ceeding from 0 to a quadrant, the common difference of the terms being an angle of one minute, or some smaller angle.

The radius being considered as an unit, the sines are ex­pressed as decimal fractions of that unit. The quadrant is generally divided into degrees and minutes, this being suf­ficient for most purposes ; but tables have been calculated which give the sines, and tangents, and secants, to every tenth second.

6. The Elements of Euclid show how to inscribe regular polygons of three, four, five, six, and fifteen sides in a circle. The sides of these polygons may be expressed in numbers by calculations derived from the geometrical constructions. Now, the side of an equilateral triangle in a circle, is the chord of 120°, which is twice the sine of 60° ; and the side of the inscribed square is the chord of 90°, which is twice the sine of 45° ; and the side of the equilateral pentagon is the chord of 72°, which is twice the sine of 36° ; and the side of the hexagon is the chord of 60°, which is twice the sine of 30° ; and the side of the quindecagon is the chord of 24°, which is twice the sine of 12°. These sines may therefore be all deduced from the properties of polygons inscribed in a circle, as delivered in the Elements. And since the sum of the squares of the chord of an arc and the chord of its supplement is equal to the square of the dia­meter (Geom. 17 of 2 and 13 of 4), the chords of the sup­plements, and thence the sines of their halves, may be found. It was in this way the trigonometrical table was first constructed by the early mathematicians, Purbach, Miiller, Copernicus, G. Joachimus Rhæticus, and others. The theory of inscribed polygons was extended by Vieta into that of angular sections, chiefly with a view to the con­struction of the table. This has again been wrought up into the *Arithmetic* or *calculus of Sines,* the formulæ of which afford great facility in the construction of the table. As an example, we shall resolve this problem.

7. Problem. To find the sines of one fifth, two fifths, three fifths, and four fifths, of a quadrant, of which the sine is equal to the radius. Putting *a* to denote an arc of 18°, one fifth of the quadrant, we are to find the sines of *a*, 2*a,* 3*a*, 4*a*. Now, from the nature of the problem, we have these simple relations :

\*\*\*\*\*\*\*Cos. *a* = sin. 4*a*, cos. 2α = sin. 3«,

Cos. 3« = sin. 2a, cos. 4a = sin. a.

And by formula 3 of (E) art. 240, Algebra,

Sin. 2α = 2 sin. *a* cos. α,

Sin. 3a — 2 sin. *2a* cos. *a —* sin. *a,*

Sin. *4a* = 2 sin. *3a* cos. a — sin. 2σ,

Sin. 5a = 2 sin. 4a cos. *a —* sin. 3«.

Therefore Sin. 2σ = 2 sin. a sin. 4a,

Sin. 3a = 2 sin. 2a sin. 4a — sin. *a,*

Sin. 4a = 2 sin. *3a* sin. 4a — sin. 2zz,

Sin. 5⅛ = 1 = 2 sin. 4a sin. 4a — sin. 3a.

To abridge, put sin. *a = ιι,* sin. 2a = *x,* sin. 3a *— y,* sin. 4a = r, and we have the following equations :

2«z = *X* A,

*2xz = y + u* B,

*2gz = z + X* C,

2zz = 1 + *y................*D.

These express symmetrical relations, which connect the four sines w, *x, y, z.*

By subtracting A from C, and dividing the remainder by *2z,* we have

y —« = J (i).

And from B, A, D ; *y + u = 2xz* = 4ι<a\* = 2a + *2ιιy.* Therefore *2uy = y — « = ⅜ ;*

*5* and hence, *(y* + w)2 = *(y—* u)2 + 4uy =

and *y* + u = yt (2).

We have now, by subtraction and addition from (1) and (2),

**V5 — 1. v,5-4-l**

sin. α — w — — , sin. 3a = *y — ÷j—.*

+ zr

Now' the arcs a and 4o being the complements of each other, also *3a* and *2a,* and the sum of the squares of the sine and cosine being equal to the square of the radius, we have ein. 2« = « = √(1 — y,) - ~~√(10-∙g√¾),~~ ⅛,⅛=,⅛√(l-⅛) = <<ll, + i^>

Our problem is now resolved.

8. We have found the sines of a series of arcs from 0 to 90°, viz. 18°, 36°, 54°, 72°, whichdiffer by 18°. From these, the sines of a series whose terms differ by 9°, may be found by formulæ, which are modifications of formulæ (1), (2) of (P), article 249, Algebra, viz.

sin. I (90° — *a) = tf- ~∙^ln' ",*

sin. ⅜ (90° + α) =

The terms intermediate to those known are,

. no /1 —sin.72° . /1 +sin. 36°

sm. 9° *= J* g > sln∙ 63 = √ ÿ ,

o\*o— /’ — sin. 36° . ∕l÷βin. 72“

sin. 2.° = *J* , sin. 81° = *J ,*

sin. 45° = yi.

By a second interpolation the number of terms in the series would be doubled, and by a third, the sines of a series of arcs which are multiples of 2° 15' would be found. In the same way, from sin. 30° = ½, sin. 60° = — , and sin. 90° = I, we may deduce the sines of 15°, 45°, 75°; and at length the sines of a series of arcs, which are multiples of 3° 45'; then, from the two series, another, the arcs of which would differ by (3° 45') — (2° 15') = 1° 30' = 90'; and again the sines of a series of arcs differing by 45'. Indeed it was in this way the table was first formed.

9. When the sines of two thirds of the quadrant, that is, from 0 to 60°, are known, the sines of the remaining third can be readily found ; for since sin. (*a* + *b*) = sin. (*a* — *b*) + 2 cos. *a* sin. *b* (Algebra, art. 240), if we make *a* = 60°, then cos. a = ½, and 2 cos. *a* = 1, and we have sin. (60° + *b) —* sin. (60° — *b)* + sin. *b.* By this formula the numbers in the table may be verified.

The radius being an unit, we have found that the semi-circumference or arc of 180° = 3·1415926535 (Algebra, article 272). This number divided by 180 × 60 = 1080, gives the arc which measures an angle of one minute = ·0002908882; and since a small arc is nearly equal to its sine, we have sin. 1'=∙0002908882 nearly. The accu­racy of this may be verified by the formula

*\*\*\*\*\*\*Qp cP*

sin. *a = a —* — + —- —, &c. (Algebra, art. 269).

The cosine of one minute is found from the sine by the formula cos. *α =* √(1 — sin.2 *a),* or otherwise, by the series cos. *a —* 1 — —, &c. ≡ ∙9999999577. Knowing

now the sine and cosine of one minute, to find the sines of 2', 3', 4’, &c. we may proceed thus,—

Sin. 2' — 2 cos. 1' sin. Γ,

Sin. 3' *= 2* cos. 1' sin. 2' — sin. P, Sin. 4' = 2 cos. 1' sin. 3'—sin. 2,, &c.

10. Since the results obtained by the successive opera-