coefficient of & vanishes. Hence on effecting these substitutions and equating that coefficient to zero, we find

⅛i,⅛4¾=o.

*a1 ~ 2ri* ' 21-pl

But by the definitions of δ and *a* we have *M≈}τ3a3 = gai,* where g is gravity, and therefore

*3mat*

*ei 2-∏l*  (6)

,3σ.

(2i + l)δ

In the particular case considered in § 11 we therefore have '-÷+s⅛⅛'

as the equation to the equilibrium tide under the potential y = 2^pî ^c°s\* 2-^'

If *σ* were very small compared with δ the attraction of the water on itself would be very small compared with that of the planet on the water; hence we see in the general case that l∕ (1^2⅛) is the factor by which the mutual gravitation of the ocean augments the deformation due to the external forces. This factor will occur frequently hereafter, and therefore for brevity we write

5⅛ <’>

and we may put (6) in the form

*3ma3 ,-λ*

*ei~W⅛i (9)*

Comparison with (5) then shows that

V≡g⅛(x)⅛Si (to)

is the potential of the disturbing forces under which

*p=a+eiSi* (11)

is a figure of equilibrium.

We are thus provided with a convenient method of specifying any disturbing force by means of the figure of equilibrium which it is competent to maintain. In considering the dynamical theory of the tides on an ocean-covered planet, we shall specify the disturb­ing forces in the manner expressed by (10) and (11). This way of specifying, a disturbing force is equally exact whether or not we choose to include the effects of the mutual attraction of the ocean. If the augmentation due to mutual attraction of the water is not included, &,· becomes equal to unity; there is no longer any necessity to use spherical harmonic analysis, and we sec that if the equation to the surface of an ocean be

p =α+∙S, where 5 is a function of latitude and longitude, it is in equilibrium under forces due to a potential whose value at the surface of the sphere (where ρ = α) is g5.

In treating the theory of tidal observation we shall specify the tide-generating forces in this way, and then by means of “ the principle of forced vibrations,” referred to in § 7 as used by Laplace for discussing the actual oscillations of the sea, we shall pass to the actual tides at the port of observation.

In this equilibrium theory it is assumed that the figure of the ocean is at each instant one of equilibrium under the action of gravity and of the tide-generating forces. Lord Kelvin has, how­ever, reasserted@@1 a point which was known to Bernoulli, but has since been overlooked, namely, that this law of rise and fall of water cannot, when portions of the globe are continents, be satisfied by a constant volume of water in the ocean. The necessary correction to the theory depends on the distribution of land and sea, but a numerical solution shows that it is practically of very small amount.

§ 13. *Development of Tide-generating Potential in Terms of Hour- Angle and Declination.—*We now proceed to develop the tide­generating potential, and shall of course implicitly (§ 12) determine the equation to the equilibrium figure.

We have already seen that, if z be the moon’s zenith distance at the point *P* on the earth’s surface, whose co-ordinates referred to A, B, C, axes fixed in the earth, and αξ, αη, af,

cos z = {Mι-(-ιιM2+fM1, where Mi, Mt, Ms are the moon's direction cosines referred to the same axes. Then, with this value of cos z,

cos%-i=2⅛MιM, + 2^Sι⅞1-~~ι~~i~ + 21ιrMiM a + 2ξ{∙M1M,

■ ∣2fM11+Ma1- 2M,1 (12)

3 3

The axis of C is taken as the polar axis, and AB is the equatorial plane, so that the functions of ξ, *η,* f are functions of the latitude and longitude of the point *P,* at which we wish to find the potential

The functions of Μι, Μ», Μ» depend on the moon’s position, and we shall have occasion to develop them in two different ways—first in terms of her hour-angle and declination, and secondly (§ 25) in terms of her longitude and the elements of the orbit.

Now let A be on the equator in the meridian of *P,* and B 900 east of A on the equator. Then, if M be the moon, the inclination of the plane MC to the plane CA is the moon’s easterly local hour-angle. Let *h0 =* Greenwich westward hour-angle; Z = the west longitude of the place of observation; λ = the latitude of the place; δ = moon's declination : then we have

Mι = cosδcos(⅛<>-Z), Ms = — cosδsin(Z⅛-Z), Mj = sin δ,

í =cos λ, *η* =0, f = sin λ.

Also the radius vector of the place of observation on the earth's surface is *a.* Whence we find

I ⅜COS1λCOS1δCOS 2‰ — Z) + S⅛ 2λ S⅛ δ COS δ COS (Z⅛ — Z) +Ki~sin2δ)(⅛-sin,λ) I (13)

The tide-generating forces are found by the rates of variation of *V* for latitude and longitude, and also for radius *a,* if we care to find the radial disturbing force.

The westward component of the tide-generating force at the earth’s surface, where *p = a,* is *dV∣a* cos λ<ZZ, and the northward component is *dV∣ad∖∙,* the change of apparent level is the ratio of these to gravity *g.* On effecting the differentiations we find that the westward component is made up of two periodic terms, one going through its variations twice and the other once a day. The southward com­ponent has also two similar terms; but it has a third' very small term, which does not oscillate about a zero value. This last term corresponds to forces which produce a constant heaping up of the water at the equator; or, in other words, the moon's attraction has the effect of causing a small permanent ellipticity of the earth’s mean figure. This augmentation of ellipticity is of course very small, but it is necessary to mention it.

If we consider the motion of a pendulum-bob under the influence of these forces during any one day, we see that in consequence of the. semi-diurnal changes of level it twice describes an ellipse with major axis east and west, and the formula when developed shows that the ratio of axes is equal to the sine of the latitude, and the linear dimensions proportional to cos’ δ. It describes once a day an ellipse whose north and south axis is proportional to sin 2δ cos 2λ and whose east and west axis is proportional to sin 2δ sin λ. Obvi­ously the latter is circular in latitude 300. When the moon is on the equator, the maximum deflexion occurs when the moon’s local hour-angle is 45°, and is then equal to

*3m ∕a∖ ’ .*

2⅛(7) c0sλ∙

This angle is equal to 0∙0174" cos λ. Attempts actually to measure the deflexion of the vertical have at length proved successful (see Seismometer).

III.—Dynamical Theory of the Tides

§ 14. *Recent Advances in the Dynamical Theory of the Tides.—* The problem of the tidal oscillation of the sea is essentially dynamical. In two papers in the second volume of *Liouvilte’s Journal* (1896) H. Poincaré has considered the mathematical principles involved in the problem, where the ocean is inter­rupted by land as in actuality. He has not sought to obtain numerical results applicable to any given configuration of land and sea, but he has aimed rather at pointing out methods by which it may some day be possible to obtain such solutions.

Even when the ocean is taken as covering the whole earth the problem presents formidable difficulties, and this is the only case in which it has been solved hitherto.@@2

Laplace gives the solution in bks. i. and iv. of the *Mécanique céleste;* but his work is unnecessarily complicated. In the 9th edition of the *Ency. Brit,* we gave Laplace’s theory without these complications, but the theory, is now accessible in H. Lamb’s *Hydrodynamics* and other works of the kind. It is therefore not reproduced here.

In 1897 and 1898 S. S. Hough undertook an important revision of Laplace’s theory and succeeded not only in intro­ducing the effects of the mutual gravitation of the ocean, but

@@@1 Thomson and Tait, *Nat. Phil.* § 807. G. H. Darwin and H. H. Turner, *Proc. Roy. Soc.* (1886).

@@@2 Lord Kelvin’s (Sir W. Thomson’s) paper on the gravitational oscillations of rotating water, *Phil. Mag.* (October 1880), bears on this subject. It is the only attempt to obtain numerical results in respect to the effect of the earth’s rotation on the oscillations of land-locked seas.