and let each side operate on ι∕r, then in virtue of (10), we have

(r∕)ftPn =pn(cos *Θ* cos 0'-∣-sin *θ* sin *θr* cos *φ-φ')*

=Pn(cos 9)Pn(cos *θ')* +2∑~~^ξ-%>fo~~(cos ff)P"(cos 9,)cos *m* (φ - ≠') (I I) which is known as the addition theorem for the function Pn.

lt has incidentally been proved that

P™ (cose) °~~⅜Z¾!~~s'>nmg j ∞s"s

\_(W-W)(n w-1) -ifl ),

2.2 m 4-2 ) z

which is an expression for Ρ» (cos 0) alternative to (4). .

10. *Legend re's Coefficients.—*The reciprocal of the distance of a point (r, *θ, φ)* from a point on the *z* axis distant *r,* from the origin is (r2-2rr, μ+r'2)-\*

which satisfies Laplace’s equation, *µ* denoting cos *θ.* Writing this expression in the forms

P I I p. 7 j I -2>+ÿ} i∙

it is seen that when *r < r',* the expression can be expanded in a convergent series of powers of *r∣r,,* and when *r, < r* in a convergent series of powers of *r,fr.* We have, when *h2(2μ-* Zt)2<1

(1—2⅛-H2) i = ι-H(2μ-½)4~--⅛2(2μ-λ)2÷...

~~1.3.5...·~~2~~n-1~~~~ÅW~~~~( ..~~

\* 2.4. . *.2n J*

and since the series is absolutely convergent, it may be rearranged as a series of powers of *h,* the coefficient of *hη* is then found to be 1.3∙5∙.∙2h-I ( nfø-i) 1 n(n-ι)(w-2)(n-3) n-4 *ì*

ï.2.3...» I 2.2W-Im ‘ 2.4.2η —1.2n—3 ” \*\*’)

this is the expression wc have already denoted by Pn(μ) ; thus

(I *-2hμ+hι)~i* =Po(μ) +⅛P1(μ) + . . +⅛nPn(μ) + ∙ . ., o3)

the function Pn(μ) may thus be defined as the coefficient of *hn* in this expansion, and from this point of view is called the Legendre’s coefficient or Legendre’s function of degree *n,* and is identical with the zonal harmonic. It may be shown that the expansion is. valid for all real and complex values of *h* and *µ,* such that mod, *h* is less . than the smaller of the two numbers mod. (μ4=√μ2 -1). We now see that

(r2-2r√μ÷√2Γi

is expressible in the form

⅛w

0

when *r* < r', or

0

when r' < r; it follows that the two expressions rnPn(μ), Γ"ft-1P√μ) are solutions of Laplace’s equation.

The values of the first few Legendre’s coefficients are

P0(μ)=I, Pι(∕i)=μ, P2(μ)=^(3M2-l), Ps(μ)=^(5Ms-3Zi) P4(μ) = ∣(35m4-3oμ2÷3), Pδ⅛)≈∣(63μ5-7oμs+15Λi) pβ (μ) = γg (231Mβ -31 5m4+1o5μ2 - 5), p7 (μ) = ⅛(429m7 - 693∕i5 ÷3i5λis-35λt)∙ We find also

Pn(ι)=≡ι, P∏(-ι) = (-ι)n P.(0) =0, or (-ι)i"~~l~~~~∙~~~~3~~~~2~~~~5~~~~4~~ ~~7^-~~~~l~~

according as *n* is odd or even; these values may be at once obtained from the expansion (13), by putting µ = 1, o, — 1.

ιι. *Additional Expressions for Legendre's Coefficients.—*The expression (3) for Pn(μ) may be written in the form

r\* z x (2n) ! nT? ∕ *n* 1 —« I ι∖

p∙°λ) =≡πr f (-≡, *~, 2~η,*

with the usual notation for hypergeometric series.

On writing this series in the reverse order

*„ , . z ..m n∖* τ,∕nn + iiΛ

-Pn(μ)=(-I)⅜" ~~∖ .7~~T∖ ,F *’ 2' μ)*

2"bn),∙bn)1

or

,>r(-⅞M+∙∙P)

2 2

according as *n* is even or odd.

From the identity

(1 — *2h* cos Θ÷Zt2)-i = (1 — Aet0)\*l(ι —⅛~l0)~b

it can be shown that

~~P~~~~’~~~~(cos 0)~~ ~~"’¾é.'.~~~~2~~~~.!?~~~~1~~ **j cos 'ιg+**~~ι.'2n-ι~~~~cos (w~~~~“~~~~2~~~~)~~~~0~~

~~+ ι.2.⅛-⅞n-~~~~3~~~~)~~**cos <-η^4)ρ+ l· <14)**

By (13), or by the formula

which is known as Rodrigue’s formula, we may prove that Pn(cosθ) = 1-⅛⅛siniξ+⅛) ~~- ι)~~~~sin~~~~∣,,~~

-F (n÷ι, *-η,* I, sin2∣) . (15)

Also that Pn(cos *θ)* =cos2"∣ j I -p∙ tan2∣+~~w~~ ~~p ~~~~~a~~~~1~~~~∙~~~~,~~ tan<∣- · · · ∣

= cos2n⅛ *n, —n,* 1, — tan20 . (16)

By means of the identity

**(l-2⅛μ+⅛η-i = (l-⅛)-> j** *l+⅜ff* **p∙**

it may be shown that

Pn(cosO) =cos-e j 1-⅜2)tan¾+~~w~~~~(~~~~w~~~~-~~~~1~~~~)^-~~~~2~~~~)(~~~~w~~~~-3)~~taπ<e-... j

= cosn0F(-⅛n, ⅛-⅛n, 1, — tan20). (17)

Laplace’s definite integral expression (6) may be transformed into the expression

**£ Γ jγ** *dφ*

τrj 0 ⅛ — V μ2 — I cos ≠) n+ν by means of the relation

(μ~h √μ2 — 1 cos φ) (µ — √μ2 - I COS ≠) = 1.

Two definite integral expressions for Pn(μ) given by Dirichlet have been put by Mehler into the forms

Pn(cosβ) =≡ P ∞⅞(n÷⅜)≠ -d 2 p sin (n÷.⅜)⅜,

V oV2 cos *φ-*2 cos 0 7Γ√ 0√2 cos *Θ — 2* COS *φ*

When *n* is large, and 0 is not nearly equal to o or to π, an approximate value of Pn(cos0) is (2∕wτr sin 0)⅜ sin ((n +1)0 + ⅛π}.

12. *Relations between successive Legendre's Coefficients and their Derivatives.—*If (ι — *2hμ~{-h2)~i* be denoted by *u,* we find

**(l-2⅛+⅛≡)g + (A-μ)w=0ι**

on substituting ∑ΛnPn for *ut* and equating to zero the coefficient of *hη,* we obtain the relation

*nρn - (2n -* I)√P^1+(n - I )Pn\_2=0. From Laplace’s definite integral, or otherwise, we find

**/?P**

0i2 ^ = η 0ipn - P"“1) = - (« + 1) - pn+1) \*

We may also show that

"⅛--≡r=wp" (n+l)P.= -μ¾+¾1 (2w+1)P->=⅛-⅛1 ^+i)⅛=("+^⅛+⅛ (2« +1) (µ’ - I)Ç = w(» +1) (Pn+1 -Pn-1) ^ = (2n-ι)P^1+(2n-5)P^s+(2n-9)Pn-5+... the last term being 3P1 or Po according as *n* is even or odd.

13. *Integral Properties of Legendre's Coefficients.—*ît may be shown that if Pn⅛) be multiplied by any one of the numbers 1, *µ, μ-, ... μ1i~l* and the product be integrated between the limits 1,— 1 with respect to *μi* the result is zero, thus

Jr √Pn(μ)⅛ =0, α=0, I, 2, . ..Μ —I. (l8)

To prove this theorem we have J11√P"(m)⅛ = 5⅛J1-X⅛k2 - I)n⅛,