where *hiht...hn* are the axes of Y∏∙ Two harmonics of the same degree are said to be conjugate, when the surface integral of their product vanishes; if Yn, Zn are two such harmonics, the addition of conjugacy is

γ"(⅛⅛⅛)z"<x-\*z>=0∙

Lord Kelvin has shown how to express the conditions that 2w + ι harmonics of degree *n* form a conjugate system (see B. *A. Report,* 1871).

16. *Expansion of a Function in α Series of Spherical Harmonics.—* It can be shown that under certain restrictions as to the nature of a function F(μ, ≠) given arbitrarily over the surface of a sphere, the function can be represented by a series of spherical harmonics which converges in general uniformly, t On this assumption we see that the terms of the series can be found by the use of the theorems (22), (23). Let F(μ, ≠) be represented by

Vo(μ, *Φ}* +Vι(μ, *φ)* + . . . +Vn(μ, *Φ) +. . . î* change *μ, φ* into *μ', φf* and multiply by

Pn(cos *θ* cos Θ'+sin *θ* sin *θ'* cos *Φ~Φ'),* we have then

yi1F(√, ≠,)Pπ(cos 0 cos 0'-J-sin *Θ* sin *θ,* cos *φ~φ')dμ'dφ'* J\*θVnG0 φ')P∏(cos 0 cos 0'-⅜-sin 0 sin 0' cos *φ-φi)dμ'dφ,* hence the series which represents F(μ, *φ)* is

co

(2"+1) f∩,r f1,F(p'. √)Pn(cθS β cos *θ'*

0

-∣-sin *θ* sin *θ,* cos *φ-φ,)dμfdφft* (24)

A rational integral function of sin *θ* cos *φ,* sin 0 sin ψ, cos 0 of degree *n* may be expressed as the sum of a series of spherical har- monics, by assuming

∕ft(χ, y, z) =Yn-hr2Yn-2+r4Yn-4 + ...

and determining the solid harmonics Yn, Yrt-2,. -. and then letting *r=* I, in the result.

Since v2(r2∙Yn-2,) =2s(2n-2s+ι)r2,~2Yn-2i, we have

V2∕ft = 2(2n-l)Yft-2+4(2M-3)r2Yn-4+6(2n-5)r4Yn-β+. . . V4∕n = 2.4(2n - 3) (2*n* - 5)Yn-4+4 · 6(2*η* - 5) (2n - 7)r≡Yft-β 4-... the last equation being

Vn∕n = w(n-}-ι)(n-2)(n-1).. .Y0, if *n* is even, or

Vn"1∕n = (w-1)(n+2)(n-3)n.. .Yi, if *n* is odd from the last equation Yq or Yι is determined,, then from the pre- ceding one Y2 or Y31 and so on. This method is due to Gauss (see *Collected Works,* v. 630).

As an example of the use of spherical harmonics in the potential theory? suppose it required, to calculate at an external point, the potential of a nearly spherical body bounded by r = α(ι -Few), the body being made of homogeneous material of density unity, and *u* being a given function of 0, *φ,* the quantity \* being so small that its square may be neglected. The potential is given by

*Jq1γJ'1-iJo I÷eM ∖r2+r'2-2rr'* cos *7C\*dr'dμ'dφ',* where *γ* is the angle between *r* and r'; now let *ut* be expanded in a scries

Y0(μ', φ')+Yι(μ', φ') + . . . +Y√√, Φ')+ . . .

of surface harmonics ; we may write the expression for the potential

+⅛P.(∞s 7) +... j *rlWdμldφ'* which is,

∫Γ∫lι í fJ<1+3\*"')+J⅛+4<u')Pι + . -.

+⅛3 ‰τ(1+n+3f",)p"(c∞ Ό *^dμ,dφ,* on substituting for *uf* the series of harmonics, and using (22), (23), this becomes

4^≡ [”+4 *fa(\* Φ)* +⅛γ><M, Φ) + · · · +^T⅛γ^≠)+-∙∣] which is the required potential at the external point (r, *θ, Φ).*

17. *The Normal Solutions of Laplace's Equation in Polars.—*If Λι, *hi, hi* be the parameters of three orthogonal sets of surfaces, the length of an elementary arc *ds* may be expressed by an equation of the form *dsi = jρdh2 + jjιdhι + jjιdh%>* where Hb H2, Hj are functions of *hi, hi, h3,* which depend on the form of these parameters; it is known that Laplace’s equation when expressed with *hi, h2, h* as independent variables, takes the form

±∕ Hi aV∖ , ∂ ∕ H2 3V∖ , *∂ ∕ Hs ∂V∖ n f s*

∂Λ1∖H2H3 *∂hj ∂hι* ∖HjHι *∂iι∙J dh$* ∖H1H2 *∂h'J \**

In case the orthogonal surfaces are concentric spheres, co-axial circular cones, and planes through the axes of the cones, the para­meters are the usual polar co-ordinates *r, θ, φ,* and in this case H1 = 1, Hi = i, Ha = -r1 thus Laplace’s equation becomes

*dr* V *dr)* sin *θ dθ∖ dθ)* ψsin20 *∂φi ^0∙*

Assume that V = RθΦ is a solution, R being a function of *r* only, θ of *θ* only, Φ of *φ* only; we then have

I *d* ∕ 2dR∖ , I *d (· iflQ∖ 1.* I d2Φ,n Rdr∖- *dr)* θ sin *Θ dθ ∖ dθ)* \* sin20 .Φ *dφ2*

This can only be satisfied is a constant, say

n(n + 1), is a constant, say—*m2,* and θ satisfies the equation

⅛J(∙"^⅛)+S-<∙+∙>-⅛iθ-"∙

if we write *u* for θ, and *μ* for sin 0, this equation becomes ⅛{cι-μ⅛i + ^w+1)-T⅛iw=0∙ · <26> From the equations which determine R, θ, *u,* it appears that Laplace’s equation is satisfied by

where *u* is any solution of (26); this product we may speak of as the normal solution of Laplace’s equation in polar co-ordinates; it will be observed that the constants *n, m* may have any real or complex values.

18. *Legendre's Equation.—*If in the above normal solution we consider the case w = 0, we see that

*rη*

*γ^n-lun*

is the normal form, where *un* satisfies the equation ⅛{<i-^⅛J+m(,i+i>=0- · · <27)

known as Legendre’s equation; we shall here consider the special case in which n is a positive integer. One solution of (27) will be the Legendre’s coefficient Pn(μ), and to find the complete primitive we must find another particular integral; in considering the forms of solution, we shall consider *µ* to be not necessarily real and between =M. If we assume

*u* — μrn+α2μw-2+α4μm-4+...

as a solution, and substitute in the equation (27), we find that *m=n,* or—η —I, and thus we have as solutions, on determining the ratios of the coefficients in the two cases,

^i<∙-⅛≡i,\*∙"+-S

and

*Q (* I , (n+1)(n÷2) 1 ■ (n + 1)(n+2)(n+3)(n+4) I . )

p ( μn+1^r 2.2∏+3 μn+3^1^ 2.4.2n+3.2n+5 μn+5 ^1^ ’ \* \* J

the first of these series is (n integral) finite, and represents Pn⅛), the second is an infinite series which is convergent when mod *µ* > 1. If we choose the constant *ß* to be 1 , ~~,~~~~^∣√r~~> the second

solution may be denoted by Qn(μ), and is called the Legendre’s function of the second kind, thus

1.2.3. *..η S* 1 « (w + ι)(M÷2) I l J

ynW~3.5...2n-H (μftH"h 2.2n+3 √^^1^∙∙∙ 5

I.2.3...W I τ√"+l W÷2 2n+3 i∖

~~3.5·. 2n÷1µ»÷~~~F (ι~,~ τ^Γ"T7∙ (28)

This function Qn(μ), thus defined for mod *µ >* 1, is of considerable importance in the potential theory. When mod *µ <* 1, we may in a similar manner obtain two series in ascending powers of *μf* one of which represents Pft(μ), and a certain linear function of the two series represents the analytical continuation of *Qn(μ)r* as defined above. The complete primitive of Legendre’s equation is u=AP4μ)+BQnω.

By the usual rule for obtaining the complete primitive of an ordinary differential equation of the second order when a particular integral is known, it can be shown that (27) is satisfied by

Pn(μ) =Jμ(μl- lf(PnGll))≈,

the lower limit being arbitrary.