\*\*\*\*\*\*Theorem III. In any right-angled triangle, the sum of the hypotenuse and one side is to their difference, as thç square of the radius to ψe s<pιare of the tangent of half the contained angle.

Let ABC be any right-an­gled triangle ; draw CD bi­secting the angle C. By Geom. 19, 4,

CAtCB = AD :BD = AB —BD: BD.

Therefore, by composition, and Cor. 3 of Theor. II. of Tr!g.

C A + CB : CB ≡ Λ B : BD = tan. C : tan. ’ C. Now CB : BA = R : tan. C, therefore CA 4- CB : BA = R : tan. ⅞C,

and (CA -f- CB), : BA\* = R- : tan.f AC·

Now BAs - CA∙i- CBe = (CA 4- CB) (CA — CB) (Gκr>M. 9 and 18 of 4), therefore

(CA + CB)’ : (CA CB) (CA — CB) = R- : tanA∣C, and C A + CB : C A — CB = R» : tan.« ⅜(!.

The first two of these pronositions are alone sufficient for the resolution of all the cases of right-angled triangles; the third, which is not commonly given in our books on trigo­nometry, is however im∣>ortant, because when the lrypo- tennse and a side are given, by its application the same lo­garithms serve for finding the remaining side and the tan­gent of half the opposite angle.

15. It has been found conducive to brevity and perspi­cuity, to represent the parts of a triangle, each by a single letter: therefore, in any triangle ABC, we shall, in what follow«, employ that notation to denote the sides and angles as follows. (See figures 2, 3, 4.)

The sides, BC =α, AC = *b,* AB = c; the angles, BAC = A, ABC = B, ACB = C.

It will also be sometimes convenient to express ratios by fractions, the numerators of which are the antecedents, and the denominators tl>c consequents, so that the expressions A C

A : B = C : D, and ⅛r = w∙, arc to be regarded as identical. B 1J

\*\*\*THEOREM IV. The sides of any plane triangle are to one another as the sines of the opposite angles. ( Figs. 3 and 4.) In the triangle ABC, draw AD perpendicular to BC; then, employing the premised notation, that is, denoting the sides by β, *b, c,* and opposite angles by A, B, C, *b:* AD = R : sin. C ; and AD : *e* = sin. B : R ;

thereforc, *ex aquali,* inversely, *b : c =* sin. B : sin. C.

\*\*\*THEOREM V. The sum of the sides of any plane triangle is to the difference, as the tangent of half the sum of the opposite angles is to the tangent of half their difference.

Continuing the same notation, we have found that

*b : c =* sin. B : sin. C ;

therefore *b* 4- *c : b —* o = sin. B -f- sin. C : sin. B — sin. C. Now (Algebkλ, 247, M),

sin. B -∣- sin.C : sin.B—sin.C=tan. A (B 4- C) : tan. ∣ (B—C) ; therefore, *b* 4- c : *b — c=,* tan. ⅛ (13 4- C) : tan. -∣ (B — C). *corollary.* Since B 4- C = 180°— A, and therefore ⅛(B + C) = 90o-⅜A,

therefore *b* -f- *c : b — c=* cot. ∣ A : tan. J (B — C).

\*\*\*THEOREM VI. In any triangle, the cosine of half the differ­ence of any two of its angles is to the cosine of half their sum, as the sum of the sides opposite these angles is to the remaining side. And the sine of half the difference of the angles is to the sine of half their sum, as the dif­ference of the opposite sides to the remaining side.

Again, denoting the sides by *a, b, c,* and opposite angles by A, B, c,

Because sin. A : sin. C = α : c,

and sin. B : sin. C == *b : c ;*

therefore sin. A + sin. B : sin. C '= *a* -f- *b* : c,

also sin. A — sin. B : sin. C = *α — b i c.*

Now, sin. A + sin. B = 2 sin. 4 (A + B) cos. A (A — B) (Algebra, 240),

and sin. A — sin. B = 2 cos. ⅜ (A + B) sin. ∣ (A — B), and sin.C= sin.(A + B) = 2sin. \* (Λ-f-B)cos.∣(A-t-B) (Algebra, 245);

therefore

sin. A + sin. B ; sin. C = cos. ⅛ (A — B) : cos. ⅜(A + B), and

sin. A — sin. B : sin. C ~ sin. ⅜ (A — B) : sin. i (A + B); hence cos. A (A — B) : c∙os.^⅜ (A 4- B) = *a \*+ b : c,* and sin. Î (A — R) : sin. ⅜ (A -j- B) = *α — b : c ;*

\*\*\*Theorem VII. If a perpendicular be drawn from an angle of a triangle on the opposite side, dividing it into two segments, the sum of these segments is to the sum of the sides as the difference of tjιc sides tq the difference of the segments.

In the triangle ACB, AC\* = AD2 4- CD5, and AB2 = AD-i + DB2, therefore AC’ — AB= = CDi — BD-. Now AC2 — AB2 = (AC + AB)(AC-r- AB), and CD'— BD2 = (CD + BD) (CD — BD) (Geometry, 12, 4) ; therefore (AC+AB) (AC— AB)-(CD÷BD)(CD-BD), and CD + BD : AC + AB = AC — AB : CD — BD.

\*\*\*TheoreM VIII. In any plane triangle, the excess of the sum of the squares of two sides which contain an angle above the square of the third side, is to twice the rect­angle contained by the sides about the angle, as the co­sine of the angle is to radius. (Figs. 3 and 4.)

In the triangle ABC, draw AD perpendicular to BC, we have, by Geometry, 14, 4, and Cor. 2, Theorem I., of this,

*c- = a\* + lf—* 2α X CD, and CD - .

therefore *ct — a" + bi — 2ab —,*

1 αs + *b° — ei* cos, C and ~⅞⅛ = ~R~ι

therefore *a° + br — c° : 2ab —* cos. C : R.

We have, in the diagram, made the angle C acute, but the proposition is equally true when C is an obtuse angle ; because, when C changes from *acute* to *obtuse,* the sign of the cosine of the angle changes from -f- to —.

\*\*\*Theorem IX. In any triangle, the rectangle contained by two sides is to the rectangle containcd by half the perimeter and the excess of half the perimeter above the base, as the square of the radius to the square of the co­sine of half the angle contained by the sides.

It has been found that *a, b, c,* being the sides of a tri­angle, and A the angle contained by *b* and *c,* then, Theo­rem VI IL,

cos. A \_ *bi* 4- e2 — α'

-R^" - 2⅛c '

To the first of these equals add ⅛5 = 1, and to the second, κ

20o

‰=15 then

R 4- cos. A \_ δ2 4- 20c-∣-c2 — *α3* \_ (ά + o)2 — α2

IÎ - 26c - 2⅛