Cos.’B cos. ^C \_sin. s \_sin. ¾ (o -⅜- ⅛ -f-c) sin. JA -sin.α- sin. a ’ Sin. IB sin. i C\_ sin. (s — *a)\_* sin. ⅛ (⅛ + c — o) sin. ^A sin. a sin. a

By adding and subtracting the sides of these equations, and observing that

Sin. AB cos. JC ± sin. JC cos. JB = sin. 1 (B =±z C),

Cos. JB cos. *!2C=p* sin. ,JB sin. *!2 C =* cos. )2 (B ∑±z C), Sin∙i *(a + h*—c)-∣-sin.J (aψc—*b) =* 2 sin.,], « cos. l(Z∣—c), Sin∙i (α-f-⅛—c)—sin. J *(a-j-c—b) =* 2cos∙iasin.⅜(0—*c),* Sin. J(a-f-Z>-pc) — sin. *,', (b-(-c—a)* = 2sin. *∖α* cos. )2 (i-∣-c), Sin. <5(6+«+a)+sin. J (Z∣-(-c—a) = 2 cos. *'2a* sin. ⅛ (⅛-f-c), Sin. *a = 2* sin. Ja cos. *⅛a,*

(as has been proved in Algebra, articles 240 and 247), we obtain the following results :

(\*)

Sin. *J2(β* -t- C)\_ cos. I (⅛ — c) ( 1 )

cos. Ü -' cos. Jα

Sin.J(B— C)\_sin.|(é — c) cos. JA - sin. ⅜a

Cos. j(B -⅜- C)\_ cos. ⅜(⅛ + c) .g,

sin. IA - cos. ∣a

Cos.⅛(B—C)\_ sin. ⅜ *(b* -f- *c)* sin. ⅜A - sin. ∣a

These formulæ were first given by Gauss, in his work *Theoria Motus corporum Caelestium.*

22. Let us now divide the sides of formula (1) by those of formula (3), also the sides of (2) by the sides of (4), and the sides of (4) by the sides of (3), and, lastly, the sides of (2) by the sides of (1) ; and put *tan.* in the results instead of —also *cot.* instead of °∙~' ; we then have

*cos. stn∙*

(∙)

Tan. j(B + C)\_ cos. ⅜ (⅛ — c) æ

cot. I A cos. *⅛ (b* -p *c)*

Tan.⅜(B-C)\_ sin, *⅛(b-c) .g.*

cot. ⅜A - sin. ∣(6 -f- c)

Tan. ⅜(6-⅜- c)\_ cos. ⅛(B—C)

tan.⅜α ~ cos. ∣(B -∣- C) Tan.⅜(⅛-c)\_ sin.⅜(B- C)

tan. ⅜α - sin. ⅜ (B -∣- C) ' ‘

These formulæ, expressed in words as proportions, are commonly called *Napiers analogies.* He however only gave the last in the concise form which it here presents. The third, as he left it, had two common divisors in the terms of a ratio, which he overlooked. His commentator Briggs removed the defect, and, in addition, deduced the first and second formulæ from the third and fourth by the theorem here demonstrated in art. 11. Thus he is en­titled to share with Napier in the honour of having dis­covered these very elegant properties of spherical triangles.

From the first and second of these formulæ, there is found, by division, this other formula:

(<)

Tan. 4(B -f- C) \_ tan. ∣(⅛ -∣- c) Tan.∣(B — C) - tan. ⅜(Z> — *c)'*

This formula is also due to Napier.

23. There are some properties of spherical triangles which may be more concisely proved by the formulæ than by geometrical reasoning. Such are those which follow.

I. In a spherical triangle, the greater angle is opposite to the greater side, and conversely.

Let α, *b, c* be the sides of a spherical triangle, and B, C the angles opposite to the sides *b, c.* It appears from for­mula (2) of *δ*, article 21, that

Sin. ⅜(B — C)\_ cos.⅜A.

Sin. ⅜(Z∕— c) - cos. ⅛α

The second member of this equation is always a positive quantity, therefore the first must also be positive. This can only be true on the hypothesis that when *b* is greater than *c*, then B is also greater than C. The truth of the converse follows from the same principle.

II. the angles at the base of an isosceles triangle are equal, and conversely.

For we have

Cos. ½ *a* sin. ½ (B — C) = cos. ½ A sin. ½(*b* — *c*). When *b* = c, the second side of the equation becomes = 0, therefore the first side must also be = 0. This re­quires that B = C. The converse follows from the same formula.

III. Any two sides of a spherical triangle are together greater than the third.

It appears from article 17, that

\*\*\*\*. sin. *b* sin. *c* cos.2 *i* A

2y ~ j sm.⅜(α -I- *b* -f- c)

The second side of this equation is always a positive quantity, therefore the first must also be positive. This requires that *b* + *c* be in every case greater than *a.* This proposition has been otherwise proved geometrically, Theo­rem III.

IV. We have found (article 21) that

Cos. ½ (*b* + *c*) sin. ½A = cos. ½ (B + C) cos. ½*a*.

Now sin. ½A and cos. ½*a* are always positive, because each arc is less than a quadrant. Therefore cos. ½(*b* + *c*) and cos. ½ (B + C) have always the same sign ; and hence ½ *(b* + *c)* and ½ (B + C) are of the same affection, and B + C, *b* + c, either both less than a semicircle, or both equal to it, or both greater.

V. The hypotenuse *a* of a right-angled spherical triangle is less or greater than 90°, according as the sides *b, c,* about the right angle A, are of the same affection, or of different affection ; that is, in the former case, both less or both greater than 90° ; and in the latter, one less and the other greater than 90°.

This follows immediately from the formula cos. *a ≡* cos. *b* cos. *c* (art. 12) ; for if *b* and *c* be both less than 90° or both greater, they have like signs, and their product has the sign +, which indicates that *a* is less than 90°. If one of the two *b, c* be less and the other greater than 90°, they will have different signs, and the sign of their product will be —, and *a* must be greater than 90°.

VI. The same rule applies to each of the sides *b, c* ; for *b* will be less than 90° if *a* and *e* be of the same affection, but greater than 90° if *a* and c are of different affection.

VII. The formula cos. *a* = cot B cot. C shows that the hypotenuse *a* will be less than 90° when B and C are of the same affection, but greater than 90° if they be of dif­ferent affection.

VIII. Also, that either of the angles B, C, will be less than 90° if *a* and the other angle be of the same affection, but greater than 90° if they be of different affection.

IX. The formula cos. B = cos. *b* sin. *c* shows that the angle B and the opposite side *b* are always of the same af­fection.

X. The formula tan. *c* = tan. *a* cot B shows that the hypotenuse *a* and either side c are of the same affection if the angle they contain be acute, but of different affection if the angle be obtuse.

XI. Lastly, the formula tan. *b =* sin. *c* tan. B proves that the side and the opposite angle are always of the same affection.

XII. In any spherical triangle ABC, draw AD an arc of a great circle perpendicular to the base BC, thus form­ing two right-angled triangles ADC, ADB, which have