The Junction Jm(p) is called Bessel’s function of order *m,* and Jo(ρ) simply Bessel’s function; the series are convergent for all finite values of p.

The equation (29) is unaltered by changing *m* into— *m,* it follows that J~m(pj is a second solution of (29), thus in general

*u~*AJm(p) 4" BJ→n(p)

is the complete primitive of (29). However, in the most important case, that in which *m* is an integer, the solutions J-m(ρ), Jm(p) are not distinct, for J→n(p) may be written in the form

Hl—\* 1 ·

M ~m Ñ? (-I)" ∕Λ2n

∖2∕ jl×√∏(n-iκ)∏(n) *∖2)*

n— 0 c0

+(-ι)m(f) ^∏(w¾)∏(i) (5) p-0

now ∏(n-*m)* is infinite when *m* is an integer, and *n< m∙* thus the first part of the expression vanishes, and the second part is (-ι)mJτnW> hence when *m* is an integer J-m(p) = (-1)mJm(p), and the second solution remains to be found.

*Bessel's Functions of the Second Kind.—*When *m* is not a real integer, we have seen that any linear function of Jm(ρ), J→n(p) satisfies the equation of order *m.* The Bessel’s function of the second kind of order *m* is defined as the particular linear function jrcw,.J→n(p) ~ ∞s *mjr ■* Jb,(p) ,

sin *2mπ*

and may be denoted by Ym(p). This definition has the advantage of giving a meaning to Yτn(p) in the case in which *m* is an integer, for it may be evaluated as a limiting form 0/0, and the limit will satisfy the equation (29). The only failing case is when *m* is half an odd integer; in that case we take cos1w7r . Ym(p) as a second finite solution of the differential equation.

When *m* is an integer, we have

Ym(p) = ( - I)1^ ξ ¾=-'- ( - I)m⅛ I e = O

on carrying out the differentiations, and proceeding to the limit we find co

Ym(p) = Jm(p) lθg≡+j5 (f) nXlt(") + t(w+w)⅛⅛ξ⅛.¾w (j) n-0

ml

\_lI *ip∖ ^m*Xλ∏(∞-n-1) ∕p∖ 2n

"1^2 ∖2∕ Z-√^ Π(») ∖2∕

n-0

where ∣(w) denotes ∏'(w)∕∏(n).

When *rn=ο* we have the second solution of (30) given by

Yo(p)=Jo(p)log≡+X⅛⅛⅛

0

21. *Relations between BesseFs Functions of Different Orders.—*Since e∙ sθr\* wφ∙‰(p) satisfies Laplace’s equation, it follows that sin *mΦ∙u∞(p)* satisfies the differential equation

a^+5∕+"=o∙ <31)

The linear character of this equation shows that if *u* is any solution .0

is also one, *f* denoting a rational integral function of the operators. Let £, *n* denote χ÷ιy, χ-ιy, then since p-T⅛(⅜) satisfies the differential equation, so also does

or

thusjνe have

Wm+p = Cp"+i>j^y5 j |

where C is a constant. If ‰(p) = Jm(p)> we have *um^p =* Jm+p(ρ), and by comparing the coefficients of *ρm+p,* we find C = ( —2)p, hence

Jm÷p(p) = (- 2)pPm+p^^2jplP-mJm(p)h

and changing *m* into — *m,* we find

*Jp-rn* (p) = ( - 2) pPp"m J-m(p) 1 ·

In a similar manner it can be proved that

Jm p(p) =2ppp^mj^p(pmJm(p)h

From the definition of Ym(p), and applying the above analysis, we prove that

Ym+p(p) = (-2)\*p→j^p(p-"∙Ym(p)) and

Ym-F (p) = m (p) !.

As particular cases of the above formulae, we find

Jp(p) = ( — 2p)pj^2pJ0(p), Yp(p) ≈ ( — 2p) j.g>)=-⅛r - γ>ω=

22. *BcsseΓs Functions as Coefficients in an Expansion.—*It is clear that eΦoo≡Φ = et≈ oreΦβiπΦ=e^ satisfy the differential equation (31), hence if these exponentials be expanded in series of cosines and sines of multiples of φ, the coefficients must be Bessel’s functions, which it is easy to see are of the first kind. To expand e\*psinΦr put *elΦ = t,* we have then to expand e⅜p(i~i~1) in powers of *t.* Multiplying together the two absolutely convergent series

<w-Σ⅛⅛)>∙≈\*',-∑⅛P(⅛)

we obtain for the coefficient of *tm* in the product

*2mrnl(* 1 ~2Γ2w+2^\*~2.4.2nz-∣-2.2∕κ-f-4- \* \* \* or JmW> hence

e⅜p(ι-i j=Jo(p)-HJi(p)÷∙ . ∙+^mJm(p)÷. . . *I ∕15∖* -rU√p) + ...-H-1)"ΓΛk(p) 5 (32)

= ∑∕mJm(p)

• — ∞

the BesseΓs functions were defined by Schlömilch as the coefficients of the powers of *t* in the expansion of e⅜p(ι-r ∖ and many of the properties of the functions can be deduced from this expansion. By differentiating both sides of (32) with respect to *ti* and equating the coefficients of Zm~l on both sides, we find the relation

Jm-1(p)+Jm+l(p)=^Jm(p),

which connects three consecutive functions. Again, by differ­entiating both sides of (32) with respect to p, and equating the coefficients of corresponding terms, we find

2⅞^-=Jm itø-KH-ltó.

l∏ (32λ let Z=etΦ, and equate the real and imaginary parts, we have then

cos (p sin φ) =J0(p)+2J2(p) cos 2φ+2J3(p) cos 3φ+... sin (p sin φ) =2J1(p) sin φ+2Js(p) sin 3φ+...

we obtain expansions of cos (p cos φ), sin (p cos ≠), by changing *φ* into *⅛~φ.* On comparing these expansions with Fourier’s series, we find expressions for Jm(p) as definite integrals, thus

*Jο(v)~~j~ο cos* (p ≡in *Φ)dΦ>* Jm(p) == ^JiQcos (p ≡ln ≠) cosmφdφ (weven) *1 fπ .*

Jm(p) = 7r I θ sin (p sin φ) sin *mφdφ (m* odd).

It can easily be deduced that when *m* is any positive integer Jm(p)=^yθc°s *(mφ~p* sin *φ)dφ.*

23. *Bessel's Functions as Limits of Legendre's Functions.—*The system of orthogonal surfaces whose parameters are cylindrical co­ordinates may be obtained as a limiting case of those whose para­meters are polar co-ordinates, when the centre of the spheres moves off to an indefinite distance from the portion of space which is con- temρlated. It would therefore be expected that the normal forms *e* “ájJm(^P)si°nw<^ woπld be derivable as limits of r-ζ-1P^(cos *iffi^mφ,* and we shalI show that this is actually the case. If O be the centre of the spheres, take as new origin a point C on the axis of *z,* such that OC = a; let P be a point whose polar co-ordinates are r, *θ, φ* referred to O as origin, and cylindrical co-ordinates p, *z, φ* referred to C as origin ; we have

*p≈r* sin *θ, z = r* cos 5 —α, hence nPn(cos0) =secn0 (1+j) Pn(cos *θ).* Now let O move off to an infinite distance from C, so that *a* becomes