P (u) ≈. (⅜ J ∣ w(tt-l)(n-2)fr"3kn-4 *l (,)*

*2nnlnt. ( μ 2.2n-iμ* ‘ 2.4-2Μ-Γ1 *.2η-*3 μ , · · J '3>

p>i⅛⅛\*-'W,k

(⅞-w)(w-m-l) . ) ω

2.2M-I μ" ‘ ) Vf√

p^ω=⅞⅛r(>-M2)∣n∙

Every ordinary harmonic of degree *n* is expressible as a linear funtion of the system of 2w-f-1 zonal, tesseral and sectorial harmonics of degree n; thus the general form of the surface harmonic is

n .

αilPn⅛) +∑(αin cos *mφ+bm* sin m≠)Pn ⅛). (5)

**W = I**

In the present notation we have

ÌW I m ∙j

Pn(μ) ÷2stm(w+^pP⅝ ⅛) cosffl(⅜~α)

if we put a = o, we thus have

» I

(cos *ΘA-ι* sin *ø* cos √>)n = Pn(cos *0)-V2∑im*~~^ψ^~~y∣Pn (cos 0) cos *mφ, m*

*from* this we obtain expressions for Pn(cos 0), Pn (cos *0)* as definite integrals

Pn(cos 0) =- ∣ (cos 0-f-ι sin *0* cos *φ)ndφ )*

*w∣ - IA* . r(6>

∣Pn (cos *ø) — - J* o (cos ö÷t sin *0* cos ≠)ncos *mφdφt* **J**

4. *Derivation of Spherical Harmonics by Differentiation.—*The linear character of Laplace’s equation shows that, from any solution, others may be derived by differentiation with respect to the variables x, y, z; or, more generally, if

*j ∖∂x ∂y ∂z∕*

denote any rational integral operator,

*J* ∖∂x *∂z∕ .*

is a solution of the equation, if V satisfies it. This principle has been applied by Thomson and Tait to the derivation of the system of any integral degree, by operating upon *ι∕r,* which satisfies Laplace’s equation. The operations may be conveniently carried out by means of the following differentiation theorem. (See papers by Hobson, in the *Messenger of Mathematics,* xxiii. 115, and *Proc. Lond. Math. Soc.* vol. xxiv.)

, p a a∖ι , τU2n)! 1 ( rV

*jn∖∂xi ∂y'∂zj r k } ~2⅛P. r2n+1( 2.2n-ι*

~~+~~~~2.4.2∕→.2n-3~~~ · · · ∖f^x' y' 2) -<7)

which is a particular case of the more general theorem

, ∕ ∂ ∂ a ∖ 17∕ χ \_ ( n <fnF f 2n-2 <Zn-1F ».

*fn ∖∂x' ∂y' ∂z)* **~ 1 2** *d(^)n+* I ! d(r≈)"~1^+ ' ’ ’

**on—2\* y√n-**íT? )

+⅜τ ~~d(r~~~~2~~~~)"-\*~~7ii,+ · · · Vn(x' y, <7\*>∙ where ∕n(x, y, 2) is a rational integral homogeneous function of degree *n.* The harmonic of positive degree *n* corresponding to that of degree—n —1 in the expression (7) is

í 1 -2-⅞⅛+~~2~~~~.4.2~~n~~-7,⅛~~-■ · · k”(\*’ \* 2>∙

It can be verified that even when *n* is unrestricted, this expression satisfies Laplace’s equation, the sole restriction being that of the convergence of the series.

5. *Maxwell's Theory of Poles.—*Before proceeding to obtain by means of (7), the expressions, for the zonal, tesseral and sectorial harmonics, it is convenient to introduce the conception, due to MaxwelI (see *Electricity and Magnetism,* vol. î. ch. ix.), of the poles of a spherical harmonic, . Suppose a sphere of any radius drawn with its centre at the origin; any line whose direction-cosines are *l, m, n* drawn from the origin, is called an axis, and the point where this axis cuts the sphere is called the pole of the axis. Different axes will be denoted by suffixes attached to the direction-cosines ; the cosine (ZιX-f-mιy+nts)∕r of the angle between the radius vector *r* to a point (x, *y,* 2) and the axis (It, *mi, nι),* will be denoted by λι; the cosine of the ang!e between two axes is *ldγ+m,mγ+neny,* which will be denoted by μt,γ. The operation

z\*⅛÷m⅛+n⅛ performed upon any function of *x, y, z,* is spoken of as differentiation with respect to the axis *(lι, mi, nff),* and is denoted by *∂∣∂hι.* The potential function Vo=^o∕r is defined to be the potential due to a singular point of degree zero at the origin ; βo is called the strength )f the singular point. Let a singular point of degree zero, and strength e0, be on an axis *hi,* at a distance α0 from the origin, and also suppose that the origin is a singular point of strength—e0; et e0 be indefinitely increased, and α0 indefinitely diminished, but so that the product e0a0 is finite and equal to *e0*; the origin is then said to be a singular point oí the first degree, of strength *ei,* the axis being *h1.* Such a singular point is frequently called a doublet, in a similar manner, by placing two singular points of degree, unity and strength, *eu — ei,* at a distance αi along an axis ⅛ and at the origin respectively, when *eι* is indefinitely increased, and αi diminished >0 that *eιαι* is finite and=β2, we obtain a singular point of degree 2, îtrength *e2* at the origin, the axes being ⅛1, *h2.* Proceeding in this nanner we arrive at the conception of a singular point of any degree ‰ of strength *en at* the origin, the singular point having any *n* given ιxes Äi, *h2,. . .hn.* lf en-ι ≠n-ι (x, *y, z)* is the potential due to a singular point at the origin, of degree *n — i,* and strength en^ι, ⅛vith axes *hi, fi2,.∙∙hn~ι,* the potential of a singular point of degree î, the new axis of which is *hn,* is the limit of

**£n-l φru-l** *(x~lna, y-mna,* **2—∏nα)-βn^ι φn-l (x,** *y, z)∖* **when**

**La—O, Lßn—1≡05, La€n—***1—en,*

his limit is

-e„ *(l* n⅜d+‰¾-l+"-⅛) . or ~i^⅛"-ι∙

since *φ<i≈ι∕rt* we see that the potential V, due to a singular point it the origin of strength *en,* and axes *hi, h2, . . Jιn* is given by

V„ = ( -1 ) *nengj^-* mj∙ (8)

6. *Expression for a Harmonic with given Poles.—*The result of oerforming the operations in (8) is that Vn is of the form

cvhere Yn is a surface harmonic of degree *n,* and wi!l appear as a ’unction of the angles which *r* makes with the *n* axes, and of the ingles these axes make with one another. The poles of the *n* ιxes are defined to be the poles of the surface harmonics, and are dso frequently spoken of as the poles of the solid harmonics Vnrn, Ynr,l'"1. Any spherical harmonic is completely specified by means of its poles.

In order to express Yn in terms of the positions of its poles, we ιpply the theorem (7) to the evaluation of Vn in (8). On putting

***r* = n**

∕n(x, *yt* z) *=U(lrx+mry+nrz),* we have

**f ='I**

γ - <2wi∖ . *1 (l esl* I *csl* ∖ y

n 2nn!ni *rn ∖* 2.2» — ι^r2.4.2n- *1.2η—*3’ ”/ ^

*η*

Π(∕rx÷wry+Mrz). \_

I

By ∑(μ∙λn~2,) we shall denote the sum of the products of 5 of the quantities *µ,* and *n~2s* of the quantities λ; in any term each suffix is to occur once, and once only, every possible order being taken. We find

Π(Zx+wy+w2) =∑(λn)rn, Δ2Π(Zx-∣-my-∣-Mz) =2∑(μ1λn^2)rn"2,

i∏d generally

*∆2m∏ (lx* -J- *my* -j- *nz)* = 2 m *m* ! S (μτnλn^m) *rn^m ;*

thus we obtain the following expression for Yn, the surface har­monic which has given poles *hi, h2, . . .hn∖*

γ **\_fn÷l(~1)"** θn , £

*nl ∂hι∂h2.. .∂hn r*

-Sb->~~v⅝r⅛~~iλ∙^,∙⅛->i ⅛>

where S denotes a summation with respect to *m* from m = o to ιw = ⅛w, or !(«“!)» according as *n* is even or odd. This is Maxwell’s general expression *(loc. cit.}* for a surface harmonic with given z>oles.

Ií the. poles on a sphere of radius *r* are denoted by A, B, C. . ., ve obtain from (9) the following expressions for the harmonics of die first four degrees:—

Yι=cos PA, Y2 = i(3 cos PA cos PB—cos AB),

^3 —⅛(15 cos PA cos PB cos PC—cos PA cos BC—cos PB cos CA

—cos PC cos AB),

^< = ⅛(35 cos PA cos PB cos PC cos PD—5∑ cos PA cos PB cos CD

+ ∑cos AB cos CD).

7. *Poles of Zonal, Tesseral and Sectorial. Harmonics.—*Let the *n* 1xes of the harmonic coincide with the axis of *z,* we have then by (8) the harmonic

**(-l)γn+l θn 1.**

*n∖ ∂Znrf*