But δM∕δ*x* is the whole shearing force Q on the section of the beam. Hence

Q *ΓVl j* 9,=∑⅞ ∕ *y-dy ;*

and this is also the intensity of vertical shearing stress at the dis­tance *y*0 from the neutral axis. This expression may conveniently be written *q =* QA*y∕z*0I*,* where A is the area of the surface AG and *y* the distance of its centre of gravity from the neutral axis. The intensity *q* is a maximum at the neutral axis and diminishes to zero at the top and bottom of the beam. In a beam of rectangular section the value of the shearing stress at the neutral axis is *q* max. =3/2Q/*bh.* In other words, the maximum intensity of shearing stress on any section is ⅜ of the mean intensity. Similarly, in a beam of circular section the maximum is 4/3 of the mean. This result is of some importance in application to the pins of pin-joints, which may be treated as very short beams liable to give way by shearing.

In the case of an **I** beam with wide flanges and a thin web, the above expression shows that in any vertical section *q* is nearly con­stant in the web, and insignificantly small in the flanges. Practi­cally all the shearing stress is borne by the web, and its intensity is very nearly equal to Q divided by the area of section of the web.

64. The foregoing analysis of the stresses in a beam, which resolves them into longitudinal pull and push, due to bending moment, along with shear in longitudinal and transverse planes, is generally sufficient in the treatment of practical cases. If, how­ever, it is desired to find the direction and greatest intensity of stress at any point in a beam, the planes of principal stress passing through the point must be found by an application of the general method given in the article Elasticity, chapter

iii. In the present case the problem is excep­

tionally simple, from the fact that the stresses

on two planes at right angles are known, and

the stress on one of these planes is wholly tan­

gential. Let AC (fig. 39) be an indefinitely

small portion of the horizontal section of a

beam, on which there is only shearing stress,

and let AB be an indefinitely small portion of

the vertical section at the same place, on which

there is shearing and normal stress. Let *q* be

the intensity of the shearing stress, which is

the same on AB and AC, and let *p* be the in­

tensity of normal stress on AB: it is required

to find a third plane BC, such that the stress

on it is wholly normal, and to find *r,* the in­

tensity of that stress. Let *θ* be the angle (to be determined) which BC makes with AB. Then the equilibrium of the triangular wedge ABC requires that

*r*BC cos *θ=p.* AB+ *q.* AC, and rBC sin *θ=q*. AB ; or *(r-p)* cos *θ* =*q*sin *θ,* and *r*sin *θ* = *q*cos *θ.*

Hence, *q2=r(r-p),*

tan *2θ = 2q∕p, r*= 1/2*p*±√*q*2 +1/4 *p*2.

The positive value of *r* is the greater principal stress, and is of the same sign as *p*. The negative value is the lesser principal stress, which occurs on a plane at right angles to the former. The equa­tion for *θ* gives two values corresponding to the two planes of principal stress. The greatest intensity of shearing stress occurs on the pair of planes inclined at 45° to the planes of principal stress, and its value is √*q2 +*1/4*p*2 (by § 5).

65. The above determination of *r,* the greatest intensity of stress due to the combined effect of simple bending and shearing, is of some practical importance in the case of the web of an Ibeam. We have seen that the web takes practically the whole shearing force, distributed over it with a nearly uniform intensity *q.* If there were no normal stress on a vertical section of the web, the shearing stress *q* would give rise to two equal principal stresses, of pull and push, each equal to *q,* in directions inclined at 45° to the section. But the web has further to suffer normal stress due to bending, the intensity of which at points near the flanges approxi­mates to the intensity on the flanges themselves. Hence in these regions the greater principal stress is increased, often by a consider­able amount, which may easily be calculated from the foregoing formula. What makes this specially important is the fact that one of the principal stresses is a stress of compression, which tends to make the web yield by buckling, and must be guarded against by a suitable stiffening of the web.

The equation for *θ* allows the lines of principal stress in a beam to be drawn when the form of the beam and the distribution of loads are given. An example has been shown in the article Bridges (§ 13, fig. 12), vol. iv. p. 290.

66. The deflexion of beams is due partly to the distortion caused by shearing, but chiefly to the simple bending which occurs at each vertical section. As regards the second, which in most cases is the only important cause of deflexion, we have seen (§ 59) that the radius of curvature R at any section, due to a bending moment M, is EI∕M, which may also be written E*y*1∕*p*1. Thus beams of uniform strength and depth (and, as a particular case, beams of

uniform section subjected to a uniform bending moment) bend into a circular arc. In other cases the form of the bent beam, and the resulting slope and deflexion, may be determined by integrating the curvature throughout the span, or by a graphic process (see Bridges, § 25), which consists in drawing a curve to represent the beam with its curvature greatly exaggerated, after the radius of curvature has been determined for a sufficient number of sections. In all practical cases the curvature is so small that the arc and chord are of sensibly the same length. Calling *i* the angle of slope, and *u* the dip or deflexion from the chord, the equation to the curve into which an originally straight beam bends may be written

<ύί\_ .. d2w *di* El *dx ’ dx2 dx~*M

Integrating this for a beam of uniform section, of span L, supported at its ends and loaded with a weight W at the centre, we have, for the greatest slope and greatest deflexion, respectively, *i*1 = WL2∕16EI, *u*1=WL3∕48EI. If the load W is uniformly distributed over L, *i*1= WL2∕24EI and *u*1 = 5WL3∕384EI. For other cases, see Bridges, §24.

The additional slope which shearing stress produces in any originally horizontal layer is *q*∕C, where *q* is, as before, the intensity of shearing stress and C is the modulus of rigidity. In a round or rectangular bar the additional deflexion due to shearing is scarcely appreciable. In an Ibeam, with a web only thick enough to resist shear, it may be a somewhat considerable proportion of the whole.

67. *Torsion* occurs in a bar to which equal and opposite couples are applied, the axis of the bar being the axis of the couples, and gives rise to shearing stress in planes perpendicular to the axis. Let AB (fig. 40) be a uniform circular shaft held fast at the end A, and twisted by a

couple applied in

the plane BB. As­

suming the strain to

be within the limits

of elasticity, a radius

CD turns round to

CD', and a line AD

drawn at any dis­

tance *r* from the axis,

and originally straight, changes into the helix AD'. Let *θ* be the angle which this helix makes with lines parallel to the axis, or in other words the angle of shear at the distance *r* from the axis, and let *i* be the angle of twist DCD'. Taking two sections at a distance *dx* from one another, we have the arc *θdx = rdi.* Hence *q,* the intensity of shearing stress in a plane of cross-section, varies as r,

*. di*

since *q*=C*θ* = Cr*di*/*dx* . The resultant moment of the whole shearing

stress on each plane of cross-section is equal to the twisting moment M. Thus

*∫2πr*2*qdr =* M*.*

Calling *r*1 the outside radius (where the shearing stress is greatest) and *q*1 its intensity there, we have *q = rq*1*/r*1*,* and hence, for a solid shaft, *q*1 = 2M∕π*r*13. For a hollow shaft with a central hole of radius *r*2 the same reasoning applies : the limits of integration are now r1 and r2, and

ιr(rι4 - r24)

The lines of principal stress are obviously helices inclined at 45° to the axis.

If the shaft has any other form of section than a solid or sym­metrical hollow circle, an originally straight radial line becomes warped when the shaft is twisted, and the shearing stress is no longer proportional to the distance from the axis. The twisting of shafts of square, triangular, and other sections has been investigated by M. de St Venant (see Elasticity, § 66-71, where a comparison of torsional rigidities is given). In a square shaft (side = *h*) the stress is greatest at the middle of each side, and its intensity there@@1 is *q*1 = M∕0∙281*h*3.

For round sections the angle of twist per unit of length is

*i* = .-γL = in solid and r,. —- in hollow shafts.

68. In what has been said above it is assumed that the stress is within the limit of elasticity. When the twisting couple is increased so that this limit is passed, plastic yielding begins in the outermost layer, and a larger proportion of the whole stress falls to be borne by layers nearer the centre. The case is similar to that of a beam bent beyond the elastic limit, described in § 57. If we suppose the process of twisting to be continued, and that after passing the limit of elasticity the material is capable of much distortion without further increase of shearing stress, the distribu­tion of stress on any cross section will finally have an approximately

*f\** y\*l

uniform value *q,,* and the moment of torsion will be ^. *2πr~qdr* = f^'(rι3-r23)∙ In the case of a solid shaft this gives for M a

@@@1 Rankine, *Applied Mechanics,* § 324.