valued; when the term spherical harmonic is used in the narrower sense, the functions may, when necessary, be termed ordinary spherical harmonics. For the treatment of potential problems which relate to spaces bounded by special kinds of surfaces, solutions of Laplace’s equation are required which are adapted to the particular boundaries, and various classes of such solutions have thus been introduced into analysis. Such functions are usually of a more complicated structure than ordinary spherical harmonics, although they possess analogous properties. As examples we may cite Bessel’s functions in connexion with circular cylinders, Lamé’s functions in connexion with ellipsoids, and toroidal functions for anchor rings. The theory of such functions may be regarded as embraced under the general term harmonic analysis. The present article contains an account of the principal properties of ordinary spherical harmonics, and some indications of the nature and properties of the more important of the other classes of functions which occur in harmonic analysis. Spherical and other harmonic functions are of additional impor- tance in view of the fact that they are largely employed in the treatment of the partial differential equations of physics, other than Laplace’s equation; as examples of this, we may refer to the θu

equation = £ V⅛ which is fundamental in the theory of con- duction of heat and electricity, also to the equation^ = ⅛V2w, which occurs in the theory of the propagation of aerial and electro-magnetic waves. The integration under given condi- tions of more complicated equations which occur in the theories of hydro-dynamics and elasticity, can in certain cases be effected by the use of the functions employed in harmonic analysis.

1. *Relation between Spherical Harmonics of Positive and Negative Degrees.—*A function which is homogeneous in x, y, z, of degree *n* in those variables, and which satisfies Laplace’s equation

∂2V 1 θ2V 1 W -9,r

‰2^+∂∕+∂zi-0' or V2V-°, (ι)

is termed a solid spherical harmonic, or simply a spherical harmonic of degree n. The degree *n* may be fractional or imaginary, but we are at present mainly concerned with the case in which *n* is a positive or negative integer. If x, *y, z* be replaced by their values *r* sin Θ cos *φ, r* sin 0 sin *φ, r* cos *θ* in polar co-ordinates, a solid spherical harmonic takes. the form *rnfn(β, Φ)* ; the factor ∕n(0, *Φ)* is calíed a surface harmonic of degree n. If Vn denote a spherical harmonic of degree *ni* it may be shown by differentiation that v2(rmVn) *= m{2n +m* ÷ ι)rw\*-2Vn, and thus as a particular case that V2(r^n~1Vn) =0; we have thus the fundamental theorem that from any spherical harmonic Vn of degree *n,* another of degree —n —1 may be derived by dividing vn by r2"+l. All spherical harmonics of negative integral degree are obtainable in this way from those of positive integral degree. This theorem is a par- ticular case of the more general inversion theorem that if F (x, y, z) is any function which satisfies the equation (1), the function

⅛√Ξ Z ≡Λ

r V2’ r2’ r2∕

also satisfies the equation.

The ordinary spherical harmonics of positive integral degree *n* are those which are rational integral functions of *χ, y, z.* The most general rational integral function of degree *n* in three letters contains j(n+1)(n+2) coefficients; if the expression be substituted in (1), we have on equating the coefficients separately to zero *}n(n-*1) relations to be satisfied; the most general spherical harmonic of the prescribed type therefore contains ∣(w+1)(w-∣-2) *— ⅛n(n-* 1), or 2w-h1 independent constants. There exist, there- fore, 2n-∣-ι independent ordinary harmonics of. degree n; and corresponding to each of these there is a negative harmonic of degree—n — i obtained by dividing by r2n+1. The three inde­pendent harmonics of degree 1 are x, *y,* z; the five of degree 2 are y2-z2, i2-x2, yz, gχ, *xy.* Every harmonic of degree *n* is a linear function of 2π-∣-1 independent harmonics of the degree; we pro­ceed, therefore, to find the latter.

2. *Determination of Harmonics of given Degree.—*It is clear that a function *f{ax+by+ez)* satisfies the equation (1), if α, *b, c* are constants which satisfy the condition a2-j-ô2+c2=o; in particular the equation is satisfied by (z⅛ιx cos α-Hy sin a)n. Taking *n* to be a positive integer, we proceed to expand this expression in a series of cosines and sines of multiples of a; each term will then satisfy (1) separately. Denoting *etα* by *k,* and y+tx by *t,* we have

(2+tx cos α-∣-ty sin a)n= ½fø)

which may be written as *(2kt)^{(z+kIp-rζ)}n.* On expansion by Taylor’s theorem this becomes

2n

o

the differentiation applying to *ζ* only as it occurs explicitly; the terms involving cos *ma,* sin *ma* in this expansion are

⅛ ∞s ma í ⅜⅜⅜⅞^-^)n+⅞+⅞ s

⅜sin>aj ⅜+⅞!a⅛^~ri),,~⅛~~t.~~~~t~~~~⅛7~~ 5

where ím = 1, 2,. . . w; and the term independent of α is

2nw! - r2)n∙

On writing

(y+tx)m = tmrm(cos *mφ-ι* sin *mφ)2* sin ∙\*0, (y÷tx)-w =

r∙fflr~m(cos *mφ+c* sin *mφ)* sin^tnfl and observing that in the expansion of (zψιx *cos* α∏-ty sin a)m the expressions cos *ma,* sin *ma* can only occur in the combination cos w(≠-a), we see that the relation

. « si∏ *mθ* 3n+m ∕ ∙> λ- - sin *~mθ ∂n~m , β β.* tmrVRqι n=t*w=^~rγ,*

must hold identically, and thus that the terms in the expansion reduce to

I ιn\* θn÷m

(w+w)! 5ι=i',m cos *ma* cos *mφ* sin

**I** *im θn+n⅛*

*l(w-∣-m)! ^=irm sin ma sin mΙ>* sin m⅛τs(z2-r2)n∙

We thus see that the spherical harmonics of degree *n* are of the form

'n⅛ ∞≠ si∏ mβ^H⅛G',-ι)n

where *µ* denotes cos ø; by giving *m* the values o, 1, *2 . .* . n we thus have the *2n-Vι* functions required. On carrying out the differen- tiations we see that the required functions are of the form A[(x+t>)m=fc (χ-ιy)m] j zn~m-——~~T'—~~~~~zn-m~~~~^~~~~i~~~~(~~~~χ2~~~~÷∕÷~~~~z2~~~~) +Ì~~~~w~~~~~~~~~OT)(~~~~”;T~~~~2~~~~I~~~~n-7”;-3~~~~(~~~~”~”~~~~t~~~~~~~~~3)~~~~^~~~~4~~~~^+^+~~~~2¾~~~~)~~~~2~~ω where *m=ο,* 1, 2, 3, .. . n.

3. *Zonal, Tesseral and Sectorial Harmonics.*—Of the system of 2n-f-1 harmonics of degree *n,* only one is symmetrical about the *z* axis; this is

r"⅛!

writing p-ω=⅛fe^-1)n∙

we observe that Pn(μ) has *n* zeros all lying between’ =\*=ι, conse­quently the locus of points on a sphere r=α, for which Pn(μ) vanishes is *n* circles all parallel to the meridian plane : these circles divide the sphere into zones, thus Pn(μ) is calíed the zonaí surface harmonic of degree *n,* and rnPn(μ), r^n~1Pn(μ) are the solid zonal harmonics of degrees *n* and — n —r. The locus of points on a sphere for which mφ.sin ffl¾^(∕12-1)n vanishes consists of *n — m* circles parallel to the meridian plane, and. *m* great circles through the poles; these circles divide the spherical surface into quadrilaterals or τeσσepα, except when *n = m,* in which case the surface is divided into sectors, and the harmonics are therefore called tesseral, except those for which *m=n,* which are called sectorial. Denoting (1 — by P„ (μ), the tesseral

surface harmonics are *mφ .* (cos 0), where Wi==ι, 2τ...w-1,

and the sectorial harmonics are nφ.PjJ(cos 0). The functions

Pn(μ), P„ (μ) denote the expressions