infinite, and at the same time let *n* become infinite in such a way that *n∣a* has a finite value λ. Then

L sec" *Θ = L (sec* 0 λ\* ≡ι, L (1÷~) "=/\*’

and it remains to find the limiting value of Pn(cos 0). From the series (15), it may be at once proved that

P.(cos fl) = I \_ ',f~,'lZι (sin θ’ψ ..

+t-,)-^ι¾⅛^±ι> (,in \_γ- where δ is some number numerically less than unity and *m* is a fixed finite quantity sufficiently large; on proceeding to the limit, we have

*τ n I* λp∖ ν λ2p2 l λ⅛4 t ∕ χwit λ2mp2m

LPn(c°s *ηJ* 22 +2242 ∙∙∙+( I) δ>22.42.(2m)l

where Ôi is less than unity.

Hence

L P√cο⅛p) =J0(λp).

*n-∞ ∖ n ∕*

Again, since

P"(cοsp)=sin-"0d⅛^.

we have \*

. . m dmPn fcos—)

Ln-«P« cos\* =L-⅞ - ~~z~~ ~~λ~~-v-~~y~~

*∖ n∕ nim j ∕ (τ ∖m*

*∖ 2nU*

**\_( -Λ ffl -m^mJfl(p)**

*-^"2> P d(p2)m*

hence

L *n~mP"* (cos∩ =Jm(p).

*n-∞ ∖ n∕*

It may be shown that Yο (p) is obtainable as the limit of Qn (cos ý- ) the zonal harmonic of the second kind ; and that

Ym(p)=Ln-"QΓ(cos^).

24. *Definite Integral Solutions of BesseVs Equation.—*Bessel’s equation of order *m,* where *m* is unrestricted, is satisfied by the

*Γ τn~⅜*

expression *pm* I *eiPt (fl—* 1) *dt,* where the path of integration is either a curve which is closed on the Riemann’s surface on which the integrand is represented, or is taken between limits, at each of which *eiPt(fl-* ι)m+i is zero. The equation is also satisfied by the expres­sion *^e⅛p^ ∖~m~1dt* where the integral is taken along a closed

path as before, or between limits at each of which *e~p^ )∕-m~1* vanishes.

The following definite integral expressions for Bessel’s functions are derivable from these fundamental forms.

J-W-~~∏(-⅜⅛(m-⅜)'~~ *© "fr c°s φ sin ≠d≠*

where the real part of w+⅛ is positive.

Yin(p) + iπi.β"-⅛ec W7Γ.Jm(p)

=π⅛⅛ι⅜) X+lπι e,p cοsh φ sinh 2mφdψ where the real parts of m+⅛, *p* are positive; if p is purely imaginary and positive the upper limit may be replaced by oo.

Ym(p)-i\*∙i-emfft Sec Wx.Jm(p)

=es""2⅛÷→i) (£) ra∫\*-‰ cos W sinh *^φdφ* under the same restrictions as in the last case; if *p* is a negative imaginary number, we may put ∞ for the upper limit.

If *p* is real and positive

2 Γ°o

Jo(p) =- lft sin (p cosh *φ)dφ*

*7ΓJ* 0

/**00**

θ cos *(ρ* cosh *φ)dφ.*

25. *Bessel,s Functions with Imaginary Argument.—*The functions with purely imaginary argument are of such importance in connexion with certain differential equations of physics that a special notation has been introduced for them. We denote the two solutions of the equation

*d?u l* I *du*

*^3t\*^~r~dr~u~0*

by I0(r)t K0(r) when

Io(r) =Jo(ιr) βl÷⅜2÷^~Js+∙ · ·

= - J \_ cosh *(r* cos *φ)dφt*

and

K0(r) = Yo(ιr) ÷^ιττJo(ιr) =*e~r cos h⅛iψ =j"0* cos *(r* sinh *ψ)dψ.*

The particular integral Kc(r) is so chosen that it vanishes when *r* is real and infinite ; it is also represented by

Λ°ο cos *υ j*

*J 0* 7⅞⅛jdl',

and by

**/00 £—ru**

1 √(tt2-ι)d"∙

The solutions of the equation

*du2* l I *du ∕ , m2∖*

V+τ^tt=ο

are denoted by lm(r), Km(r), where

~2m∏(w) ( 1 *+2T2m+2+2.4.2m*+2.*2m* +4,,, I

= (2'∙)m2^pr1"(0.

when *m* is an integer, and

Km(r) = (2r)wg~~^2)~~~~,~~~~nι~~K0(r) *=e* Ytn(ιr) ÷^wrjm(ιr) .

We find also

U(r) =~~i .~~~~3~~~~.~~~~5~~~~.∕(~~~~2w~~~~-ι)~~ *½T0* cosh <r cos ≠) ≡ini",Φ⅜ Km(r) "E3'.∙~~h⅞m-ι)∕~~o"g^τc°sh φ sînh 2mφdφ

= (~ 1)m3· 5 · ·

26. *The Asymptotic Series for Bessers Functions.—*It may be shown, by means of definite integral expressions for the BesseΓs functions, that

J-ω-∖⅛i ∙∙∞⅛+5--)+δ∙'√⅞'+r') !

Ym(p) = sec *mπ* ∣ P sin “Q ∞s

where P and Q denote the series

p-1 (4^-Ia)(4>w,-3i)

I.2.(8p)\*

. (4>"i - li) (4wi - 3i) (4wt - 5i) (4>"3 ~ 7i) \_ ^1^ ι.2.3∙4(8p)< ∙∙∙

(4mi-li)(4mi-38)(4>ni-52) ,

w ι.8p 1.2.3.(8p)3 ,^∙∙∙

These series for P, Q are divergent unless *m* is half an odd integer, but it can be shown that they may be used for calculating the values of the functions, as they have the property that if in the calculation we stop at any term, the error in the value of the function is less than the next term; thus in using the series for calculation, we must stop at a term which is small. In such series the remainder after *n* terms has a minimum for some value of *n,* and for greater values of *n* increases beyond all limits; such series are called semi- convergent or asymptotic.

We have as particular cases of such series:—

J∙ω=√g∞(Hi-⅛fc+⅛⅜5⅛.-! -√⅛-i√H⅛-.-⅛¾⅛+-l when *m* is an integer,

κ.w.<-.>-√≡-s .+⅞ι+~~<ι--¾y-^>~~ +.,. j ι-w-⅛∙j -⅞≤+~~^-¾--a->~~-.. j

27. *The Bessers functions of degree half an odd integer* are of special