with the forced lunar or solar wave, the actual tide is largely aug­mented. Thus, for,example, for an ocean of 29,000 ft. in depth the solar semi-diurnal tide would have a height at the equator 235 times as great as the equilibrium height, and would be inverted so that low water would agree with the high water of the equilibrium theory.

The general outcome of the discussion is that it is impossible to foresee the height of any forced tide-wave by mere general inspection. If this is so in the simple case of an ocean of uniform depth, how much more must it be true of oceans of various depths interrupted by continents?

§ 21. *Stability of the Ocean.—*Imagine a globe of density δ, sur­rounded by a spherical layer of water of density *σ.* Then, still maintaining the spherical figure, and with water still covering the nucleus, let the layer be displaced 'sideways. The force on any part of the water distant *r,* from the centre of the water and *r* from the centre of the nucleus is jιrστ, towards the centre of the fluid sphere and jτr(δ-σ)r towards the centre of the nucleus. If *δ* be greater than *σ* there is a force tending to carry the water from places where it is deeper to places,where it is shallower; and therefore the equili­brium, thus arbitrarily disturbed, is stable. If, however, δ is less than *σ* (or the nucleus lighter than water) the force is such that it tends to carry the water from where it is shallower to where it is deeper and therefore the equilibrium of a layer of fluid distributed over a nucleus lighter than itself is unstable. As Lord Kelvin remarks,@@1 if the nucleus is lighter than the ocean, it will float in the ocean with part of its surface dry. Suppose, again, that the fluid layer be disturbed, so that its equation is r=α(ι-f-Si), where s< is a surface harmonic of degree t; then the potential due to this deformation is ∙φι+i ⅛, and the whole potential is

4πδα\* t *4πσ at+i ~3r i~2i-j-1 r\*+1 si'*

If, therefore, σ∕(2i-∣-ι) is greater than ⅜δ, the potential of the forces due to deformation is greater than that due to the nucleus. But we have seen that a deformation tends to increase itself by mutual attraction, and therefore the forces are such as to increase the deformation. If, therefore, σ = ⅛(2i4-ι)δ,, all the deformations up to the ïth are unstable, but the (i⅛ι)th is stable.@@2 If, however, σ be less than *S,* then all the deformations of any order are such that there are positive forces of restitution. For our present purpose it suffices that the equilibrium is stable when the fluid is lighter than the nucleus.

§ 22. *Precession and Nutation.—*Suppose we have a planet covered with a shallow ocean, and that the ocean is set into oscillation. Then, if there are no external disturbing forces, so that the oscilla­tions are u free,” not “ forced,” the resultant moment of momentum of the planet and ocean remains constant. And, since each particle of the ocean executes periodic oscillations about a mean position, it follows that the oscillation of the ocean imparts to the solid earth oscillations such that the resultant moment of momentum of the whole system remains constant. But the mass of the ocean being very small compared with that of the planet, the component angular velocities of the planet necessary to counterbalance the moment of momentum of the oscillations of the sea are very small compared with the component angular velocities of the sea, and therefore the disturbance of planetary rotation due to oceanic reaction is negligible. If now an external disturbing force, such as that of the m∞n, acts on the system, the resultant moment of momentum of sea and earth is unaffected by the interaction between them, and the precessional and nutational couples are the same as if sea and earth were rigidly connected together. Therefore the additions to these couples on account of tidal oscillation are the couples due to the attraction of the moon on the excess or deficiency of water above or below mean sea-level. The tidal oscillations are very small in height compared with the equatorial protuberance of the earth, and the density of water is -⅛ths of that of surface rock; hence the additional couples are very small compared with the couples due to the moon’s action on the solid equatorial protuberance. Therefore pre­cession and nutation take place sensibly as though the sea were congealed in , its mean position. If the ocean be regarded as frictionless, the principles of energy show us that these insensible additional couples must be periodic in time, and thus the corrections to nutation must consist of semi-diurnal, diurnal and fortnightly nutations of absolutely insensible magnitude. We shall have much to say below on the results of the introduction of friction into the conception of tidal oscillations as a branch of speculative astronomy, .

§ 23. *Some Phenomena of Tides in Rivers.—*As a considerable part of our practical knowledge of tides is derived from observations in estuaries and rivers, we shall state the results of an investigation of waves which travel along a shallow canal, and we refer the reader to the article Waves for the mathematical investigations on which they are based.

It must be premised that when the profile of a wave does not pre­sent the simple harmonic form, it is convenient' to analyse its shape into a series of partial, waves superposed on a fundamental wave; and generally the principle of harmonic analysis is adopted in which the actual wave is regarded as the sum of a number of simple waves.

⅛se that the water is contained in a straight and shallow canal m depth Ä, and that at one end the canal debouches on to the open sea. Suppose further that in the open sea there is a forced oscillation of water level, given by this formula

*η = H* sin *nt* where *η* is the elevation of the water at time *t* above its mean level, *21r∣n* the period of the oscillation and *H* the amplitude of the oscillation.

Waves will clearly be transmitted along the canal, and the problem is to obtain a formula which shall represent the oscillations of level at any distance *x* measured from the mouth of the canal.

The mathematical investigation shows that, if g denotes gravity, the formula for the oscillation of water level at the point defin⅛ by *x* is

*η=H* sin *n* (\*~⅛) +⅛⅛≈ ≡ ≡n (i-ς⅛) ·

The second of these terms is proportional to *x,* and if the canal were infinitely long it would become infinite. The difficulty thus occasioned may be eluded by supposing the canal to debouch on a second sea in which a second appropriate oscillation is maintained. In actuality friction gradually annuls all motion, and no such diffi­culty arises.

The first term of the formula is called the fundamental tide, the second gives what is called the first over-tide; and further approxi­mation would give second and third over-tides, &c. All the over- tides travel up the river at the same rate as the fundamental, but they have double, treble, quadruple frequencies or ” speeds,” and the ratio of the amplitude of the first over-tide to the fundamental is *3H nx 4h ^gh'*

As a numerical example, let the range of tide at the river mouth be 20 ft., and the depth of the river 50 ft. The “ speed ” of the semi­diurnal tide, which is an angular velocity, is 28∙980 per hour or ι∕ι ∙9 radians per hour; √gλ = 27 miles per hour; hence—-2-x. Therefore 34. miles up the river the over-tide is ι∕ιoth of the funda­mental ana has a range of 2 ft. If the river shallows very gradually, the formula will still hold, and we see that the height of the over- tide varies as (depth).

Fig. 6@@3 read from left to right exhibits the progressive change of shape. The steepness of the advancing crest shows that a shorter

•

time elapses between low to high water than inversely. The same investigation shows that the law of the ebb and flow of currents, mentioned in § 2, must hold good.

The second law of waves in rivers to which we draw attention relates to the effects produced by the simultaneous propagation into shallow water of two waves of different periods. It appears that the effect is not simply the summation of, the two separate wτaves.

Suppose that at the mouth of the river the oscillation of the open sea is represented by

*V=Hλ* sin n√+Hι sin *(n2t-∖-t).*

Then we find that at distance *x* from the river’s mouth the wave is given by the formula

*η* = ¾ sin w, (<- *~r.) + H,* sin [na (∕- +1]

+2fi⅛≈--∙[<∙∙ + '≈>(-⅛)+i -2¾i∙5S⅛∙>"[⅛+∙∙)(-⅛)+∙]' The first two terms give us the two waves just as if each existed by itself. The third and fourth terms give the results of their combina­tion, and are called “compound ” tides, the first being a summation tide and the second a difference tide.

As a numerical example, suppose at the mouth of a river 50 ft. deep that the solar semi-diurnal tide has a range 2ΣΓ1 = 4 ft., and the lunar semi-diurnal tide has a range 2∑Zι = I2j then = 59/57 radians per hour, and «i—«3 = 1/57 radians per hour, and as before √g⅛=27 miles per hour.

With these figures

¾gl⅞ «l+«¾ ï λλ

@@@1 Thomson and Tait, *Nat. Phil.* § 816.

@@@2 Compare an important paper by H. Poincaré, in *Acta math.* (1885), 7; 3, 4∙

@@@8 From Airy’s *Tides and Waves*, with omission of part which was erroneous.