importance in connexion with the differential equations of physics. The two equations

¾ = ⅛V⅛,⅛ = ⅛Vu,

are reducible by means of the substitutions *u = e~ktυ, u≈eiktυ* to the form v2ι>÷t> = o. If we suppose *υ* to be a function of *r* only, this last differential equation takes the form

*^p-+vr≈o,*

so that *υ* has the values

sin *r/r,* cos *r/r;*

in order to obtain more general solutions of the equation ν⅛÷ν=o, we may operate on

sin *r/r,* cos *r∣r*

with the operator

γ *(JLτ* A

ïnVã7 *∂y' ∂z∕ ’*

where Yn (x, *y, ζ)* is any spherical solid harmonic of degree *n.* The result of the operation may be at once obtained by taking Yn(x, *y,* J≡) for *f∏(x, yt ζ)* in the. theorem (7'), we thus find as solutions, of V¾4-ν=o, the expressions

Λ7 ∕ χ *dn* sin r λ7 z *.λ dn cosr* γ^(\*> y. . y√≈> y. z)^5)ι *~r~*

By recurring to the definition of the function Jm(r), we see that jtw-√≡j.-⅛+r⅛5-...μ√r⅛i thus

r\*Ji(r)-Λ∕≡⅛i∙

v 7Γ *r*

Using the relation between Bessel’s functions whose orders differ by an integer, we have

J„,w - hi∙√⅛∙⅛ ⅛

It may be shown at once that

*dn* cos *r*

*r —*

is a second solution of Bessel’s equation of order w+J; thus the differential equation ν2ν⅛ν = o is satisfied by the expression

Yn(x, *y,* z)~~j~~~~"~~~~r~~~~⅛4∖~~

and by the corresponding expression with a second solution of Bessel’s equation instead of Jfn-i(r) ; if Sn(μ, *φ)* denotes a surface harmonic of degree *n,* the expression

Sn(μ, *φ)* ~ψjn+l(r)

is a solution of the equation ν⅛τ^ = o.

The Bessel’s functions of degree half an odd integer are the only ones which are expressible in a closed form involving no trans- cendental functions other than circular functions, lt will be observed that in this case the semi-convergent series for Jτn becomes a finite one as the expressions P, Q then break off after a finite number of terms.

28. *The Zeros of BesseIs Functions.*—The determination of the position of the zeros of the Bessel’s functions, and the values of the argument at which they occur, have been investigated by Hurwitz *!Math. Ann.* vol. xxxiii.), and more completely by H. M. Macdonald *Proc. Lond. Math. Soc.* vols, xxix.,xxx.). lt has been shown that the zeros of Jn(2)∕2n are all real and associated with the singular point at infinity when *n* is real and > — 1, and that all the real zeros of Jn(2)∕2\* when *n* is real and <— 1, and not an integer, are associated with the essential singularity at infinity. When n is a negative integer — m, Jn(z)∕sn has, in addition, *2m* real zeros co­incident at the origin. When *n==-m-v, m* being a positive integer, and ι>v>o, Jn(z)∕zn has a finite number *2rn* of zeros which are not associated with tne essential singularity. If *n* is real, and starts with any positive value, the zeros nearest the origin approach it as *n* diminishes, two of them reaching it when n=—1, and two more reach it whenever *n* passes through a negative integral value; these zeros then become complex for values of *n* not integral.\* The zeros of Jn(z)∕zn are separated by those of *J^ι(ζ)∕sin,* one zero of the latter, and one only, lies between two consecutive zeros of Jn(z)∕sn. When *n* is real and >—ι,all the zeros of Jn(z)∕zn are given.by a formula due to Stokes; the rnih positive zero in order of magnitude is given by

4^-1 4(4n2-ι)i28n2-31) o\_ α^-8∑ ∏⅛p &c·’

where a = ir(2nT4m — 1). It has been shown by Macdonald that the function Kn(a) has no real zeros unless *n = 2kA-⅛* where *k* is an integer, when it has one real negative zero; and that l<n(z) has no purely imaginary zeros, and no zero whose real part is positive, other than those at infinity. When i>n>o,, Kn(z) has no zeros other than those at infinity, when 2>n>ι,it has one zero whose real part is negative, and when *m-∖-ι>n>m* where *m* is an integer, there are *m* zeros whose real parts are negative. When *n* is an integer, Kn(z) has *n* zeros with negative real parts.

29. *Spheroidal Harmonics.—*For potential problems in which the boundary is an ellipsoid of revolution, the co-ordinates to be used are *r, Θ, φ* where in the case of a prolate spheroid

x = c√r2- I sin 0 cos *φ, y=c>∕ri-ι* sin 0 sin *φ, z≈cr* cos 0,

the surfaces *r = rο, 0=θ0, Φ=Φο* are confocal prolate spheroids, confocal hyperboloids of revolution, and planes passing through the axis of revolution. We may suppose *r* to range from 1 to 00 , *θ* from o to π, and *φ* from o to 2jγ, every point in space has then unique co-ordinates *r, θ, φ.*

For oblate spheroids, the corresponding co-ordinates are r, *θ, φ* given by

x = c√r2-H sin0cosφ, y = c√r2 + ι sin0sinφ, z = crcos0, where

0≤r <0° ’ 0 ≤0 =7r, o = 2îr’

these may be obtained from those for the prolate spheroid by chang- ing *c* into — *ιe,* and *r* into *ιr.*

Taking the case of the prolate spheroid, Laplace’s equation becomes

M ∕ , .χ5V ) 1 I *∂ ( . λ∂V∖ ,* r2-cos20 a2V *dr* i ^∙r *1) dr* J ÷sin *θ ∂θ* V5ιn *θdθ)* +(rs-l)sin20 3≠2 -°,

and it will be found that the normal solutions are

PΓ(r))p∏cosfl))cos .

QnmωjQ7(∞sfl) Jsin"l0∙

For the space inside a bounding spheroid the appropriate normal forms are P7(r)PΓ(cos0)≡∣^swiφ, where *n, m* are positive integers, and for the external space

QΓ(r)PΓ(cos⅛)=ionswlφ.

For the-case of an oblate spheroid, Ρ„(σ), Q^Gr)» take the place of PΓ(r), Q?(r).

30. *Toroidal Functions.—*For potential problems connected with the anchor-ring, the following co-ordinates are appropriate: If A, B are points at the extremities of a diameter of a fixed circle, and P is any point in the plane PAB which is perpendicular to the plane of the fixed circle, let P = log(AP∕BP), *0 —* ∆APB, and let *φ* be the angle the plane ÄPB makes with a fixed plane through the axis of the circle. Let *θ* be restricted to lie between —ττ and 7r, a discontinuity in its value arising as we pass through the circle, so that within the circumference 0 is π on the upper side of the circle, and *-π* on the lower side; *θ* is zero in the plane of the circle outside the circumference; *p* may have any value between — ∞ and ∞, and *Φ* any value between o and 2jγ. The position of a point is then uniquely represented by the co-ordinates p, *θ, φ,* which are the parameters of a system of tores with the fixed circle as limiting circle, a system of bowls with the fixed circle as common rim, and a system of planes through the axis of the tores. If *x, y, z* are the co-ordinates of a point referred to axes, two of which *x, y* are in the plane of the circle and the third along its axis, we find that

r α s\*nl1 *P λ* a sinh *P -* x a s\*n &

cosh p —cos 0cos cosh p—cos0sιn 3~cosh p—cos*Θ\**

where *a* is the radius of the fixed circle.

Laplace’s equation reduces to

∂ ( sinh *p* ôV ) l ∂ ( sinh *p ∂V ) l* 1 θ2V

*dp(* 1« ¾>s+∂0( 1« *dθ* J +l,2sinhp *0φ1 ~θ'*

when P denotes √(cosh *p—* cos 0). It can be shown that this equation is satisfied by

l f . zλ p Γ-i(cosh p) cos „a COS

√ (cosh p—cos 0)cr (cosh ⅛ sin *nθ sinmφ,*

the functions P„\_|(cosh p), Qrw^∣(cosh p) required for the potential problems, are associated Legendre’s functions of degree n —⅜, half an odd integer, of integral order rn, and of argument real and greater than unity; these are known as toroidal functions. For the space external to a boundary tore the function Q^(cosh *p)* must be used, and for the internal space Pj χ(cosh p).