From this form it can be shown that

*QnW* =1P.(μ) log

where Wn-ι(μ) is a rational integral function of degree *n —* ι in µ; it can be shown that this form is in agreement with the definition of Qn(μ) by series, for the case mod μ>ι. ln case mod μ<ι it is convenient to use the symbol Qn(μ) for

5p"<M) log ⅛-W^1(μ),

which is real when *µ* is real and between ±ι, the function Qn(μ) in this case is not the analytical continuation of the function Qn(μ) for mod μ>ι, but differs from it by an imaginary multiple of Pn(μ). It will be observed that Qn(1), Qn(~ ι) are infinite, and Qn(∞) =o. The function Wn-ι⅛) has been expressed by Christoff el in the form

^pM(/1)+§=Sp^)+gEÎpM((.)+...,

and it can also be expressed ìn the form 1Po(µ)P»-l(µ)+^îP1(µ)PB-2(µ) + . . . +PB\_l(µ)P0(µ).

It can easily be shown that the formula (28) is equivalent to

which is analogous to Rodrigue’s expression for Pft(μ).

Another expression of a similar character is

**QnG\*) » (- I)"⅛ξl ⅛ j Û\*S- 1)"*f*√C√i)n÷l j ·**

It can be shown that under the condition mod (u —√ (w2-1)) >mod (µ —√(μ2-1)b the function *ι∣(μ-uy)* can be expanded in the form Σ(2n + 1)Pn(u)Qn(u) ; this expansion is connected with the definite integral formula for Qn(μ) which was used by F. Neumann as a definition of the function Qn(μ), this is

0∙ω=5β1⅞⅛^

which holds for all values of *µ* which are not real and between ≡\*= I. From Neumann’s integral can be deduced the formula

iZ≠

1 {μ + √ (μ2 — I ). cosh ≠ l"+1,

which holds for all values of *µ* which are not real and between =fc J, provided the sign of √(μ2-1) is properly chosen; when *µ* is real and greater than 1, √ (μ2-1) has its positive value.

By means of the substitution.

{μ÷√ (μ2- 1) .cosh ≠)(μ-√ (μ2-1).cosh χ) = I,

the above integral becomes

Q∏(μ)≈J^0(μ-√(μ2-1).coshχ)⅛, where Xo=-loge^⅛∣.

This formula gives á simple means of calculating Qn(μ) for small values of n; thus

Qo(M)= ∫>=5'o⅛

Qι(μ)=MXo~ √ (μi-l).sinh Xo=μ.jlθg^⅛∣-1.

Neumann’s integral affords a means of establishing a relation between successive Q functions, thus

nQn-(2n - J )μQn-ι-f-(π-i)Qλ-2

= ∣ ∩ wP.(\*⅛+,⅛\*-1JFζ\*<w)~^2w~^μPn-ι(⅜)dtz

= - j∕√2n- l)P»-l(«) =0.

Again, it may similarly be proved that

⅛-⅛=(2w+1>e- ·

19. *Legendre Associated Functions.—*Returning to the equation (26) satisfied by the factor in the normal forms fIn-1 ⅛θs *mφ. u™,* we shall consider the case in which n, *m* are positive integers, and *n>m.* Let *u = (μi-ι)^mυ,* then it will be found that *υ* satisfies the equation

(ΐ-μ2)^-2(»»+ΐ)μ^+(»-»ϋ(»+”ί+»)’'=0·

If, in Legendre’s equation, we differentiate *m* times, we find *z i∖dm^Hu λ∕ i ∖ dm+lu . f w r , flmu* (1-M2)^ra-2(in+1)μ^+l+(w-wi)(n+w÷1)^=0i it follows that o=jjjs>hence «” = (μ\*-ι)4mg^∙

The complete solution of (26) is therefore

when *µ* is real and lies between ±1, the two functions

(I

are called Legendre’s associated functions of degree *n,* and ord?r *m,* of the first and second kinds respectively. When *µ* is not real and between ≡ι, the same names are given to the two functions

in either case the functions may be denoted by PT(μ), Q^(μ).

It can be shown that, when *µ* is real and between d= 1 ™-⅛¾ (ï¾) 4ι>-1)"+m(m+i)-} =F5⅛ (ffi) 4"⅛i∣(∕'-ι)"-(A∙+ι)^+"}∙ In the same case, we find

Pn+2(cos 0)-2(w + 1) cot *θ* Pn+1(cos 0)

+ (n - *m) (n+m*+1 ) P„ (cos 0) = o,

(n-?n+2)P^.2(cos 0)-(2n-∣-3)μPΓ+1(cos *Θ)*

÷(w-∣-wz-∣-1)PΓ(cos 0) =0.

20. *BesseΓs Functions.—*If we take for three orthogonal systems of surfaces a system of parallel planes, a system of co-axial circular cylinders perpendicular to the planes, and a system of planes through the axis of the cylinders, the parameters are z, pf *φ,* the cylindrical co-ordinates; in that case H1 = 1, H2 = i, H3 = i∕p, and the equation (25) becomes

∂2ν , 02ν , I av t 1 a2v

*∂z2 ‘ ∂ρ2 ‘ ρ ∂ρ~'p2 ∂φi* θ’

To find the normal functions which satisfy this equation, we put V = ZRΦ, when Z is a function of z only, R *of p* only, and Φof *φ,* the equation then becomes

I d2Z , I ∕<i2R . I dR∖ , τ 1 <Z2Φ~

Z *dz2* , R ∖dp2 ‘ *ρ dp* ∕ \*^p2 Φ *dφ2* θ\*

1 (∕2Z

That this may be satisfied we must have *2~d^* constant, say *=k2,* I d3Z

Z *d¾* constant, say = — w2, and R, íor which we write *u,* must satisfy the differential equation

*diu* I *du ∕ ,2 rn2∖*

*d?+-p Tp+ ∖\*-f)u~0'*

it follows that the normal forms are *e ^kzc^rnφ.u(kp),* where w(p) satisfies the equation

*d2u ,* 1 *du , ∕ rn2∖ f .*

37+p <⅛ + (1 - 77 “=0∙ <29)

This is known as Bessel’s equation of order *m∖* the particular case *d2u ,* 1 *du 1 z .*

*tf+-pTp+u=0'* (30)

corresponding to m = o, is known as Bessel’s equation.

lf we solve the equation (29) in series, we find by the usual process that it is satisfied by the series

*ì*

pm(1 "-2.2w-∣-2 ' 2.4.2m-∣-2.2ot+4~ · \* · J »

the expression

*Pm* ∖ τ P2 1 *p! I*

2m∏(m) ( 2.2m-f-2"r2.4.2m+2.2w+4 \* \* J

or

Σ(^I)⅞pm+⅞n

2m÷anll(w4-n)∏(n)

is denoted by Jm(p)∙ n"0>

When *m=οi* the solution

*τ P2 1* **P4**

of the equation (30) is denoted by Jo(p) or by J (p).