applying the theorem (7) to evaluate this expression, we have

**(—ι)nrn+ι** a⅛ I \_ *(2n)l* I ( r2y2 r4y4 )

*n∖ ∂zη r ~2nnlnl rη ( 1* 2 . 2tt-l"Γ2.4.2η —1 . 2n—3 ) =-fe⅜i μn-⅛-1i.tn→+ ,

*2nn[n∖ ∖ m* 2.2η —im ~ \* \* \* V

the expression on the right side is Pn(μ), the zonal surface har­monic ; we have therefore

P ΛΛ-(~l)n'n+1 ∂n J

*r"w n∖ ∂z^ f*

The zonal harmonic has therefore all its poles coincident with the *z* axis. Next, suppose *n — m* axes coincide with the *z* axis, and that the remaining *m* axes are distributed symmetrically in the plane of *x, y* at intervals *π/m,* the direction cosines of one of them being cos α. sin a, o. We have

∏ I cos (a÷^)⅛ + Sin (a+3⅛ j ≈⅜∏ ∣ ⅛~⅛)

**+e-'G⅛)(⅛+lg }.**

Let £ = *η≈x-ιyt* the above product becomes ∏μ⅛<∙+¾!∙ which is equal to

j ; when α = o, ⅛ this becomes

**i<->ψj (⅛)--l-,>∙(⅛)- p ',-,⅛'[(⅛)-+<->^(⅛r ].**

From (7), we find

*∂zn~m* \óx *∂y∕ r* 2nn!r2ft+1L 2.2Zi-r,,,J '

= ("1)n¾¾\* F∏<cos w≠rfct sin w≠) sinτn0 J cosn-m0 *(« — m){n-m~***1) w m 9λ i )**

⅛ ⅛os-≡0+... J, hence

⅛=s θxdz⅛) 7 = (~1)"⅛π^-i(∞s w<⅛±i sin rnψ)P"(cos *θ),*

as we see on referring to (4) ; we thus obtain the formulae 5≡H⅜)'+(⅜)Ι,i=<->-≈⅛⅛'~ »>1 , ,

*ε⅛***(⅞)∙-(⅜)^jj-<->∙a⅛⅛'.i. ’**

It is thus seen ,that the tesseral harmonics of degree *n* and order *m* are those which have *n—m* axes coincident with the *z* axis, and the other *m* axis distributed in the equatorial plane, at angular intervals *π∣m.* The Sectorial harmonics have all their axes in the equatorial plane.

8. *Determination of the Poles of a given Harmonic.—*It has been shown that a spherical harmonic Yn(x, *yt* 2) can be generated by means of an operator

*- Id d d∖ .* **I**

**Afø'ã? ⅛)actm≡uP°n<**

the function *fn* being so chosen that

Yn(x, y, z) = (- 1)n⅛⅛ j I -2T2⅛ι+ · · · *r∙ β) i*

this relation shows that if an expression of the form

(x2+y2+2‰(x, *y, z)*

is added to ∕n(x, *y, z}t* the harmonic Yn(x, *yi z)* is unaltered; thus if Yn be regarded as given, *fn{xt y, z)* = 0τ is not uniquely deter­mined, but has an indefinite number of values differing by multiples of *x2+y2A∙z2.* In order to determine the poles of a given harmonic, *fn* must be so chosen that it is resolvable into linear factois; it will be shown that this can be done in one, and only one, way, so that the poles are all real.

If x, y, *z* are such as to satisfy the two equations Yn(x, *yt z)* =0, x2÷-y2÷z2≈0, the equation ∕n(x, y, *z)* is also satisfied; the problem of determining the poles is therefore equivalent to the algebraical one of reducing Yn to the product of linear factors by means of the relation x2-j-y2-∣-22=O, between the variables. Suppose

*m*

Yn(x, *y>* 2) =AII(Z,.τ+w√+n42)+(x2+y+z2)Vn-2⅛, *y, z),*

**J-I** we see that the plane *ltx+msy+n\*z = 0* passes through two of the *2n* generating lines of'the imaginary cone x2+y2-∣-z2 = 0, in which that cone is intersected by the cone Yn(x, *y,* z)=0. Thus a pole *‰ m8, n∙)* is the pole with respect to the cone x2+y2-∣-22=O, of a plane passing through two of the generating lines; the number of systems of poles is therefore *n{2n-*1), the number of ways of taking the *2n* generating lines in pairs. Of these systems of poles, however, only one is real, viz. that in which the lines in each pair correspond to conjugate complex roots of the equations Yn≈0, x2 \_|\_22 \_ θt Supp0sc

*χ \_ y z*

**<⅛+i3ι α2+i∕⅜ α3+^3**

gives one generating line, then the conjugate one is given by

*x \_ y z*

αι — *ιβι* α2 — *íßì* a3 — *ιβ3t ·* and the corresponding factor *lxA~my-∖-nz* is

*x y z*

**αι+\*βι α2 + ^2 α3+433 , C⅛-ιβι a2-ι^2 as —** *tβ3*

which is real. It is obvious that if any non-conjugate pair of roots is taken, the corresponding factor, and therefore the pole, is imaginary. There is therefore only one system of real poles of a given harmonic, and its determination requires the solution of an equation of degree 2n. This theorem is due to Sylvester *{Phil. Mag.* (1876), 5th series, vol. ii., “ A Note on Spherical Harmonics ”).

9. *Expression for the Zonal Harmonic with any Axis.—*The zonal surface harmonic, whose axis is in the direction

r' √ *zf . n Ixxf-∖-yy, A-zzf∖*

**r∕, ‘T, IS P„( rr, )**

or Pn(cos *θ* cos'Θ'+ sin *θ* sin *θf* cos ≠-≠') ; this is expressible as a linear function of the system of zonal, tesseral, and sectorial har- monics already found. It will be observed that it is symmetrical with respect to (x, y, z) and (x∖ y', z'), and must thus be capable of being expressed in the form

α0Pn(cos θ)Pn(cos *θ'^)* +∑αmPJ,(cos 0)P^(cos 0')cos *m(φ-φf),*

and it only remains to determine the co-efficients αo, αι, *...aτη...aη.* To find this expression, we transform (√x+y'7+z'z)n,. where

x, *y, z* satisfy the condition x2+y2+z2 = 0; writing ξ = x+t3½ *η=x-ιy, ξf≈xfA∙ιyf, ηr=X,-Ly,,* we have

*(xx'+yy'+zz,)n ≈(⅛ifξ+*⅛ξ⅛ ÷zzx)ft

which equals

n «I *( faffbfa 6* I *fbt-∕aJ, a ∖*

*(^∖ lL· I (^fλn-a-b*

*k 7 a∖b∖{n-a-b)∖ (* Γ '

the summation being taken for all values of *a* and *bf* such that *aA~b≤nf a>b;* the values α = 0, ö = 0 corresponding to the term (zz')n. Using the relation *ξη≈ -z2,* this becomes (x√+w'+zz')" = (≈')"+∑2^⅛-δ ~~q⅛~~l~~(~~~~w~~~~i~~~~i~~~~α~~~~,⅛)t~~⅛V)⅛'n~o~ιl

((√0a^6+(i'η)°-6)zn-o+δ, putting *a-b = m,* the coefficient of *ξmzη~mt* on the right side is X⅛⅛δ ~~¾!(~~~~w+~~~~¾)i⅞L~~~~w~~~~-~~~~2~~~~⅜)!~~⅛⅛0t√^"-→∖

from *b = 0* to *b = ⅛(n-m)t* or ⅛(n-w —1), according as *n—m* is even or odd. This coefficient is equaI to

(n-m)(w-m-l)(n-TO-2)(¾->n-3) w . w,, ) .

2.4.2m+2.2m+3 κ ' 7 ' } t

in order to evaluate this coefficient, put 3 = 1, *x, =* t cos α,

*y, ≈ι* sin a, then this coefficient is that of (t cos a+sin *a)m,* or of

in the expansion of *{zf A∙ιx,* cos a-Hy' sin a)n in powers of er"ta and eta, this has been already found, thus the coefficient is c⅛-→'P^(cos0')∙r'..

Similarly the coefficient of ηtn2n-m is

*, -111 e+imfVΡτη (rcπ Q,∖τ,n∙* (w÷w)!e rntcosσ)r , hence we have z

^i(xx'+^z+z2z)n = 2nPn(cos 0')4-n!∑PJl(cos 0')(cos *mφ'⅛m+vm)*

*r>n—m*

÷t sin W'⅛m-Γ)l(w+mji∙

In this result, change *xt y, z* into

*d\_ d\_ d\_*

*∂xt ∂y, ∂zi*