equilibrium tide *t,* which is as we know equal to *EtPt* cos (2np+<1). Whence we find

b = ~4⅛ ⅛3∙5AW,s+3∙5.7.9^Λ+3∙5... *ι3KiKtKtP,... ).* The number/ is a fraction such that its reciprocal is twice the number of sidereal days in the period of the tide. The greatest value of ∕ is that appertaining to the lunar fortnightly tide (Mf in notation of harmonic analysis), and in this case/ is in round numbers 1/28, or more exactly ∕2 = ∙00133. The ratio of the density *σ* of sea-water to δ the mean density of the earth is -18093; which value gives us ⅛ = 1-^=∙89144.

The quantity *m* is the ratio of equatorial centrifugal force to gravity, and is equal to 1/289. Finally, *y/a* is the depth of the ocean expressed as a fraction of the earth’s radius.

With these numerical values Mr Hough has applied the solution to determine the lunar fortnightly tide for oceans of various depths. Of his results we give two :—

*First,* when 7 = 7260 ft. = 1210 fathoms, which makes *y∣4ma-l∣40,* he finds

b=p^∣∙2669P5--ι678P4+∙0485Pe-∙∞81Ps+∙∞09P10—o∞ιPι2...). If the equilibrium theory were true we should have b=⅛P2∣1

thus we see how widely the dynamical solution differs from the equilibrium value.

*Secondly,* when 7 = 58080 ft. =9680 fathoms, and 7∕4mα = ι∕5, he finds

b=∙p-(∙72θ8P2--0973∙P4+∙0048Pt-∙oooιPβ. ■ ■ 1-

From this we see that the equilibrium solution presents some sort of approximation to the dynamical one; and it is clear that the equilibrium solution would be fairly accurate for oceans which arc still quite shallow when expressed as fractions of the earth's radius, although far deeper than the actual sea.

The tides of long period were not investigated by Laplace in this manner, for he was of opinion that a very small amount of friction would suffice to make the ocean assume its form of equilibrium. In the arguments which he adduced in support of this view the friction contemplated was such that the integral effect was propor­tional to the velocity of the water relatively to the bottom. It is probable that proportionality to the square of the velocity would have been nearer the truth, but the distinction is unimportant.

The most rapid of the oscillations of this class is the lunar fort­nightly tide, and the water of the ocean moves northward for a week and then southward for a week. In oscillating systems, where the resistances are proportional to the velocities, it is usual to specify the resistance by a " modulus of decay,” namely the time in which a velocity is reduced by friction to *e~1 or 1/2-78 of* its initial value. Now in order that the result contemplated by Laplace may be true, the friction must be such that the modulus of decay is short compared with the semi-period of o<∙',lation. It seems certain that the friction of the ocean bed would not reduce a slow ocean current to one-third of its primitive value in a day or two. Hence we cannot accept Laplace’s discussion as satisfactory, and the inves­tigation which has just been given becomes necessary. (See § 34).

§ 18. *Tesseral Oscillations -* -The oscillations which we now have to consider are those in which the form of surface is expressible by the tesseral harmonics. The results will be applicable to the diurnal and semi-diurnal tides— Laplace's second and third species.

If we write <r = s∕∕ the equation (22) becomes *j* ∣^ (sin 0g⅛+<r cos ø ) ΣZ>iKil *(a* cos 0-^j+s2 cosec lîj ∑δ,w> d0L s2- σ2 cos- *θ* I *st-* σ2 cos2 0

sin β∑⅛4=θ. (29)

If we write *D* for the operation sin 0f⅞, the middle term may be arranged in the form

*a* cot *0* (D+σ cos *θ)(,ΣbjUj) \_ ΣbiUi* s2-σ2cosi0 sin *0'*

Therefore on multiplying by sin 0 the equation becomes *(,D-σ* COS0) ~~[~~~~cp~~~~⅛~~~~c~~~~σ~~~~θ~~~~s~~~~c~~~~θ~~~~s~~~~⅞~~~~ιιl~~~~'~~~~l~~] -(∑δ.-w,∙)+^fsin20∑‰=o. (30)

We now introduce two auxiliary functions, such that *∑b{(thi-e∕) ss∑biiit*

*= (D-a* cos 0)Φ+(s2-σ2 cos2 0)φ. (31)

It is easy to prove that

(O-∣-σ cos 0)(P-*a* cos 0) = Z)2-s2+σ sin2 0+(s2-σ2 cos2 0), ) z ⅛ (D-σ cos 0)<r>4-σ cos 0) =P2-s2-<r sin2 0+(s2-σ2 cos2 0). f ' ''3j

Also

(2>+σ COS 0)(s2-σ2 COS2 0) φ = (i2-σ2 cos2 0)(Z>+<r cos 0)φ +2σ2 sin2 0 cos 0 Φ. (33) Now perform D+σ cos 0 on (31), and use the first of (32) and (33), and we have

(Z>+σ cos 0)(∑¼tti) = *(Di-*s2+<r sin2 0-∣-s2-σ2 cos20)ψ.

+ (i2-σ2 cos2 0)(Z>-j-σ COS 0)φ-∣-2σ2 si∏20 COS 0 Φ. (34) The functions Ψ and Φ are as yet indeterminate, and we may impose another condition on them. Let that condition be *(Dt-*s\*+σ sin2 0)Φ = — *2σt* sin, 0 cos 0 Φ. (35)

Then (34) may be written

(Z>+σ cos 0)(ΣZ⅛Wi) τ , zrι 1

— w⅛⅞- = H(β÷σ cos 0)Φ.

Substituting from this in (30), and using the second of (32), the function Ψ disappears and the equation reduces to '

(P2-*st-σ* sin2 0)Φ+7~⅛r sin2 0 ∑⅛ = o. (36)

Since by (35) —σ2 cos2 0 Φ = J-iif-^(Γ2-r24-σ sin2 0)Φ, (31) may be written

∑⅛,∙ι¾ = jjp—σ cos 0÷i^=-r∣(D2-J2+<r sin2 0)Jψ-∣-s2Φ. (37)

The equations (35), (36) and (37) define Ψ and Φ, and furnish the equation which must be satisfied.

If we denote cos 0 by *µ* the zonal harmonics are defined by p\*=⅛½)^2-1>i∙

The following are three well-known properties of zonal harmonics : ⅛[<ι-Ms)⅛]+2∙σ+>)Λ=o, (38)

(» + l)P⅛l - (2i+l)µPi+ÏPí\_i = O, (39)

¾u-⅛,-⅛<+∙>Λ∙ <4<,)

If *P,i c.w, sφ* are the two tesseral harmonics of order « and rank s, , sin

it is also known that

Λ'=(l-M2)i∙⅛ (41)

Let us now assume

*hi = C∙P', ei=EtiP∙, V=Σa'P∙, Φ=Σβ∙P'.*

These must now be substituted in our three equations (35), (36), (37), and the result must be expressed by series of the *P’.* functions. It is clear then that we have to transform into *P\*i* functions the following functions of *P,i,* namely

-~^(P2-i2±σ sin2 0)PJ, cos 0P\*, [p-σ cos 0+⅛^∣(D2-s∙+o- sin2 0)]p,,.

If we differentiate (38) 5 times, and express the result by means of the operator *D,* we find

(Z>2-s2)Pj-J-l(t-∣-l)P∙ sin2 0 = 0. (42)

Again, differentiating (39) í times and using (40), we find (i-s-∣-ι)Pj+1-(2i + ι) cos 0 P5+(i-H)P\*j=0. (43)

Lastly, differentiating (41) once and using (38), (40) and (43) r,r,, ι(i-s+ι)n, (i+l)(i+i)n<, z..λ

dp<~ 2Ï+ì—p∙\* Γ^+l—p\*->∙ (44)

By means of (42), (43) and (44) we have ^(P2-s2±σ sin2 *0)P∙ = [-i(i+*1) =tσ]P∙, \_\_\_ Λ ps p∙ t I *n∙*

cosθ j > 2i-H\* i-1+ 21 +1 p∙'+l'

Γ τ⅜ I jCOS0∕γι, I \_·\_» λ∖^1 τ>\* (^\* "~ l)ln∙

P~σ+⅛0<p ~i +σsltl 0)JP< - P‘«

\_ (\* +^)[<r+(t+l)(t +2)l *p,*

2(2i+i) i"i'

Therefore the equations (35), (36), (37) give ∑[α'(-i(i+l)+<rlP∙+2σ2ft∙∣ ⅛Λ'-ι+⅛¾iΛ\*÷1 j ] *=0,* 2[tfl -t(i+1)-σ)P∙+^C!pf∣ =0, ∑[δi(C∙-P∙)P'+α∙ j g^,¾ff~⅛ +0'+^⅞¾κf+⅜ I -^] =o∙