and by applying the first principle to this property, we de­duce the fundamental equation,

time down PQR= time down PSR *ultimately.*

Hence,

\*\*\*\*\*PQ , QR PS SR , . ,

velocity at Q v,. at R v,. at Q -\*^ v,. at R 't *t',nαte lI ’*

PQ—PS SR—QR Qe Sc , .

√NS - √OR ,°r √NS-√OR

, . cos. ^≤rSQe cos. ^c≤QSc , . ,

**~~that~~ ~~'\* ' √NS~~** = - √OR ≠,wβ"⅛"

sin. which curve makes with vertical at Q √NS °r do. do. R- √OR,

a well-known property of the cycloid.

3. Of the two principles employed in this solution, the first is equivalent to the theorem demonstrated in Fluxions, art. 62, and is the basis of all solutions. The second, al­though in the case in question it is actually true, is an as­sumption not warranted by the nature of the problem. It does not appear that any geometrical solution is exempt from this objection, and we cannot in consequence date the existence of the Calculus of Variations earlier than 1744, when Euler published his treatise, entitled *Methodus inve­niendi Lineas curvas proprietate maximi minimive gau­dentes.* This work was followed by a memoir from the pen of Lagrange, published in the second volume of the *Mis­cellanea Taurinensia.* By the introduction of the symbol δ to express that change which is termed a *variation,* La­grange may be said to have perfected the calculus ; for al­though certain extensions were afterwards made by himself and others, no change was afterwards introduced into the general process. We shall therefore conclude our brief sketch at this point, and refer the reader for further infor­mation to the following works: the *Acta Eruditorum* for 1696 and the following years ; the collected works of James and John Bernoulli; *Memoires de l'Acad. des Sci.* 1706, 1718, &c.; Brook Taylor’s *Methodus Incrementorum;* the Pe­tersburg Commentaries, vols. vi. and viii. ; which, together with Euler’s tract, quoted above, and his last memoir in the New Petersburg Com. vol. x. contain Euler’s writings on the subject: the Turin Miscellanies, vols. ii. and iv. and the *Theorie des Fonctions Analytiques,* contain Lagrange’s perfecting of the calculus. The reader will find in Wood­house’s Isoperimetrical Problems a complete history of the calculus, as well as an admirable digest of the different me­thods employed.

4. We proceed to the investigation of the theorem which is the basis of the calculus of variations.

To find the change or variation of a formula compre­hended under the integral sign, when the variables *x* and *y,* on which it depends, receive the increments δ*x* and δ*y.*

Let *u=∫ydx* be the formula under consideration ; in which y involves \*\*\*\* *x, y, ~,* &c.

*\*\*\*\*\*\*\*\*\*\*dx dr*

*dl∕ fFq*

Denote by *p, -⅛3* by *q,* &c. ; the partial differential coefficients ⅛ by M, ⅛ by N, ⅛ by P, &c.; the increment *dx dy j dp*

of*fydx* by *Dfydx,* &c.

Then *Du=fy*4.⅛γ) *(dx*4- *dDx)—-fydx,*

*—fDydx +fydix,*

*—J'Dydx -∣- yDx—fDxdy.*

Now *fdxDy=fdx(JΛDx* 4. Nδy 4- P⅛> 4-...), and *J'Dxdy-∣'Dx(lΛd.τ* 4- *Sdy* 4. Pψ4- ■·.),

.∙. δa=γβz4-∕[N(<⅛8y—*Dxdy)*4-*P(dxDp—Dxdp)+*... j, *=V8x+f{N(Dy-*p&r)4-P(⅞d—*qDx)* 4-. ]*dx.*

Also if δy—*pDx* be denoted by ω, we have

*j 7 d(x + 6x) dx dx ’*

*dDu dDx dp , ^-cdχ-p-dx~disx'*

*dhy d d da>*

*= -cbt-Tx^=d-χ^y-p^=dχ--*

hence putting *p* for *y* and .∙. *q for p,* we obtain

&C. — &C.

By substitution,

βu=γftr4-*f\*{* N“4-P^~ *~^^^dxu ^\*^'" } djc'*

*I‘ d^iω dω dQ Γ dlQ*

*J Q d^* **ώ** *+∕ “ dxi dx'* **&C. = &C.**

.∙. 8li=yδi4-ω(P-^ 4- ^-&C.)

*dω* <7R

+ ώ(0\_^+···)

4- &c. &c.

The last line of this very elegant expression is due to Euler ; the rest of the formula to Lagrange.

Let *xo x*1*, y*0 *y*1*,* be the values of *x* and *y* at the two li­mits; then the whole value of δ*u* between the limits is, \*\*\*\*\*\*δu=y,ta1—γ08x,4-ω1(P1— &c.)—α><,(Po— & c.)

*β r\*' dP d1Q „ v,*

4- &c. +yu,(N- — 4. — \_ &c.)dx

5. To satisfy the conditions of a maximum or minimum value of *u,* the value of δ*u,* expressed only as far as to the first powers of δ*x*0*,* δ*y*0, &c., must be equal to zero.

But this quantity consists of two parts ; the one an in­tegrated expression, depending only on the values of δ*x*0*,* δ*y*0*,* &c. ; the other an unintegrated expression, depending on the general values of δ*x,* δ*y.* Now it is evidently pos­sible to make the latter expression assume an infinity of different values, corresponding to an assigned value of the former. It is therefore impossible to render the whole ex­pression equal to zero for all values of ω, without making the two parts separately zero.

Our conditions are therefore,

\*\*\*\*\*κτ rfP d2Q o

N-S + Sf-fe=»· <1>

γ1δx,—γΛr<,-∣- ω1(Pl— &c.) — ω0(P<,— &C.) =0. (2.)

6. It may be remarked, that the object of our investiga­tion is to discover a mode of satisfying certain conditions, by means of establishing a relation between *y* and *x.* To effect this, we seek to separate these quantities from the quantities δ*x* and δ*y.* This will explain why the uninte­grated part of the expression has been reduced to the form in which we left it. Had ω been implicated with *x* and *y,*