Thus i§ miles up the river the quater-diurnal tide (called MS in harmonic analysis) would have a range of 1/60 of an inch. Where the two interacting compound tides are nearly of the same “ speed ” the summational compound tide is much the larger of the two. As before, when the river shallows gradually this formula will still hold true.

It is interesting to note the kind of effect produced by these com­pound tides. When the primary tides are in the same phase (as at spring tide)

Wιf = t⅛f ·f“€>

and we may write the formula in the form

-,-rx . 1-τv . *∕. x ∖ . 3tH∖Hι* nι÷n2 . Γ\_ j (λi+w2)x-]

”=sιn ⅛ v~√ix) + 4⅛-√ir\*sm L2wιi—√⅛ !

I 3⅞⅞ ⅜-⅝-, -,∙n (”■ - \*⅞)\*. ‘ 4⅛ √g⅛ x *∖gh*

Hence the front slope of the tide-wave is steeper.at springs than at neaps, and the compound tide shows itself at springs in the form of an augmentation of the.first over-tide; the converse holds at neaps. Also mean water-mark is affected to a slight extent as we go up the river by an inequality represented by the last term.

IV.—Harmonic Analysis

§ 24. *Outline of the Method.—*We have seen in §13 that the poten­tial of the tide-generating force of the moon consists of three terms, one being approximately semi-diurnal, one approximately diurnal, and one varying slowly. In consequence of the irregular motion of the moon in right ascension and in declination and the variability of parallax, none of these three classes of terms is simply harmonic in time. The like is also true of the potential of the sun’s tide­generating force. In the method of harmonic analysis we conceive the tidal forces or potential due to each disturbing body to be developed in a series of terms each consisting of a constant (deter­mined by the elements of the planet’s orbit and the obliquity of the ecliptic) multiplied by a simple harmonic function of the time. Thus in place of the three terms of the potential as developed in § 13 we have an indefinitely long series of terms for each of the three terms. The loss of simplicity in the expression for the forces is far more than counterbalanced by the gain of facility for the dis­cussion of the oscillations of the water. This facility arises from the dynamical principle of forced oscillations, which we have ex­plained in. the historical sketch. Applying this principle, we. see that each individual term of the harmonic development of the tide­generating forces corresponds to an oscillation of the sea of the same period, but the amplitude and phase of that oscillation must depend on a network of causes of almost inextricable complication. The analytic or harmonic method, then, represents the tide at any port by a series of simple harmonic terms whose periods are determined from theoretical considerations, but whose amplitudes and phases are found from observation. Fortunately the series representing the tidal forces converges with sufficient rapidity to permit us. to consider only a moderate number of harmonic terms in the series.

Now it seems likely that the corrections which have been applied in the use of the older synthetic method might have been clothed in a more satisfactory and succinct, mathematical form had investigators first carried out the harmonic development. In this article we shall therefore invert history and come back on the synthetic method from the analytic, and shall show how the formulae of correction stated in harmonic language may be made comparable with them in synthetic language. One explanation is expedient before proceeding with, the harmonic development. There are certain terms in the tide-generating forces of the moon, depending cn the longitude of the moon’s nodes,, which complete their revolution in .18∙6 years. Now it has been found practically convenient, in the application of the harmonic method, to follow the syn­thetic plan to the extent of classifying together terms whose periods differ only in consequence of the movement of the moon’s node, and at the. same time to conceive that there is a small variability in the intensity of the generating forces.

§ 25. *Development of Equilibrium. Theory of Tides in. Terms of the Elements of the Orbits.—*Within the limits at our disposal we cannot do more than indicate the processes to be followed in this development. We have already seen in (2) that the expression for the moon’s tide-generating potential is

z-⅜),

and in (12) that

M12-M∏2

cos2 *z-* i=2^M1Mi+2≤^∙i^-τ^+2^M2M1 + 2^M1M,

,3 1 M12+M22-2M32

"t2 3 3

where Mi, M2, M> and *ξ, ηt* f are respectively the direction cosines referred to axes fixed in the earth of the moon and of a place on the earth’s surface at which the potential *V* is to be evaluated. At such a place the radius vector *p* is equal to *a* the earth’s radius.

Let the axes fixed in the earth be taken as follows : the axis *C* the north polar axis; the axis *A* through the earth’s centre and a point on the.equator on the same meridian as the place of observa­tion; the axis *B* at right angles to the other two and eastward of *A,* Then if λ be the latitude of the place of observation

f = cos λ, *η=o, f* = sin λ.

With these values we have

cos2 z —i = i cos2 λ(Mι2-M22) +sin 2λ.M1M2 +i(i-sin2 λ) (M12+Ms2-2Mj2).

Ift fig. 7 let ABC be the axes fixed in the earth; XYZ a second set of axes, XY being the plane of the moon’s orbit; M the projection of the. moon in her orbit; *∕≈>ZC,* the obliquity of the lunar orbit to the equator; χ=AX=BCYj Z = MX, the moon’s longitude in her orbit measured from X, the descending node of the equator on the lunar orbit, hereafter called the “ intersec­tion.”

Then

Mi = cos *I* cos χ+sin *I* sin χ cos *I* =cos2 *⅛I* cos (χ—Z)

÷sin2 ∣Z cos (χ+Z), M2= —cos *I* sin χ÷sin *I* cos χ cos Z= —cos2 *⅛I* sin (χ—Z)

-sin2 ∣Z sin (χ-W), Mj=sιn Z sm *I* -2 sin JZ cos JZ sin Z.

When these expressions are substituted in Mi2—M22, MjM8, Mi2÷M22-2Ms2, it is clear that the first will have terms in the cosines of 2(χ-Z), .2χ,. 2(χ+Z) ; the second.in sines of χ-2Z, χ, xd-2/; and the third in cos *2lt* together with a term depending only on Z.

Now let *c* be the moon’s mean distance, *e* the eccentricity of her orbit, and let

χ = 5M11 γ = 5Msi Z = p⅛≤-\*] iM1,

andr=∣∣∙

Then we have for the lunar tide-generating potential at the place of observation

*v=co≡2 λ · (χ2 “ γ2) + s'n 2λ ■xz*

+J(⅛-sin\*λ) (X≈÷Y≡-2Z≈) (47)

The only parts of this expression which are variable in time are the functions of X, Y, Z.

To complete the development the formulae of elliptic motion are introduced in these functions, and terms which appear numerically negligible are omitted. Finally, the three X-Y-Z functions are obtained as a series of simple time-harmonics, the arguments of the sines and cosines being linear functions of the earth⅛ rotation, the moon’s mean motion, and the longitude.of the moon’s perigee. The next step is to pass, according to the principle of forced oscillations, from the potential to the height of tide generated by the forces corresponding to that potential. The X-Y-Z functions being simple time-harmonics, the principle of forced oscillations allows us to con­clude that the forces corresponding to *V* in (47) will generate oscilla­tions in the ocean of the same periods and types as the terms in *Vt* but of unknown amplitudes and phases. Now let .X2-\*^2., XZ, (X2+322-2≡2) be three functions having respectively similar forms to those of

X2-Y2 XZ (X2+Y2-2Z2)

√ι-fψ,(Γ√p,anα (ι-β2)3 '

but differing from them in that the argument of each of the simple time-harmonics has some angle subtracted from it, and that the term is multiplied by a numerical factor. Then, if g be gravity and h the height of tide at the place of observation, we must have

h = y li cos2 λ (X2-O +sin 2λXZ + ! (⅛ - sin2λ) ⅛(X2 +⅛ - 2≡2)]. (48) The factor *τa2∣g* may be more conveniently written^^ (c)fl, w^ιere *M* is the earth’s mass. It has been so chosen that, if the equilibrium theory of tides were fulfilled, with water covering the whole earth, the numerical factors in the functions would

be each unity and the alterations of phase would be zero. The terms in (X2+¾2- *2Zt)* require special consideration. The function of the latitude being

sin2 λ, it follows that, when in the northern hemis­phere it is high-water north of a certain critical latitude, it is low- water on the opposite side of that parallel ; and the same is true of the southern hemisphere. It is best to adopt a uniform system for the whole earth, and to regard high-tide and high-water as