on integrating the expression *k* times by parts, and remembering that *(μ2-* i)n and its πrst *n —* I derivatives all vanish when *μ=* ≡\*=ι, the theorem is established. This theorem derives additional importance from the fact that it may be shown that APft(μ) is the only rational integral function of degree *η* which has this property; from this arises the importance of the functions Pn in the theory of quadratures.

The theorem which lies at the root of the applicability of \* the functions Pft to potential problems is that if *n* and *n'* are unequal integers

J^ιPo∕(μ)Pn(μ)⅛=0, (19)

which may be stated by saying that the integral of the product of two Legendre’s coefficients of different degree taken over the whole of a spherical surface with its centre at the origin is zero ; this is the fundamental harmonic property of the functions. It is immediately deducible from (18), for if *nf <n,* Pn√∕χ) is a linear function of powers of *μt* whose indices are all less than n.

When n'=n, the integral in (19) becomes J\*1 JPn(ju)]2⅛G to evaluate this we write it in the form

on integrating *n* times by parts, this becomes ⅛⅛Jl1Gx2-1)n⅛(μ2-l)n⅜1 or ⅛⅛J'1(I -μ')nM, which on putting

i4=∣(ι-*μt* becomes *-u)ηdut*

hence

J>.W⅛=⅛. (≡0)

14. *Expansion of Functions in Series of Legendre’s Coefficients.—* If it be assumed that a function ∕(μ) given arbitrarily in the interval µ= —I to 4-1, can be represented by a series of Legendre’s co- efficients α04-α1P1(μ)+α2P2(μ)÷. · .+c⅛Pn(μ)+.. .and it be assumed that the series converges in general uniformly within the interval, the coefficient *a* can be determined by using (19) and (20) ; we see that the theorem (19) plays the same part as the property í ∞sw^cosw ^~θ, (nφn') does in the theory of the expansion of ∙∕ o functions in series of circular functions. On multiplying the series by Pn (µ), we have

Onj"^ι(Pn⅛)]2⅛ = J\*^∕(μ)Pn(μ)⅛ hence

hence the series by which ∕(μ) is in general represented in the interval is

2⅛ip^)Jl∕^')pn0x')⅜,∙ (21)

The proof of the possibility of this representation, including the investigation of sufficient conditions as to the nature of the function ‰), that the series may in general converge to the value of the function requires an investigation, for. which we have not space, similar in character to the corresponding investigations for series of circular functions (see Fourier’s Series). A complete investi- gation of this matter is given by Hobson, *Proc. Loηd. Math. Sοc.,* 2nd series, vol. 6, p. 388, and vol. 7, p. 24. See also Dini’s *Serie di Fourier.*

The expansion may be applied to the determination at an external and an internal point of the potential due to a distribution of matter of surface density ∕(μ) placed on a spherical surface *r = a.* If V1=2A' sSiPn(μ), V.=2A„^P„Gu),

we see that Vt, Vo have the characteristic properties of potential functions for the spaces internal to, and external to, the spherical surface respectively; moreover, the condition that Vι is continuous with Vo at the surface r=α, is satisfied. The density of a surface distribution which produces these potentials is in accordance with a known theorem in the potential theory, given by

I *(dVt,* aν0∖ σ 4π ∖ *dr ∂r ) r a,* hence

σ = ^-2∑(2n4-1)AftPn(μ) ; on comparing this with the series (21), we have *An≈21raiJ'τ^f(μf)Ρn(μf)dμf,*

hence

Vι = 2τrαΣ^Pn(μ) *j∖τf(j\*')* Pn(√) *dμr*

λΛ+1 *r'* I

Vo = 2TaS^P„Gz)J *J{μf)Vn{μf)dμ,* are the required expressions for the internal and external potentials due to the distribution of surface density ∕(μ).

15. *Integral Properties of Spherical Harmonics.—*The fundamental harmonic property of spherical harmonics, of which property (19) is a particular case, is that if Yn(x, y, 2), Zn<(r,y,z) be two (ordinary) spherical harmonics, then,

jJγn(x, *y,* z)Zn,(x, *y,* x)dS = 0, (22)

when *n* and *nr* ar≠- unequal, the integration being taken for every element dS of a spherical surface, of which the origin is the centre.

Since V2Yn = 0f V2Zf√ = 0, we have *fff* (Yn√‰ - *Zη,Wηi)dxdydz*=0, the integration being taken through the volume of the sphere of radius *r* ; this volume integral may be written

*CC( ∖ d (v öZ“' 7* aγΛ . a ÍV öZ"' 7 ôYA

*J J J 1 ∂^x* c⅛-z^'‰7 +ãÿ √n^0F -z"'∙^r}

*+ ⅛(γ-%'-z∙∙⅛) ∖dxdydz=0∙,* by a well-known theorem in the integral calculus, the volume integral may be replaced by a surface integral over the spherical surface; we thus obtain ∕∫i ; (γ-⅛-z-⅛) +? (γ∙⅝-z-¾)

+Kν⅛-7~¾) j∙≈-"i on using Euler’s theorem for homogeneous functions, this becomes whence the theorem (22), which is due to Laplace, is proved.

The integral over a spherical surface of the product of a spherical harmonic of degree n,.and a zonal surface harmonic Pn of the same degree, the pole of which is at *(x', y', z,)* is given by

jjγn(x, *y,* 2)P^S=⅛-α^Yn(√, *y',* s'); (23)

thus the value of the integral depends on the value of the spherical harmonic at the pole of the zonal harmonic.

This theorem may also be written

Jì·f 1Vft(Θ, φ)Pn(cos 0 cos 0’4-sin *d* sin *θf* cos *φ- φ')dμdφ* =⅛vn(θ',φ'). To prove the theorem, we observe that Vft is of the form

*» m* α0Pn(μ)+2(αm cos *mΦ+bm* sin *mΦ)Ρn ⅛) ;*

I to determine αo we observe that when *μ = ιt*

Pn(μ) = I,P-(μ)=0j hence α0 is equal to the value Vn(0) of Vn *(Θ, Φ)* at the pole 0 = 0 of Pn⅛). Multiply by Pn(μ) and integrate over the surface of the sphere of radius unity, we then have

=⅛>=⅛v"(°)> if instead of taking *µ* = 1 as the pole of Pn(μ) we take any other point *{μ,, φ')* we obtain the theorem (23).

If *f(x,* y, 2) is a function which is finite and continuous through- out the interior of a sphere of radius R, it may be shown that jfJγ.(χ, *y, Z)f(,x, y,* ≡)<\*S=4\*R4"+⅛¾d 1+r¾ ~~+~~~~2~~~~.~~~~4~~~~.2n+‰~~~~w~~~~+~~~~5~~+∙∙∙jγ"⅛ ⅛ where x, y, *z* are put equal to zero after the operations have been performed, the integral being taken over the surface of the sphere of radius R (see Hobson, “ On the Evaluation of a certain Surface Integral,” *Proc. Lond. Math. Soc.* vol. xxv.).

The following case of this theorem should be remarked: If *fn(x, y1* z) is homogeneous and of degree *n ffΥn(x, y, z)fn(x, yl z-)dS=4^^^n(fχ∙* ⅛ ⅛)∕n⅛J.≡) if ∕n(x, y1 z) is a spherical harmonic, we obtain from this a theorem, due to Maxwell *(Electricity,* vot i. ch. ix.),

jJγ.(x, *y, z)fn(x, y, z)dS=^^ ~~∂h^7^Ji^~~~~x~~~~'~~ ~~z~~~~)~~*