of necessity a single curve: it may be, for instance, a skew polygon of four or more sides.

The partial differential equation was dealt with in a very remarkable manner by Riemann. From the second form given *qdx— pdy*

above it appears that we have ι =¾ ≡ a complete differential, or, putting this=*dζ* we introduce into the solution a vari­able *ζ*, which combines with z in the forms *z=iζ* (*i*=√-1). The boundary conditions have to be satisfied by the determination of the conjugate variables *η, η'* as functions of *z+iζ*, *z—iζ*, or, say, of Z, *Z'* respectively, and by writing *S, S'* to denote *x+iy, x—iy* respectively. Riemann obtains finally two ordinary differ­ential equations of the first order in 5, *S', η, η'', Z, Z',* and the results are completely worked out in some very interesting special cases.

(A. Ca.)

Part II.

We proceed to treat the differential geometry of surfaces, a study founded on the consideration of the expression of the lineal element in terms of two parameters, *u, v,*

*ds2 = Edu2+2Fdudυ+Gdυ2,*

w=const, *v=const*, being thus systems of curves traced on the surface. This method, which may be said to have been inaugurated by Gauss in his classical paper published in 1828, *Disquisitions generales circa superficies curves,* has the great advantage of dealing in the most natural way with all questions connected with geodetics, geodetic curvature, geodetic circles, &c.—in fact, all relations of lines on a surface which can be formulated without reference to anything external to the surface. All such relations when deduced for any particular surface can be at once generalized in their application, holding good for any other surface which has the same expression for its lineal element; *e.g.* relations involving great circles and small circles on a sphere furnish us with corresponding relations for geodetics and geodetic circles on any synclastic surface of constant specific curvature.

I. Gauss begins by introducing the conception of the integral curvature *(curvatura integra)* of any portion of a surface. This he defines to be the area of the corresponding portion of a sphere of unit radius, traced out by a radius drawn parallel to the normal at each point of the surface; *i.e.* it is *ff* *ds/RR* where R, R, are the principal radii of curvature. The quotient obtained by dividing the integral curvature of a small portion of the surface round a point by the area of that portion, that is 1/RR', he naturally calls the measure of curvature or the specific curvature at the point in question. He proceeds to establish his leading proposition, that this specific curvature at any point is expressible in terms of the E, F and G which enter into the equation for the lineal element, to­gether with their differential coefficients with respect to the variables, *u* and *v.*

It is desirable to make clear the exact significance of this theorem. Of course, for any particular surface, the curvature can be expressed in an indefinite variety of ways. The speciality of the Gaussian expression is that it is deduced in such a. manner as to hold good for all surfaces which have the same expression for the lineal element. The expression for the specific curvature, which is in general somewhat elaborate, assumes a very simple form when a system of geodetics and the system of their orthogonal trajectories are chosen for the parameter curves, the parameter *u* being made the length of the arc of the geodetic, measured from the curve, *u=o* selected as the standard. If this be done the equation for the lineal clement becomes *ds2 = du2 + P2dv2,* and that for the specific curvature (RR1)-1 = -P-1 *d2P/du2.* By means of this last ex­pression Gauss then proves that the integral curvature of a triangle formed by three geodetics on the surface can be expressed in terms of its angles, and is equal to A+ B+ C—*π*.

This theorem may be more generally stated :—

*The integral curvature of any portion of a surface = 2π—∑di round the contour of this portion,* where *di* denotes the angle of geodetic contingence of the boundary curve. The angle of geodetic contingence of a curve traced on a surface may be defined as the angle of intersection of two geodetic tangents drawn at the ex­tremities of an element of arc, an angle which may be easily proved to be the same as the projection on the tangent plane of the ordinary angle of contingence. the geodetic curvature, p-1, is thus equal to the ordinary curvature multiplied by cos *φ, φ* being the angle the osculating plane of the curve makes with the tangent plane.

. Gauss's theorem may be established geometrically in the following simple manner: If we draw successive tangent planes along the curve, these will intersect in a system of lines, termed the *conjugate* tangents, forming a developable surface. If we unroll this develop­able then *di=dθ-dψ,* where *di* is the angle of geodetic con­tingence, *dθ* the angle between two consecutive conjugate tangents, *ψ* the angle the conjugate tangent makes with the curve. There­fore, as *ψ* returns to its original value when we integrate round the curve, we have *∑di = Σdθ.* This equation holds for both the curve on the given surface and the representative curve on the sphere. But the tangent planes along these curves being always parallel, their successive intersections arc so also; therefore *∑dθ* is the same for both; consequently *∑di* for the curve on the surface *=∑di* for the representative curve on the sphere. Hence integral curvature of curve of surface = area of representative curve on sphere,

= 2*π*- *∑di* on sphere by spherical geometry,

= 2*π*-*∑di* for curve on surface.

A useful expression for the geodetic curvature of one of the cutves, v=const, can be obtained, if a curve receive a small displacement on any surface, so that the displacements of its two extremities are normal to the curve, it follows, from the calculus of variations, that the variation of the length of the curve =*∫p-1δnds* where *ρ-1* is the geodetic curvature, and *δn* the normal component of the dis­placement at each point. Applying this formula to one of the *v* curves, we find

*δ∫Pdv = ∫(dP/du)δudv = δ* length of curve = *∫ρ*-1 *SuPdv,* and as *δu* is the same for all points of the curve, *ρ*-1 = P-1*d*P∕*du.*

We can deduce immediately from this expression Gauss’s value for the specific curvature. For applying his theorem to the quadri­lateral formed by the curves *u*, *u*1, *v, v*1, and remembering that *∑di* along a geodetic vanishes, we have

*∫∫* (RR)-1P*dudv* = — *∑di* for curve BC— *∑di* for curve DA,

*= — ∑ρ-1ds* for curve BC + *∑p-1ds* for curve AD,.

*= ~ fγ>* i°r curve ^or curve AD,

-√∣⅛-(S>

therefore passing to the limit P∕RR' = — *d2P∣du2.*

Gauss then proceeds to consider what the result will be if a surface be deformed in such a way that no lineal element is altered. It is easily seen that this involves that the angle at which two curves on the surface intersect is unaltered by this deformation; and since obviously geodetics remain geodetics, the angle of geodetic contin­gence and consequently the geodetic curvature are also unaltered, It therefore follows from his theorem that the integral curvature of any portion of a surface and the specific curvature at any point are unaltered by non-extensional deformation.

*Geodetics and Geodetic Circles.*

A geodetic and its fundamental properties are stated in part 1., where it is also explained in that article within what range a geodetic possesses the property of being the shortest path between two of its points. The determination of the geodetics on a given surface depends upon the solution of a differential equation of the second order. The first integral of this equation, when it can be found for any given class of surfaces, gives us the characteristic property of the geodetics on such surfaces. The following are some of the well- known classes for which this integral has been obtained: (1) quadrics; (2) developable surfaces ; (3) surfaces of revolution.

I. *Quadrics.—*Several mathematicians about the middle of the 19th century made a special study of the geometry of the lines of curvature and the geodetics on quadrics, and were rewarded by the discovery of many wonderfully simple and elegant analogies between their properties and those of a system of confocal conics and their tangents *in piano.* As explained above, the lines of curvature on a quadric are the systems of orthogonal curves formed by its inter­section with the two systems of confocal quadrics. Joachimsthal showed that the interpretation of the first integral of the equation for geodetics on a central quadric is, that along a geodetic *p*D con­stant (C,) *p* denoting the perpendicular let fall from the centre on the tangent plane, and D the semidiameter drawn parallel to the element of the geodetic, the envelope of all geodetics having the same C being a line of curvature. In particular, all geodetics passing through one of the real umbilics (the four points where the indicatrix is a circle) have the same C.

Michael Roberts pointed out that it is an immediate consequence of the equation *p*D=C, that if two umbilics, A and B (selecting two not diametrically opposite), be joined by geodetics to any point P on a given line of curvature, they make equal angles with such line of curvature, and consequently that, as P moves along a line of curvature, either PA+PB or PA —PB remains constant. Or, conversely, that the locus of a point P on the surface, for which the sum or difference of the geodetic distances PA and PB is constant, is a line of curvature. It follows that if the ends of a string be fastened at the two umbilics of a central quadric, and a style move over the surface keeping the string always stretched, it will describe a line of curvature.

Another striking analogue is the following: As, *in piano,* if a variable point or an ellipse be joined to the two foci S and H,