also in determining the nature and periods of the free oscil­lations of the sea.@@1 A dynamical problem of this character cannot be regarded as fully solved unless we are able not only to discuss the “ forced ” oscillations of the system but also the *“ free.”* Hence we regard Mr Hough’s work as the most important contribution to the dynamical theory of the tides since the time of Laplace. We shall accordingly present the theory briefly in the form due to Mr Hough.

The analysis is more complex than that of Laplace, where the mutual attraction of the ocean was neglected, but this was perhaps inevitable. Our first task is to form the equations of motion and continuity, which will be equally applicable to all forms of the theory.

§15. *Equations of Motion.—*Let r, 0, φ be the radius vector, colatitude and east longitude of a point with reference to an origin, a polar axis and a zero-meridian rotating with a uniform angular velocity *n* from west, to east. Then if R, H, Sbe the radial, colati- tudinal and longitudinal accelerations of the point, we have R=Sr-r(ï)í-rsinîø(^+”)1 ≡=7 ⅛ (r⅞) - r sin θ cos θ +n) ’

If the point were at rest with reference to the rotating meridian we should have

R = *-ntr* sin 0, Ξ = — n’r sin *θ* cos *θ,* H =0.

When these considerations are applied to the motion of an ocean relative to a rotating planet, it is clear that these accelerations, which still remain when the ocean is at rest, are annulled by the permanent oblateness of the ocean. As then they take no part in the oscillations of the ocean, and as we are not considering the figure of the planet, we may omit these terms from R and S. This being so we must replace (^7⅞+fi) a≡ >t occurs in R and *Ξ* by (⅛) +2n^∙

Now suppose that the point whose accelerations are under consideration never moves far from its zero position, and that its displacements ξ, *η* sin *θ* in colatitude and longitude are very large compared with *ρ* its radial displacement. Suppose, further, that the velocities of the point are so small that their squares and products are negligible compared with *n,rt∙,* then we have

2J = 2j, a very small quantity;

*r sin e Tt=ít (’ siπ 0) ’* d0-<⅛

*rdt ~ dt'*

Since the radial velocity always remains very small it is not neces­sary to concern ourselves further with the value of R, and we only require the two other components which have the approximate forms,

≡=^-2wsιn0cos03i' I (l4)

H =sin *θug^+2n* cos *θ,^-∙*

We have now to consider the forces by which an element of the ocean is urged in the direction of colatitude and longitude. These forces are those due to the external disturbing forces, to the pressure of the water, surrounding an element of the ocean, and to the attraction of the ocean itself.

If e denotes the equilibrium height of the tide, it. is a function of colatitude and longitude, and may be expanded in a series of spherical surface harmonics e<. Thus we may write the equation to the equilibrium tide in the form.

r = α-f-e — α+∑ε<∙

Now it appears from (10) and (11) that the value of the potential, at the surface of the sphere where ρ = α, under which this is a figure of equilibrium, is

V = ∑gδ,∙ei.

We may use this as specifying the external disturbing force due to the known attractions of the moon and sun, so that e» may be regarded as known.

But in our dynamical problem the ocean is not a figure of equi­librium, and we may denote the elevation of the surface at any moment of time by 6. Then the equation to the surface may be written in the form

r = α+b = α+∑t⅛,

where ⅛ denotes a spherical harmonic just as e,∙ did before.

The surface value of the potential of the forces which would main­tain the ocean in equilibrium in the shape it has at any moment is *∑gb(b{.* Hence it follows that in the actual case the forces due to fluid pressure and to the attraction of the ocean must be such as to balance the potential just determined. Therefore these forces are those due to a potential—*∑gbibi.* If we add to this the potential of the external forces, we have a potential which will include all the forces, the expression for which is —g∑i⅛(bj-e<). If further we perform the operations *d∣adθ* and *d∣a* sin *θd<l>* on this potential, we obtain the colatitudinal and longitudinal forces which are equal to the accelerations Ξ and H.

It follows, then, from (14) that the equations of motion are g-2nsin0cos⅛=-f∑ft4(bi~e∙) | (is) sιn ⅛+2κ cos 0S = -Γ⅛∙fl2δ⅛ (bi~e) J

It remains to find the equation of continuity. This may be de­duced geometrically from the consideration that the volume of an element of the fluid remains constant ; but a shorter way is to derive it from the equation of continuity as it occurs in ordinary hydro­dynamical investigations. If Φ be a velocity potential, the equation of continuity for incompressible fluid is

*ir7r (r^s'm 9 iθ i≠) +δ0⅛ (r sin 0^δrtψ)*

+δφ⅛(rF⅛⅛Sδri0) =°·

The element referred to in this equation is defined by *r, θ, φ, r-}-6r,* 0+δ0, *φ+δφ.* The colatitudinal and longitudinal velocities are the same for all the elementary prism defined by *θ, φ, β+δθ, Φ+δφ,* and the sea bottom. Then —⅛5=sin⅛

and, since the radial velocity is *db∣dt* at the surface of the ocean, where r = α-f-γ, and is zero at the sea bottom, where *r=a,* , dΦ *r— a db ττ ...*

we have 37=—^” fâ· Hence, integrating with respect to *r* from *r~a+y* to *r—a,* and again with respect to *t* from time *t* to the time when b, ξ, 17 all vanish, and treating γ and ⅛ as small compared with *a,* we have

*ba* sin 0+^(-yi sin 0)+^(γ4 sin 0) =0. (16)

■This is the equation of continuity, and, together with (15), it forms the system which must be integrated in the general problem of the tides. The difficulties in the way of a solution are so great that none has hitherto been found, except on the supposition that γ, the depth of the ocean, is only a function of latitude. In this case (16) becomes

bo+≡⅛^0⅛^sin0)+τ⅞=o∙ ('7)

§T6. *Adaptation to Forced Oscillations.—*Since we may suppose that the free oscillations are annulled by friction, the solution re­quired is that corresponding to forced oscillations. Now we have seen from (13) that e (which is proportional to V) has terms of three kinds, the first depending on twice the moon’s (or sun’s) hour-angle, the second, on the hour-angle, and the third independent thereof. The coefficients of the first and second vary slowly, and the whole of the third varies slowly. Hence e has a semi-diurnal, a diurnal and a long-period term. We shall see later that these terms may be expanded in a series of approximately semi-diurnal, diurnal and slowly varying terms, each of which is a strictly harmonic function of the time.

Thus according to the usual method of treating oscillating systems, we may make the following assumptions as to the form of the solution

e = ∑e,∙ = ∑e,∙ c0s(2n∕∕+sφ+<χ)

b = b∑,∙ = ∑⅛,∙ c0s(2n∕f+s<∕>+a) \_ z «■,

*ξ = ΣbiXi c0s(,2nft+sφ+<i)* Γ '1 '

*η=∑b<yi sin(,2nft+sφ+a)*

where e,∙, *hi,* x,∙, y,∙ are functions of colatitude only, and e,∙, *hi* are the associated functions of colatitude corresponding to the harmonic of order » and rank s.

For the semi-diurnal tides *s = 2* and *f* is approximately unity; for the diurnal tides *s = ι* and *f* is approximately ⅛ ; and for the tides of long period 5=0 and/is a small fraction.

Substituting these values in (17) we have

2[si⅛0 *ãø(ybiXi sin* =0. (19)

Then if we write «,· for *hi—a,* and put *m = n,a∕g,* substitution from (18) in (15) leads at once to

*f∑bixi-f* sin 0 cos 0∑δiy.'=⅛ *^iUi, .*

∕5 sin 0Σδ,∙y,∙+∕ cos0∑⅛,∙⅞t∙= -4fft *g∑biui.*

*@@@1 Phil. Trans.,* 189 A, pp. 201-258 and 191 A, pp. I39~i85.