Solving (20), we have

(26l∙χs)(f-cos’ *e-) =⅛* 1

*(∑i>iyi)* sin’ β(∕2-cos≡ β) = -∙j‰ [≡y-β 3⅛26itii+sιnβ2δl"i].

Then substituting from (21) in (19) we have

1 d 7(sin 0⅛∑¼ul +-7∙cos 0∑¼t⅛) sinSdø ÉL\_ *J*

**L — <l>a β**

**57 Γ, 22-0 ∙‰¼s,∙+-r^-g∑6jU,∙l**

~~--⅛Ar~oΛ√~~~~1~~~~-~~~~l +4~~~~≡≡^~~~~+~~~~\*)-~~~~0~~~~∙ . (≡≡)~~

This is almost the same as Laplace's equation for tidal oscillations in an ocean whose depth is only a function of latitude. If indeed we treat *bi* as unity (thereby neglecting the mutual attraction of the water) and replace ∑u,∙ and ∑e,∙ by *u* and *e,* we obtain Laplace’s equation.

When *Ui* is found from this equation, its value substituted in (21) will give *Xi* and y,∙.

§17. *Zonal Oscillations.—*We might treat the general harmonic oscillations first, and proceed to the zonal oscillations by putting s=o. These waves are, however, comparatively simple, and it is well to begin with them. The zonal tides are those which Laplace describes as of the first species, and are now more usually called the tides of long period. As we shall only consider the case of an ocean of uniform depth, *y* the depth of the sea is constant. Then since in this case r = o, our equation (22), to be satisfied by t⅛ or A,—e;, becomes

d∣^sin⅛⅛^∣ *+Hna, m.=o fi-*cos’0 I y

This may be written

*j-ΣbiUi+^* sin 3d0+A =0∙ t(23)

where *A* is a constant.

Let us assume

*hi — C{Pij ct= E{P i*

where *Pi* denotes the íth zonal harmonic of cos *β.* The coefficients *Ci* are unknown, but the *E,* are known because the system oscillates under the action of known forces.

If the term involving the integral in this equation were expressed in terms of differentials of harmonics, we should be able to equate to zero the coefficient of each *dPi∣dθ* in the equation, and thus find the conditions for determining the *C’s.*

» cos^0 /"ø φ β φ

The task then is to express I *Pt* s⅛ Ödø in differentials

of zonal harmonics.

It is well known that *Pi* satisfies the differential equation

**⅛ (s'ln 0⅛) +\*'(\*'+1 )Pi sin β=0∙ (24)**

Therefore *jPi* sin *θdβ ≈ ~* ~~j~~~~,~~~~(^~~~~1~~~~l~~~~'ψ i)'~~~~s~~~~\*~~~~n~~~~-~~~~e~~~~~⅞~~~~j, aπc~~~~\*~~

*~~fl~~~~~~~~~s~~~~^θ~~ ~~9~~~~f~~***pi sinθdβ J(⅛j^-cos, 0>⅛**

—i⅛η(∕,- 1)⅛r~i⅛jsbs 0¾∙

Another well-known property of zonal harmonics is that

≡in⅛⅞=¾T<p^-p∙-)∙ (≡5)

If we differentiate (25) and use (24) we have

1⅛T (¾d*~~Ιr)* +»(»+I)Λ∙ sin 0=o. (26)

Multiplying (25) by sin *θ,* and using (26) twice over,

• ∙^dpi -\*’(\*'÷1) ( I *idPi+i dPi∖ ,* I *∕dPi dPi.i∖ )* slπ<7β 2i4-ι ( 2⅛3 ∖ *dθ dβ ) ~'2i-τ∖3β' di)* ∕ J,

**τherefore⅛^p. siπ gdø=(2.\_ljI(2.÷l) ¾s**

J ∕,-t , \_ *2* j *dPi I dPjt,*

( ’(«+Ό r(2\*-ι)(2t+3) ) *'fió* +(2i+ι)(2i+3) *dθ '*

This expression, when multiplied by *4ma∣y* and by *Ci* and summed, is the second term of our equation.

The first term is

≡f⅛(C<-

In order that the equation may be satisfied; the coefficient of each *dPi∣dθ* must vanish identically. Accordingly we multiply the whole by *y∣⅛ma* and equate to zero the coefficient in question, and obtain

4m<∕c\* ∙e\*j÷(2¼- l)(,2\*-^3j- i + (2» -1 j (,2t+3) í Cí

+ (≡i+3)(≡\*+5) " °'

This equation (27) is applicable for all values of í from 1 to infinity, provided that we take Co, *E<,,* C-ι, £\_i as being zero.

We shall only consider in detail the case of greatest interest, namely that of the most important of the tides generated by the attraction of the sun and moon. We know that in this case the equilibrium tide is expressed by a zonal harmonic of the second order ; and therefore all the E,∙, excepting £j, are zero. Thus the equation (27) will not involve *Ei* in any case excepting when »=2.

If we write for brevity

*τf,-'l* I . 2 , 6ιγ

∙u i(i+l)+(2i-l)(21'+3) 4»;<i’

the equation (27) is

(2t+3)'(2i+5)- L·+(2í - 3) (21 -1 ∫=°∙ (28)

Save that when i=2, the right-hand side is i⅛7½∕4mα, a known quantity *ex hypothesi.*

The equations naturally separate themselves into two groups in one of which all the suffixes are even and the other odd. Since our task is to evaluate all the *C’s* in terms of *Et,* it is obvious that all the C’s with odd suffixes must be zero, and we are left to consider only the cases where 1=2, 4, 6, &c.

We have said that Co must be regarded as being zero ; if however we take

C0--3⅛γB√4ntα,

so that Co is essentially a known quantity, the equation (28) has complete applicability for all even values of *i* from 2 upwards.

The equations are

Co r z- , C⅜ —X1G+—-o

G *τ r∙* I Co

**r^-IoC<+-=0.**

&c. &c.

It would seem at first sight as if these equations would suffice to determine all the C's in terms of C∣, and that Ct would remain in­determinate; but we shall show that this is not the case.

For very large values of 1 the general equation of condition (28) tends to assume the form

C,⅛¾÷C,-t ! *ily Ci* "’m

By writing successively 1+2, 1+4, \*-f-6 for 1 in this equation, and taking the differences, we obtain an equation from which we see that, *unless Ci∣Ci+t tends to become infinitely small,* the equa­tions are satisfied by C< = C{+j in the limit for very large values of í.

Hence, if C, does not tend to zero, the later portion of the series for *h* tends to assume the form C,(P<+P<+s-l-Λ+4...). All the *P's* are equal to unity at the pole; hence the hypothesis that C, does not tend to zero leads to the conclusion that the tide is of infinite height at the pole. The expansion of the height of tide is essentially convergent, and therefore the hypothesis is negatived. Thus we are entitled to assume that C,∙ tends to zero for large values of *i.*

Now writing for brevity

αi = ι∕(2i+ι)(2i+3)8(2t+5),

we may put (28) into the form

*Ci-PC< -r‰*

By successive applications of this formula we may write the right­hand side in the form of a continued fraction.

Let

*v a∣-t 0i ai+t*

*-tii≈J* — 7- — w—- —...

*■L'l* **— Z√i+4-**

Then we have

Cua∕C⅜∙ fl⅜-¾

(21- (21-1.) “T? or C,/Ci\_« = (2i —i)(2í+i)£<.

Thus

Cs=3∙5∙^iCo! *Ct=3∙5∙7 ∙9^sKtCt',*

*Ct=3.5,7.<).11.13KtKtKtCll,* &c.

If we assume that any of the higher C’s, such as Cu or Cu,,is of negligible smallness, all the continued fractions λ'2, *Kt, Kt,* &c., may be computed; and thus we find all the C’s in terms of Co, which is equal to— *3btyEt∕4ma.* The height of the tide is therefore given by

h = ∑A,cos (2 *nft*+α)

= -^^¾{3.SKΛ+3-5∙7∙9^i^Λ + . . .) cos (2κ∕∕ + a).

It is however more instructive to express ⅛ as a multiple of the