value greater than it has when the stress in the outermost layer only reaches the intensity *q'*, in the ratio of 4 to 3.@@1 It is obvious from this consideration that the ultimate strength of a shaft to resist torsion is no more deducible from a knowledge of the ultimate shearing strength of the material than the ultimate strength of a beam to resist bending is deducible from a knowledge of *ft* and *fc*. It should be noticed also that as regards ultimate strength a solid shaft has an important advantage over a hollow shaft of the same elastic strength, or a hollow shaft so proportioned that the greatest working intensity of stress is the same as in the solid shaft.

69. *Twisting combined with Longitudinal Stress.—*When a rod is twisted and pulled axially, or when a short block is twisted and com­pressed axially, the greatest intensity of stress (the greater principal stress) is to be found by compounding the longitudinal and shearing stresses as in § 64. In a circular rod of radius *r*1, a total longitudi­nal force P in the direction of the axis gives a longitudinal normal stress whose intensity *p*1=P∕π*r*12. A twisting couple M applied to the same rod gives a shearing stress whose greatest intensity *q*1=2M∕π*r*13. The two together give rise to a pair of principal stresses of intensities r = P∕2π*r*12±∖∕(2M∕π*r*13)2+(P∕2π*r*13)2, their inclinations to the axis being defined by the equation tan 20 = 2M∕r1P, and the term under the square root is the greatest intensity of shearing stress.

70. *Twisting combined with Bending.—*This important practical case is realized in a crank-shaft (fig. 41). Let a force P be applied at the crank-pin A at right

angles to the plane of the crank.

At any section of the shaft C

(between the crank and the

bearing) there is a twisting mo­

ment M1 = P . AB, and a bend­

ing moment M2=P. BC. There

is also a direct shearing force P, but this does not

require to be taken into account in calculating

the stress at points at the top or bottom of the

circumference (where the intensity is greatest),

since (by § 63) the direct shearing stress is dis

tributed so that its intensity is zero at these points. The stress there is consequently made up of longitudinal normal stress (due to bending), *p*1 = 4M2∕πr13, and shearing stress (due to torsion), *q*1 = 2M1∕π*r*13. Combining these, as in § 64, we find for the prin­cipal stresses r=2(M2±∖∕M12 + M22)∕π*r*13, or r = 2P(BC±AC)∕π*r*13. The greatest shearing stress is 2P . AC∕πr13∙, and the axes of principal stress are inclined so that tan 20 = M1∕M2 = AB∕BC. The axis of greater principal stress bisects the angle ACB.

71. *Long Columns and Struts—Compression and Bending.—*A long strut or pillar, compressed by forces P applied at the ends in the direction of the axis, becomes unstable as regards flexure when P exceeds a certain value. Under no circumstances can this value of P be exceeded in loading a strut. But it may happen that the intensity of stress produced by smaller loads exceeds the safe com­pressive strength of the material, in which case a lower limit of load must be chosen. If the applied load is not strictly axiai, if the strut is not initially straight, if it is subject to any deflexion by transverse forces, or if the modulus of elasticity is not uniform over each cross-section,—then loads smaller than the limit which causes instability will produce a certain deflexion which increases with increase of load, and will give rise to a uniformly varying stress of the kind illustrated in figs. 26 and 28. We shall first consider the ideal case in which the forces at the ends are strictly axial, the strut perfectly straight and free from transverse loads and perfectly symmetrical as to elasticity. Two conditions have to be distinguished—that in which the ends are left free to bend, and that in which the ends are held fixed. In what follows, the ends are supposed free to bend. The value of the load which causes instability will be found by considering what force P applied to each end would suffice to hold an originally straight strut in a bent state, supposing it to have received a small amount of elastic curvature in any way. Using *u* as before to denote the deflexion at any part of the length, the bending moment is *Pu,* and (taking the origin at the middle of tho chord) the equation to the elastic curve is

*d*2*u \_ - Pu*

<‰Γ2-^~ËÎ’

from which, for a strut of uniform section, *u*=*u*1 cos *x*√P∕EI, *u1* being the deflexion at the centre, Now *u* = 0 when *x* =1/2L (the half length), and therefore 1/2L∖∕P∕EI = 1/2πr or an integral multiple of 1/2π. The smallest value (1/2π) corresponds to the least force P. Thus the force required to maintain the strut in its curved state is P = π2EI∕L2, and is independent of *u*1*.* This means that with this particular value of P (which for brevity we shall write P1) the strut will be in neutral equilibrium when bent ; with a value of P less than P1 it will be stable : with a greater value it will be unstable. Hence a load exceeding P1 will certainly cause rupture. The value

π2EI∕L2 applies to struts with round ends, or ends free to turn. If the ends are fixed the effective length for bending is reduced by one half, so that P1 then is 4π2EI∕L2. When one end is fixed and the other is free P1 has an intermediate value, probably about 9π2EI∕4L2.

72. The above theory, which is Euler’s, assigns P1 as a limit to the strength of a strut on account of flexural instability ; but a stress less than P1 may cause direct crushing. Let S be the area of section, and *fc* the strength of the material to resist crushing. Thus a strut which conforms to the ideal conditions specified above will fail by simple crushing if *f*cS is less than P1, but by bending if *f*cS is greater than P1. Hence with a given material and form of section the ideal strut will fail by direct crushing if the length is less than a certain multiple of the least breadth (easily calculated from the expression for P1), and in that case its strength will be independent of the length ; when the length is greater than this the strut will yield by bending, and its strength diminishes rapidly as the length is increased.

But the conditions which the above theory assumes are never realized in practice. The load is never strictly axial, nor the strut absolutely straight to begin with, nor the elasticity uniform. The result is that the strength is in all cases less than either *f*cS or P1. The effect of deviations from axiality, from straightness, and from uniformity of elasticity may be treated by introducing a term expressing an imaginary initial deflexion, and in this way Euler’s theory may be so modified as to agree well with experimental results on the fracture of struts,@@2 and may be reconciled with the observed fact that the deflexion of a strut begins gradually and passes through stable values before the stage of instability is reached. In consequence of this stable deflexion the stress of compression on the inside edge becomes greater than P∕S, the stress on the outside edge becomes less than P∕S, and may even change into tension, and the strut may yield by one or the other of these stresses becoming greater than *fc* or *ft* respectively. As regards the influence of length and moment of inertia of section on the deflexion of struts, analogy to the case of beams suggests that the greatest deflexion consistent with stability will vary as L2∕*b*, *b* being the least breadth, and the greatest and least stress, at opposite edges of the middle section, will consequently be

J√1,∈U∖ n s(1± &a *j,*

where *a* is a coefficient depending on the material and the form of the section. This gives, for the breaking load, P = S*f*c∕(1 + *a*L2∕*b*2) or -S*ft* ∕(1 *-a*L2*/b2),* the smaller of the two being taken.

This formula, which is generally known as Gordon’s, can be made to agree fairly with the results of experiments on struts of ordinary proportions, when the values of *f* as well as *a* are treated as empirical constants to be determined by trial with struts of the same class as those to which the formula is to be applied. Gordon’s formula may also be arrived at in another way. For very short struts we have seen that the breaking load is *f*cS, and for very long struts it is π2EI∕L2. If we write P=∕S∕(1+ *f*cSL2∕π2EI), we have a formula which gives correct values in these two extreme cases, and inter­mediate values for struts of medium length. By writing this P=√S∕(1 *+ c*SL2/I), and treating *f* and *c* as empirical constants, we have Gordon’s formula in a slightly modified shape. Gordon’s formula is largely used; it is, however, essentially empirical, and it is only by adjustment of both constants that it can be brought into agreement with experimental results.@@3 For values of the constants, see Bridges. In the case of fixed ends, *c* is to be divided by 4.

73. *Bursting Strength of Circular Cylinders and Spheres.—*Space remains for the consideration of only one other mode of stress, of great importance from its occurrence in boilers,

pipes, hydraulic and steam cylinders, and guns.

The material of a hollow cylinder, subjected to

pressure from within, is thrown into a stress of

circumferential pull. When the thickness *t* is

small compared with the radius R, we may treat

this stress as uniformly distributed over the

thickness. Let *p* be the intensity of fluid pres­

sure within a hollow circular cylinder, and let

*f* be the intensity of circumferential stress.

Consider the forces on **a** small rectangular plate

(fig. 42), with its sides parallel and perpen­

dicular to the direction of the axis, of length *l*

and width Rδ*θ*, δ*θ* being the small angle it

subtends at the axis. Whatever forces act on this plate in the direction of the axis are equal and opposite. The remaining forces, which are in equilibrium, are P, the total pressure from within, and a force Tat each side due to the circumferential stress. P=*pl*Rδ*θ*

@@@1 See Elasticity, §§ 10-20.

@@@2 See papers by Profs. Ayrton and Perry, *The Engineer,* Dec. 10 and 24, 1886, and by T. C. Fidler, *Min. Proc. Inst. C.E.,* vol. lxxxvi. p. 2C1.

@@@3 For experiments on the breaking strength of struts, see papers by Hodgkin­son, *Phil. Trans.,* 1840; Berkeley, *Min. Proc. Inst. O.E.,* vol. xxx.; Christie, *Trans. Amer. Soc. Civ. Eng.,* 1884.