In like manner, the whole elevation at *a* above the inſcribed ſphere is M — S.

Hence we ſee that the whole tide, when the moon is in quadrature, is the difference of S and Μ. We alſo ſee, that if M exceeds S, the water will be higher at *a* than at *s.* Now it is a matter of obſervation, that in the quadra­tures it is high water under the moon, and low water under the ſun. It is alſo a matter of obſervation, that in the free ocean, the ebb tide, or the water at *s,* immediately under the ſun, is below the natural ſurface of the ocean. Hence we muſt conclude, that 2/3S is leſs than ⅓ M, or that M is more than double of S. This agrees with the phenomena of nutation and preceſſion, which ſeem to make S = ⅖ of Μ.

In all other poſitions of the ſun and moon, the place of high water will be different. It is high water where the sum of the elevations produced by both luminaries above the natural ocean is greateſt ; and the place of low water is where the depreſſion below the natural ocean is greateſt. Therefore, in order that it may be high water, we muſt have S × coſ.2x + M × coſ.2 y a maximum ; or, neglecting the constant quantity (S + M)/3, we muſt have

S × coſ.2x + M × cos.2y a maximum.

In like manner, to have low water in a place where the zenith diſtances of the ſun and moon are *v* and w, we muſt have S ∙ sin.2v + M × ſin.2*w* a maximum.

*Lemma* I. If we consider the sines and coſines of angles as numeral fractions of the radius 1, then we have cos.2Z = ½ + ½coſ.2*Z,* and ſin.2 Z = ½ — ½ coſi2 Z.

Let *ams* (fig. 3.) be a quadrant of a circle of which O is the centre, and Os is the radius. On Os deſcribe the ſemicircle OMS, cutting *Om* in Μ. Draw sM, and pro­duce it till it cut the quadrant in *n.* Alſo draw MC to the centre of the ſemicircle, and MD and *nd* perpendicular to Os.

It is plain that *s*M is perpendicular to OM ; and if O*s* be radius, *s* M is the sine of the angle *s*OM, which we may call Z ; OM is its coſine : and becauſe Os ; OM = OM : OD, and O*s* : OD = Os2 : OM2, and OD may repreſent coſ.2Z. Now OD = OC + CD. If Os = 1, then OC = ½. CD CM × coſ. MCD, — CM × coſ. 2 MOD, = ½ × cos. 2Z. Therefore cos.2Z — ½ + ½ coſ. 2 Z.

In like manner, becauſe Os : *s*M = *s*M : *s*D, sD is = ſin.2Z. This is evidently = ½ — ½ coſ.2Z.

*Lemma* 2. Coſ.2Z — sin.2Z = coſi.2Z. For, becauſe sM is perpendicular to OM, the arch s*n* is double of the arch sm, and becauſe MD is parallel to *nd, sd* is = 2sD, and dD — sin.2 Z Therefore *Od =* coſ.2Z — sin.2Z. But O*d* is the coſine of *ns, =* coſ. 2Z, and coſ.2Z — ſin.2Z = coſ. 2Z.

By the first Lemma we ſee, that in order that there may be high water at any place, when the zenith diſtances of the ſun and moon are *x* and *y,* we muſt have S × cos.2x + M× coſ.*2y* a maximum.

That this may be the case, the fluxion of this formula muſt be = 0. Now we know that the fluxions of the co­sines of two arches are as the sines of thoſe arches Therefore we muſt have S × ſin2*x* + M × ſin. 2y = 0, or S × sin. 2 *x =* — M × ſin.*2y,* which gives us ſin.2*x* : ſin. *2y* = M : S.

In like manner, the place of low water requires sin. 2*v :* ſin. 2w = M : S.

From this laſt circumstance we learn, that the place of low water is 0, removed 90⁰ from the place of high water ; whereas we might have expected, that the ſpheroid would have been moſt protuberant on that side on which the moon is : For the sines of 2*v* and of 2*w* have the ſame propor­tion with the sines of 2x and of 2y. Now we know that the sine of the double of any arch is the ſame with the sine of the double of its complement. Therefore if low water be really diſtant 90⁰ from high water, we ſhall have ſin.2*x :* ſin. 2y = ſin.2*v* : ſin.2w. But if it is at any other place, the sines cannot have this proportion.

Now let *s* be the point of the earth’s ſurface which has the ſun in the zenith, and *m* the point which has the moon in the zenith. Let *h* be any other point. Draw O*h* cutting the ſemicircle OMs in H. Make CM to CS as the diſturbing force of the moon to that of the ſun ; and draw S*v* parallel, and St, Mr perpendicular to HH'. Join MH and MH'. The angle HCs is double of the angle HOs, and MCH is double of MH'H, or of its equal MOH. Becauſe HMH is a ſemicircle, HM is perpendi­cular to MO. Therefore if HH' be conſidered as radius, HM is the sine, and H'M is the cosine of ΜΗΉ. And C*r* is = MC × coſ. *2y,* — M × coſ.*2y.* And C *t* is SC × coſ.2x*.* Therefore *tr* or S'*ν* is = S × coſ.2*x +* M × coſi.*2y.* Therefore *tr or Sν* will express the whole difference of elevation between h and the points that are 90 degrees from it on either side (by *Lemma* 2.); and if *h* be the place of high water, it will expreſs the whole tide, becauſe the high and low waters were ſhown to be 90⁰ aſunder. But when *h* is the place of high water, S*v* is a maximum. Be­cauſe the place of the moon, and therefore the point M, is given, Sv will be a maximum when it coincides with SM, and CH is parallel to SΜ.

This ſuggeſted to us the following new, and not inele­gant, ſolution of the problem for determining the place of high water.

Let sQoqs (fig. 4. and 5.) be a ſection of the terra­queous globe, by a plane paſſing through the ſun and moon, and let O be its centre. Let *s* be the point which is imme­diately under the ſun, and *m* the place immediately under the moon. Biſect Os in C, and deſcribe round C the circle OMsLO, cutting *Om* in Μ. Take C*s* to repreſent the diſturbing force of the moon, and make Cs to CS as the force of the moon to that of the ſun (ſuppoſing this ratio to be known). Join MS, and draw CH parallel to it. Draw OHh*,* and lOLl' perpendicular to it. And laſtly, draw CI perpendicular to SM. Then we say that *m* and its op­poſite *m'* are the places of high water, *l* and l' are the places of low water, MS is the height of the tide, and MI, SI are the portions of this tide produced by the moon and ſun.

For it is plain, that in this caſe the line S*v* of the laſt proposition coincides with MS, and is a maximum. We may alſo obſerve, that MC : CS ≡ sin. MSC ; ſin. SMC, = ſin. HCS : sin. MCΗ, = sin. 2hOs ; ſin. 2*h*O*m, =* ſin.2x : ſin. *2y,* or M : S — ſin.*2x* : sin. *2y,* agreeably to what was required for the maximum.

It is also evident, that MI = MC × coſ.CMI, = M∙ coſi.*2y,* and SI — SC × coſi ISC, — S × coſ.2x ; and therefore MS is the difference of elevation between *h* and the points *l* and *l',* which are 90⁰ from it, and is therefore the place of low water ; that is, MS is the whole tide.

The elevation of every other point may be determined in the ſame way, and thus may the form of the ſpheroid be completely determined.

If we ſuppoſe the figure to repreſent a ſection through the earth’s equator (which is the caſe when the ſun and moon are in the equator), and farther ſuppoſe the two lu­minaries to be in conjunction, the ocean is an oblong spheroid, whoſe axis is in the line of the ſyzigies, and whole equator coincides with the six hour circle. But if the moon be in any other point of the equator, the figure of the ocean will be very complicated. It will not be any figure of re­volution ; becauſe neither its equator (or moſt depressed