The following expressions may be given for the toroidal functions:—

*τ,m f* , v ( — 1)ot ∏(n-⅛) *Çπ* cos *mφ .*

P«-Pcos *pì π* n(n— *m —* i)Jo (cosh p +sinh *p* cos ≠)n+l

~~=~~~~π~~ ~~π~~~~∏(¾ — ⅜)~~ **<cosh p + s\*nh p cos ≠)n^\* cos**

**Pn-j(cosh p) =-** I *-.. flφ.*

π√ 0√2 cosh P — 2 cosh *φ*

**∕ 1 x ∕ λtn∏(n-hm-⅛) flog coth Jp**

Qn-4(cosh p) = (-1)*~~m~~ ~~∏(⅞~-∣) J~~~~0~~*(cosh p

— sinh *p* cosh w)n\*^l cosh *mwdw*

-(-ι)ψ\*Π(m-j)Π(-i) sinh mpf⅞∙--c-s-h∙~~p~~~~c~~~~-~~~~2~~~~n~~~~c~~~~0s~~~~⅜)-∙÷^≠∙~~ The relations between functions for three consecutive values of the degree or the order are ·

*2n* cosh pP^\_,(coshp) — (n ~m+⅛)P^h, (cosh p)

-(n-bm — J)P^\_I (cosh p) = o.

P^\*i2(cosh p) + *2{m* + i) coth pP^1(cosh p)

- (n - *m* - ⅛)(n + w + i)P^Li (cosh p) = o,

with relations identical in form for the functions Q^\_, (cosh p).

The function Qrt-j(cosh p) is expansible in the form ~~n(~~~~”~n¾y~~~~~¾e~~~(w÷n<>F(ì’ n + i. » + I. e-⅛). which is useful for calculation of the function when *p* is not small. Prt-j(cosh p) can also be expressed in terms *of e~fl* by a somewhat complicated formula.

31. *Ellipsoidal Harmonics.—*In order to treat potential problems in which the boundary surface is an ellipsoid, Lamé took as co­ordinates the parameters pf µ, *v* of systems of confocal ellipsoids, hyperboloids of one sheet, and of two sheets; these co-ordinates are three roots of the equation

**75÷** *P — ht* **^^\*∕2 \_ ⅛2= 1 ’** *≥ h)* **ï**

we thence find that

*pµv pi—* Ä2 V μ2 — λ2 Vλ2-p2 V p2 — ⅛2V⅛2— μ2V *k2 —* p2

**X “ ΉΓ’ ? ”** *hy∕k2-h2 ,z~ k^/kt-ht*

where *∞≥p2≥h2t k2<μ2<h2,* and ⅛2>p2>0.

We find from these values of x, *y, z*

*<\*V + «,). + (\*).* ~~⅞~⅞⅞ I⅞⅛^⅞Z¾⅞L'√⅛~~~~1~~~~' ÷⅜Ξ⅞⅛~~~~,~~~~Ξ¾~~**W. .**

and on applying the general transformation of Laplace’s equation that equation becomes

**, 2 I < 2 ?2V - f 2** 2λ02V λ

**(/i ÷k** *v )lhjt+(p μ )~∂ζt* **= θr**

where £; *η, ζ^* are defined by the formulae

t = fp *jp ri- Cµ\_ dµ\_*

5 J \*√p2-⅛2√p2-⅛2' ' Λ √μ2-A2√⅛2-μ2'

*Γv dv*

which are equivalent to

*p = kdnifiξ,* ⅛ι), *μ≈kdn(K-kη, ⅛1), v≈ksn(kξ, ⅛∕),* where ‰2, *kf2* denote the quantities 1 —Λ2∕Λ2, *h2∣k2* and K denotes the complete elliptic integral

Γ⅛1r *dψ*

*J* ο V I — ⅛12 sin2≠

It can now be shown that Laplace’s equation is satisfied by the product E(p)E(μ)EW, where E(o) satisfies the differential equation ^i-(n(n+l)pi-(⅛J+⅛η∕>)E(p)=Oj

and E(μ), E(√) satisfy the equations

*^^+[η(η4* ι)μ≈-∕>(⅛≈+⅛∙)]E(μ)=o, ^i-[n(n+l^-p(⅛s+⅛s)∣EW =0, where *n* and *p* are arbitrary constants. On substituting the values of the parameters *ξ, ηt ξ^* in terms of *p, µ, vl* we find that the equation satisfied by E(p) becomes

(p2 -Ä2) (?- ⅛s)d-^d+p(2P2 -⅛s -⅛2)⅛1

+((⅛≈+⅛^-n(n+ι)p≈)E(p)=o,

and E(μ), E(ν) satisfy equations in *µ, v* respectively of identically the same form; this equation is known as Lamé’s equation.

If *n* be taken to be a positive integer, it can be shown that it is possible in 2n-}-ι ways so to determine *p* that the equation in E(p) is satisfied by an algebraical function of degree w, rational in P» √ (p2-∕ι2)i √(p2-⅛2)∙ The functions so determined are called Lamé’s functions, and the *2n-j-ι* functions of degree *η* are of one of the four forms.

K(ρ) = αopn + *aιpn~2 + . . .,*

**L(p) =√p^(α√pn-l + α⅛--3+ . . .), M(p) = √√-½2(αo\*pn-l+α\*ιpn-3+ t t N(p) =** *Jpt-kt √ f-ht(aQfPn^+a{"pn-i+...).*

These are the four classes of Lamé’s functions of degree *n;* of the functions K there are ι+∣n, or i(n-bι), according as *n* is even or odd; of each of the functions L, M, there are Jn, or ⅛(w-1), and of the functions N, there are ∣n, or i(n-∣-ι).

The normal forms of solution of Laplace’s equation, applicable to the space inside the ellipsoid, are the 2w4-1 products E(p) E(μ) E(^). lt can be shown that the 2n+ι values of *p* are real and unequal.

It can be shown that, subject to certain restrictions, a function of *µ* and p, arbitrarily given over the surface of the ellipsoid *p=pιt* can be expressed as the sum of products of Lamé’s functions of *µ* and *v,* in the form

co 2rt⅛I

Σ Σ ⅛E∙(μ)Ei(,)i I j-ι

the potential function for the space inside the ellipsoid, which has the arbitrarily given value over the surface of the ellipsoid, is consequently

Xa Va ..Eì(p)EA(µ)E„(i>)

Z√ 2√ n ëíSû

It can be shown that a second solution of Lamé’s equation is Fn(p) where

F,ω - ⅛∙f>∣wJ7~~∣E~~~~ιgl~~~~⅛∕⅛√,>-~~~~lli~~

this function Fn(p) vanishes’at infinity as p~y^1, and is therefore adapted to the space outside the bounding ellipsoid. The external potential which has at the surface *p — pi,* the value

√, r ,⅛E⅞(μ)E⅞(y) is~y^ -^c⅞f¾¾(\*t)¾(y)∙

32. *History and Literature.—*The first investigator in the subject was Legendre, who introduced the functions known by his name, and at present also called zonal surface harmonics; he applied them to the determination of the attractions of solids of revolution. Legendre’s investigations are contained in a memoir of the Paris Academy, *Sur l'attraction des sphéroides,* published in 1785, and in a memoir published by the Academy in 1787, *Recherches sur la figure des planètes;* his investigations are collected in his *Exercices,* and in his *Traite des functions elliptiques.* The potential function was introduced by Laplace, who also first obtained the equation which bears his name; he applied spherical surface harmonics to the determination of the potential of a nearly spherical solid, in his memoir, *Théorie des attractions des sphéroides et de la figure des planètes,* published by the Paris Academy in 1785. Laplace was the first to consider the functions of two angles, which functions have consequently been known as Laplace’s functions; his investigations on these functions are given in the *Mécanique céleste,* tome ii. livre iii., tome v. livre xi., and in the supplement to vol. v. The notation P(n) was introduced by Dirichlet (see Crelle’s *Journal,* vol. xvii., “ sur les séries dont le terme général dépend de deux angles” &c. ; see also his memoir, “ Ueber einen neuen Ausdruck zur Bestimmung der Dichtigkeit einer unendlich dünnen Kugelschale,” in the *Abhandlungen* of the Berlin Academy, 1850). The name “ Kugel- functionen ” was introduced by Gauss (see *Collected Works,* vi. 648). A direct investigation of the expression for the reciprocal of the distance between two points in spherical surface harmonics was given by Jacobi (Crelle’s *Journal,* vol. xxvi., see also vol. xxxii.). The functions of the second kind were first introduced by Heine (see his “ Theorie der Anziehung eines Ellipsoïdes,” Crelle’s *Journal,* vol. xlii., 1851). The above-mentioned investigators employed almost entirely polar co-ordinates; the use of Cartesian co-ordinates for the expression of spherical harmonics was introduced by Kelvin in his theory of the equilibrium of an elastic spherical shell (see