double point, and when this is so the plane is a tangent plane having the double point for its point of contact. The double point is either an acnode (isolated point), then the surface at the point in question is convex towards (that is, concave away from) the tangent plane ; or else it is a crunode, and the surface at the point in question is then concavo-convex, that is, it has its two curvatures in opposite senses (see *infra,* No. 16). Observe that in either case any line whatever in the plane and through the point meets the surface in the points in which it meets the plane curve, namely, in the point of contact, which *qua* double point counts as two intersections, and in *n* - 2 other points ; that is, we have the preceding definition of the tangent line.

5. The complete enumeration and discussion of the singularities of a surface is a question of extreme difficulty which has not yet been solved.@@1 A plane curve has point singularities and line singularities ; corresponding to these we have for the surface isolated point singularities and isolated plane singularities, but there are besides -con­tinuous singularities applying to curves on or torses circum­scribed to the surface, and it is among these that we have the non-special singularities which play the most important part in the theory. Thus the plane curve represented by the general equation (*x, y, z*)*n*, = 0*,* of any given order *n,* has the non-special line singularities of inflexions and double tangents ; corresponding to this the surface represented by the general equation (*x, y, z, w*)*n*, = 0, of any given order *n,* has, not the isolated plane sin­gularities, but the continuous singularities of the spinode curve or torse and the node-couple curve or torse. A plane may meet the surface in a curve having (1) a cusp (spinode) or (2) a pair of double points ; in each case there is a singly infinite system of such singular tangent planes, and the locus of the points of contact is the curve, the envelope of the tangent planes the torse. The reciprocal singularities to these are the nodal curve and the cuspidal curve : the surface may intersect or touch itself along a curve in such wise that, cutting the surface by an arbitrary plane, the curve of intersection has at each intersection of the plane with the curve on the surface (1) a double point (node) or (2) a cusp. Observe that these are singularities not occurring in the surface represented by the general equation (*x, y, z, w*)*n*, = 0 of any order ; observe further that in the case of both or either of these singularities the definition of the tangent plane must be modified. A tangent plane is a plane such that there is in the plane section a double point in addition to the nodes or cusps at the intersections with the singular lines on the surface.

6. As regards isolated singularities, it will be sufficient to mention the point singularity of the conical point (or cnicnode) and the corresponding plane singularity of the conic of contact (or cnictrope). In the former case we have a point such that the consecutive points, instead of lying in a tangent plane, lie on a quadric cone, having the point for its vertex ; in the latter case we have a plane touching the surface along a conic ; that is, the complete intersection of the surface by the plane is made up of the conic taken twice and of a residual curve of the order *n* - 4.

7. We may, in the general theory of surfaces, consider either a surface and its reciprocal surface, the recipro­cal surface being taken to be the surface enveloped by the polar planes (in regard to a given quadric surface) of the points of the original surface; or—what is better—we

may consider a given surface in reference to the reciprocal relations of its order, rank, class, and singularities. In either case we have a series of unaccented letters and a corresponding series of accented letters, and the relations between them are such that we may in any equation inter­change the accented and the unaccented letters ; in some cases an unaccented letter may be equal to the correspond­ing accented letter. Thus, let *n, n* be as before the order and the class of the surface, but, instead of immediately defining the rank, let *a* be used to denote the class of the plane section and *a*' the order of the circumscribed cone ; also let *S*, *S'* be numbers referring to the singularities. The form of the relations is *a* = *a*'( = rank of surface); *a*' = *n* (*n*- 1)-*S*; *n'=n(n-*1)2-*S*; *a* = *n'* (*n'* - 1) - *S'*;

*n'* = *n'*(*n'* - 1)2 - In these last equations *S, S'* are merely written down to denote proper corresponding com­binations of the several numbers referring to the singular­ities collectively denoted by *S, S'* respectively. The theory, as already mentioned, is a complex and difficult one, and it is not the intention to further develop it here.

8. A developable or torse corresponds to a curve in space in the same manner as a cone corresponds to a plane curve : although capable of representation by an equation *U=*(*x, y, z, w*)*n*, = 0, and so of coming under the foregoing point definition of a surface, it is an entirely distinct geo­metrical conception. We may indeed, *qua* surface, regard it as a surface characterized by the property that each of its tangent planes touches it, not at a single point, but along a line ; this is equivalent to saying that it is the envelope, not of a doubly infinite series of planes, as is a proper surface, but of a singly infinite system of planes. But it is perhaps easier to regard it as the locus of a singly infinite system of lines, each line meeting the consecutive line, or, what is the same thing, the lines being tangent lines of a curve in space. The tangent plane is then the plane through two consecutive lines, or, what is the same thing, an oscu­lating plane of the curve, whence also the tangent plane intersects the surface in the generating line counting twice, and in a residual curve of the order *n* - 2. The curve is said to be the edge of regression of the developable, and it is a cuspidal curve thereof ; that is to say, any plane section of the developable has at each point of intersection with the edge of regression a cusp. A sheet of paper bent in any manner without crumpling gives a developable ; but we cannot with a single sheet of paper properly exhibit the form in the neighbourhood of the edge of regression : we need two sheets connected along a plane curve, which, when the paper is bent, becomes the edge of regression and appears as a cuspidal curve on the surface.

It may be mentioned that the condition which must be satisfied in order that the equation *U*= 0 shall represent a developable is *H(U)* = 0 ; that is, the Hessian or functional determinant formed with the second differential coefficients of *U* must vanish in virtue of the equation *U=* 0, or—what is the same *thing—H(U*) must contain *U* as a factor. If in Cartesian coordinates the equation is taken in the form *z -f(x, y)* = 0, then the condition is *rt -s2 =* 0 identically, where *r, s, t* denote as usual the second differential co­efficients of *z* in regard to *x, y* respectively.

9. A ruled surface or regulus is the locus of a singly infinite system of lines, where the consecutive lines do not intersect; this is a true surface, for there is a doubly infinite series of tangent planes,—in fact any plane through any one of the lines is a tangent plane of the surface, touching it at a point on the line, and in such wise that, as the tangent plane turns about the line, the point of con­tact moves along the line. The complete intersection of the surface by the tangent plane is made up of the line counting once and of a residual curve of the order *n-*1*.* A quadric surface is a regulus in a twofold manner, for

@@@1 In a plane curve the only singularities which need to be considered are those that present themselves in Plücker’s equations, for every higher singularity whatever is equivalent to a certain number of nodes, cusps, inflexions, aud double tangents. As regards a surface, no such reduction of the higher singularities has as yet been made.