definitely small ; by taking *m* sufficiently large the sum ‰+ι + ‰+2 . . . . + ttnι+r (where *r* is any number how­ever large) can be made as small as we please ; or, as this may also be stated, the sum of the infinite series ‰+ι + mot+2 ÷ · · · can be made as small as we please. If the terms are all positive (but not otherwise), we may take, instead of the entire series ‰+ι+ ‰+2+. ∙, any set of terms (not of necessity consecutive terms) subse­quent to *um',* that is, for a convergent series of positive terms the sum of any set of terms subsequent to *um* can, by taking 7?i sufficiently large, be made as small as we please.

13. It follows that in a convergent series of positive terms the terms may be grouped together in any manner so as to form a finite number of partial series which will be each of them convergent, and such that the sum of their sums will be the sum of the given series. For instance, if the given series be *u0 + u1 + u2* +.. ., then the two series τzθ ÷ m2 + w4 + · · · and *u1 + u3* +.. will each be convergent and the sum of their sums will be the sum of the original series.

14. Obviously the conclusion does not hold good in general for series of positive and negative terms : for in­stance, the series l-\* + g- ^ + .. is convergent, but the

two series 1 +1 +1 +.. and -1 - 7 - . . are each diver-

gent, and thus without a sum. In order that the conclusion may be applicable to a series of positive and negative terms the series must be “ absolutely convergent,” that is, it must be convergent when all the terms are made posi­tive. This implies that the positive terms taken by them­selves are a convergent series, and also that the negative terms taken by themselves are a convergent series. It is hardly necessary to remark that a convergent series of positive terms is absolutely convergent. The question of the convergency or divergency of a series of positive and negative terms is of less importance than the question whether it is or is not absolutely convergent. But in this latter question we regard the terms as all positive, and the question in effect relates to series containing positive terms only.

15. Consider, then, a series of positive terms *u0 + ul* + τz2 + .. ; if they are increasing—that is, if in the limit *un-ιr∖∣un* be greater than 1—the series is divergent, but if less than 1 the series is convergent. This may be called a first criterion ; but there is the doubtful case where the limit =1. A second criterion was given by Cauchy and Raabe ; but there is here again a doubtful case when the limit considered =1. A succession 'of criteria was estab­lished by De Morgan, which it seems proper to give in the original form ; but the equivalent criteria established by Bertrand are somewhat more convenient. In what follows *lx* is for shortness written to denote the logarithm of *x,* no matter to what base. De Morgan’s form is as follows :—

1 *J,cb'je*

λ∖ riting *υll = ψ,ny* put *Po — ~~ψχ~ >* if for æ = 00 the limit α0

of *p0* be greater than 1 the series is convergent, but if less than 1 it is divergent. If the limit α0 = l, seek for the limit of *p1, = (p0-* l)/·r; if this limit <z1 be greater than 1 the series is convergent, but if less than 1 it is divergent. If the limit σ1 = l, seek for the limit *p2, =* (p1 - *V)llx* ; if this limit *a2* be greater than 1 the series is convergent, but if less than 1 it is divergent. And so on indefinitely.

16. Bertrand’s form is :—If, in the limit for *n = ∞. I—tin*

*un*

be negative or less than 1 the series is divergent, but if

greater than 1 it is convergent. If it = 1, then if *l-^—Illn nunl*

be negative or less than 1 the series is divergent, but if

greater than 1 it is convergent. If it =1, then if *I—-j-∣llln*

be negative or less than 1 the series is divergent, but if greater than 1 it is convergent. And so on indefinitely.

The last-mentioned criteria follow at once from the theorem that the several series having the general terms

—, - .. .a, .... -<-, -7 „ .z,, .α,.. respectively are each

*na- n(ln)a nln{Un)a nlnlln(llln)a*

of them convergent if α be greater than 1, but divergent if *a* be negative or less than 1 or =1. In the simplest case, series with the general term —α, the theorem may be

proved nearly in the manner in which it is shown above (cf. § 9) that the harmonic series is divergent.

17. Two or more absolutely convergent series may be added together,{*u0* + w1 + *u.2* . .} + {⅞÷q÷¾∙. =} (w0÷*vo)* + (m1 + v1) . . ; that is, the resulting series is absolutely con­vergent and lias for its sum the sum of the two sums. And similarly two or more absolutely convergent series may be multiplied together {7∕θ + w1 + *u2 . .*} × {ι>θ + v1 + *v2 . .}*

*= uovo* ÷ (⅜vι+ wι⅞)+ (wot½ + *u1v1 ÷ u2v0) ÷ · ·}* that is, the resulting series is absolutely convergent and has for its sum the product of the two sums. But more properly the multiplication gives rise to a doubly infinite series—

«A tt0l>1, m0¾ ...

m1v0, *uivn ulv2*

—which is a kind of series which will be presently con­sidered.

18. But it is in the first instance proper to consider a single series extending backwards and forwards to infinity, or say a back-and-forwards infinite series . ..∙m-2 + m-i *+ u0 + u1 + u2. .. ;* such a series may be absolutely con­vergent, and the sum is then independent of the order of the terms, and in fact equal to the sum of the sums of the two series τzθ + M1 + 7∕2 . . and «\_ι + τζ\_2 + ?ζ\_3 · · respectively. But, if not absolutely convergent, the ex­pression has no definite meaning until it is explained in what manner the terms are intended to be grouped together ; for instance, the expression may be used to denote the foregoing sum of two series, or to denote the series *u0* ÷ (τz1 + *u* \_ 1) + (τz2 + *u* \_ 2) + .. and the sum may have different values, or there may be no sum, accordingly. Thus, if the series be..-∣-T + 0 + ∣ + ^+ .., in the

former meaning the two series 0 +1 + +.. and — ∣ -1 ..

are each divergent, and there is not any sum. But in the latter meaning the series is 0 + 0 + 0 -I-.., which has a sum = 0. So, if the series be taken to denote the limit of (ttθ + *Uχ + U2 .* . +Mm) + («\_1 + 2Z\_2 . . . +M-n√), where 777, 7τz' are each of them ultimately infinite, there may be a sum depending on the ratio *m : m,,* which sum conse­quently acquires a determinate value only when this ratio is given.

19. In a singly infinite series we have a general term *un,* where *n* is an integer positive in the case of an ordinary series, and positive or negative in the case of a back-and- forwards series. Similarly for a doubly infinite series we have a general term τzw,re, where *m, n* are integers which may be each of them positive, and the form of the series is then

mo.o , ⅜4 , ⅝,2 · · ·

,i1.0 » ttl,l > *111J*

or they may be each of them positive or negative. The latter is the more general supposition, and includes the former, since *,um,n* may = 0 for *m* or *n* each or either of them negative. To put a definite meaning on the notion of a sum, we may regard *m, n* as the rectangular coordi­nates of a point in a plane ; that is, if *m, n* are each of