(whether the number of terms be definite or indefinite) the sum is independent of the order of arrangement.

*Infinite series.*

8. We consider an infinite series m0 + *u-i + u2* + . . . of terms proceeding according to a given law, that is, the general term ‰ is given as a function of *n.* To fix the ideas the terms may be taken to be positive numerical magnitudes, or say numbers continually diminishing to zero; that is, *un>un+ι,* and *un* is, moreover, such a function of *n* that by taking *n* sufficiently large *un* can be made as small as we please.

Forming the successive sums >Sθ = *u0,* >S'1 = ωθ + M1, \*S'2 *= υ0 + ul + uc,, .* . these sums λS,0, λS,1, \*S,2 .. ., will be a series of continually increasing terms, and if they increase up to a determinate finite limit \*S' (that is, if there exists a determinate numerical magnitude $ such that by taking *n* sufficiently large we can make N - *Sn* as small as we please) 0' is said to be the sum of the infinite series. To show that we can actually have an infinite series with a given sum \*S,, take m0 any number less than >S,, then \*S' - wθ is positive, and taking *uχ* any numerical magnitude less than *S - u0,* then *S -u0-u1* is positive. And going on continually in this manner we obtain a series wθ + -w1 *+ u2+ . . .* such that for any value of *n* however large >S' - «Q - ∙w1. . . — *un* is positive ; and if as *n* increases this difference diminishes to zero, we have m0 + m1 + -m2 + . . . , —an infinite series having *S* for its sum. Thus, if *s* = 2, and we take *u0<2,* say *u0 =* 1 ; *ul<2 -1,* say *= -; u2*<2

-1 -say ‰ = and so on, we have 1 +1 +1 + ... = 2 ; or, more generally, if *r* be any positive number less than

1, then 1 + *r* + r2 + ... = that is, the infinite geo­metric series with the first term = 1, and with a ratio

*r<l,* has the finite sum = . This in fact follows from

1 - r

1 — *τn*

the expression 1 + *r* + r2 ... + *rn ~*1 = ∣—- for the sum of

the finite series; taking *r<l,* then as *n* increases rn de­creases to zero, and the sum becomes more and more neariy =

9. An infinite series of positive numbers can, it is clear, have a sum only if the terms continually diminish to zero ; but it is not conversely true that, if this condition be satis­fied, there will be a sum. For instance, in the case of the harmonic series 1 +~ + }, + .. . it can be shown that by tak­ing a sufficient number of terms the sum of the finite series may be made as large as we please. F or, writing the series in the form 1 +’ + (i + i) + (j + 1 + 1- + >) + . . , the

number of terms in the brackets being doubled at each successive step, it is clear that the sum of the terms in any bracket is always > ∣ ; hence by sufficiently increas­ing the number of brackets the sum may be made as large as we please. In the foregoing series, by grouping the terms in a different manner 1 + (I + i) + Q +j + l + ∣) + ..j

the sum of the terms in any bracket is always < 1 ; we thus arrive at the result that (n = 3 at least) the sum of 2n terms of the series is > 1 + *~n* and < *n.*

10. An infinite series may contain negative terms; sup­pose in the first instance that the terms are alternately positive and negative. Here the absolute magnitudes of the terms must decrease down to zero, but this is a suffi­

cient condition in order that the series may have a sum. The case in question is that of a series v0 - v1 -(- v2 - . . , where vθ, v1, ι,2, . . are all positive and decrease down to zero. Here, forming the successive sums >S,θ = vθ, Λ'l = *v0* - ∙υ1, ∕S'2 = *v0 - v1 + v2, .* . >S'θ, N1, *S2, . . .* are all positive, and we . have \*S'0 > <S'1, N1 < *S2, S2 >* \*S'3, . . and Nn+1 - *sn* tends con­tinually to zero. Hence the sums \*Srθ, \*S'1, *S2, .* . tend con­tinually to a positive limit *S* in such wise that >S'θ, >S2, \*S4, . . . are each of them greater and N1, >S,3, *S5, . .* are each of them less than *S* ; and we thus have *S* as the sum 111

of the series. The series l-0 + s- τ+ · · will serve as 2 o 4

an example. The case just considered includes the appar­ently more general one where the series consists of alternate groups of positive and negative terms respectively ; the terms of the same group may be united into a single term *± vn,* and the original series will have a sum only if the resulting series *v0 - v1 + v2 . . .* has a sum, that is, if the positive partial sums *v0, v1, v2,.*. decrease down to zero.

The terms at the beginning of a series may be irregular as regards their signs ; but, when this is so, all the terms in question (assumed to be finite in number) may be united into a singL term, which is of course finite, and instead of the original series only the remaining terms of the series need be considered. Every infinite series whatever is thus substantially included under the two forms,—terms all posi­tive and terms alternately positive and negative.

11. In brief, the sum (if any) of the infinite series ⅝ + wι + ⅝+ · · is the finite limit, (if any) of the succes­sive sums *u0, Uq + u1, u0 + u1 + u2, . .* . ; if there is no such limit, then there is no sum. Observe that the assumed order *u0, u1, u2 . .*. of the terms is part of and essential to the definition ; the terms in any other order may have a different sum, or may have no sum. A series having a sum is said to be “ convergent ” ; a series which has no sum is “ divergent.”

If a series of positive terms be convergent, the terms cannot, it is clear, continually increase, nor can they tend to a fixed limit : the series 1 + 1 + 1 + . . is divergent. For the convergency of the series it is necessary (but, as has been shown, not sufficient) that the terms shall decrease to zero. So, if a series with alternately positive and nega­tive terms be convergent, the absolute magnitudes cannot, it is clear, continually increase. In reference to such a series Abel remarks, “ Peut-on imaginer rien de plus horrible que de débiter 0 = lm - 2n +3w - 4w +, &c., où n est un nombre entier positif?” Neither is it allowable that the absolute magnitudes shall tend to a fixed limit. The so- called “ neutral ” series 1-1 + 1-1.. is divergent : the successive sums do not tend to a determinate limit, but are alternately + 1 and 0 ; it is necessary (and also suffi­cient) that the absolute magnitudes shall decrease to zero.

In the so-called semi-convergent series we have an equa­tion of the form *s — Uo- U1 + U2-.. .,* where the positive values *Uo, U1, U2, . . .* decrease to a minimum value, suppose *Up,* and afterwards increase ; the series is divergent and has no sum, and thus *S* is not the sum of the series. N is only a number or function calculable approximately by means of the series regarded as a finite series terminating with the term ± *Up.* The successive sums *Uo, Uo — Ul, Uo- U1 + U2, .* . up to that containing ± *Up,* give alter­nately superior and inferior limits of the number or function N.

12. The condition of convergency may be presented under a different form : let the series *u0 + u1 + u2* + . . be convergent, then, taking *m* sufficiently large, the sum is the limit not only of *u0 + u1 +. . + um* but also of *u0 + 1ιl... + um+r,* where *r* is any number as large as we please. The difference of these two expressions must therefore be in-