a relation would have been assigned, not between *x* and *y,* but between *x, y,* and ω. We may add, that if *∫ydx* be an integrable expression, the equation (1) is an identical equa­tion.

7. In the above investigation it has been assumed that the other quantities which enter into the function γ, toge­ther with *x* and *y,* are absolute constants. If this hypothe­sis be not correct, our formulæ require to be modified.

Suppose *a, b,* &c., to enter the function *γ.*

Let δγ=Mδ*x+*Nδγ+Pδ*p*+&c.+Aδ*a*+&c..., then the addition to be made to the expression for δ*u* is *∫(Aδa+...*)*dx.* But δ*a* &c*.* depend not on the general va­riation, but only on the variation at the limits, (see Prob. 4); hence they are independent of δ*x* and δ*y*, and consequently form part of equation (2), which thus becomes

*\*\*\*\*\*rx'*

γ∣8r1-γoδx<,+ω1(P∣-&c.)-ω0(P.-&C.) *+\*α* Ad⅛-)-fec.≡O.

(3.)

8. Had our object been merely to determine the *form* of a function which should satisfy certain conditions, we might have rested content with causing one of the quantities, as *x* only, to vary. By this means equation (1) would have been obtained. There are however many problems which depend on the relation between δ*x* and δ*y,* at least at the limits, (see Prob. 3.) At the commencement of the varia­tion, for instance, we cannot remove to any point we please, from the circumstance that we are compelled to *begin* in a certain manner. Hence equation (2) is a most important part of the result ; and by assigning a relation between the things given and sought at the extremities, shews us the position of the extremities of the curve, and thereby re­stricts the solution from a *kind* of curve (as a cycloid) to a *specific* curve, (a cycloid of known dimensions and position).

9. From equation (1) we might easily deduce a number of expressions, for particular cases, easy of application. Thus, if γ contains only *y* and *p,* the formula (1) gives \*\*\*\*\* -s=°∙

Hence *dy=Ndy±Pdp= dy-}-Pdp=dP .p-}-Pdp ax*

*-d{Pp)∖* and γ=Py>-pc. (4.)

But as the reader will readily supply such formulæ for himself, we proceed to exhibit the solution of a few prob­lems.

*\*\*\*\*\*\*\*\*\*\*\*\*\*\*Prob.* 1. To find the curve of quickest descent from one given point to another.

Let *x* and *y* be the co-ordinates of a point in the curve, *y* being measured vertically downwards from the upper point.

Then fc= - ? — ∕\*A *+p dx.* jn w]1jc)1 ⅛e quantity √2√ √j

corresponding to *y* contains only *y* and *p.*

Ifγ= —, P≡ ; ÷ andformula(4)gives

√,y √1+p2∙√.J'

¼∕ 1 *+ pt p"l ds \_* 1

√y *~ Ji^+p2 -Jy+c' °r dx~cJy*

which is the differential equation to a cycloid. We may solve the problem by taking *x* as the vertical co-ordinate, in which case *y* will contain only *p,* and equation (1) will *dV*

give -r \_0, or P=c.

α⅛

But P, in this case, is *”, "' r"- . a — \_*

*Λj∖+pτ .,y∕x dg C/Jx*

the same result as before, with the exception that *x* and *y* are interchanged. The solution of this equation shews that the cusp of the cycloid is at the highest point, and its axis vertical.

*Prob.* 2. Required the curve which by revolution round its axis generates the solid on which the resistance pro­duced by motion in a fluid shall be less than on any other curve having the same extreme co-ordinates.

The resistance varies \*\*\*\*\*\*\*\*\*\* *dx∖ ∙'.* by formula (4)

W, \_(W W ∖ ,

1+P\*-V+P, (>+∕∕∕p

orθ= 2⅛+c.

(1+P4),ψ

This equation expresses the same property as that which Newton, without demonstration, assigned to the curve.

*Prob.* 3. To find the shortest distance between two given curves.

The expression for *y* is here √1+*p*2 where, as the line on which the distance is measured is a right line, *p=c.* But by formula (2)

γ1δx1—γ0δ*x*0+P1ω1—P0ω0=0, when δ*x*1*,* &c*.* are inde­finitely small. Also, if the curves have no connexion, we may make δ*x*1 δ*y*1 assume what values soever we please, without affecting δ*x*0*,* δ*y*0;we must therefore have sepa­rately

\*\*\*\*\*\*\*\*γ1 ⅛rl4-P1ω∣=0, and *γ,* ∂r0+Poωo=0.

The latter gives √l +∕√ *δx0 + -∙p° (δy,t-p,,δxο)=0,*

*\*l* 1 *+p:*

or δ∙ri 4- *pοδyc∙=0,* or ~ = ι = —, when *δxa* and

°aiu *Pa c*

*δy*0 are indefinitely diminished.

But, since the measurement must begin *in* the first cuτve, the limit of expresses the tangent of the angle in

\*\*\*Oj'.

which the tangent to the first curve cuts the axis. Our equation, then, shews that the line of shortest distance cuts the first curve at right angles. For the same reason it cuts the second curve also at right angles. Its position is consequently determined.

*Prob.* 4. To find the curve of quickest descent from one given curve to another.

Let *h* be the vertical ordinate of the point at which mo­tion begins :

\*\*\*\*1 ∕,√ 1 *-PP2*

then *tχx-i=z ! — ■' - dx,* from which (by Prob. 1.) *V2gJ t∕y-Λ*

it is evident that the curve is a cycloid. To determine the angles at which the cycloid cuts the given curves, we have the following equation, \*\*\*(3): γ∣ &r,—yβδx04-P1ω1—P,<0,4- *∙√⅛ dx=0.* The problem admits of several cases.

1. If the body starts from a given point in the first curve, that curve performs no part in the problem. Here \*\*\**δx,, ωa,* and *hh,* are all zero.

∙,∙ γ1Λτι + Pι(ty,-p,δar,)=O.

But γ1—P1p1=c, (see Prob. 1.)

1 *, r, Pi*

.∙. , -=c, and P1 = —-**~~-5=π- —; — .,~~** = r>∣c,

1 +Pι, *—b ∙J* 1 +∕>12 V*y1—h*

*.∙. cixl -∣- pl cδyl* = 0 ; or ⅛ for the second curve, at the