for a congruence there is in general a surface having the congruence lines for bitangents, for a complex there is not in general any surface having the complex lines for tangents ; the tangent lines of a surface are thus only a special form of complex. The theory of complexes first presented itself in the researches of Malus on systems of rays of light in connexion with double refraction.

15. The analytical theory as well of congruences as of complexes is most easily carried out by means of the six coordinates of a line ; viz., there are coordinates (a, *b, c, f, g, h)* connected by the equation *af+ bg + ch =* 0, and there­fore such that the ratios *a :b :c :f ∙.g : h* constitute a system of four arbitrary parameters. We have thus a congruence of the order *n* represented by a single homogeneous equa­tion of that order (\*)(*a, b, c, f, g, h*)*n*, = 0 between the six coordinates ; two such relations determine a congruence. But we have in regard to congruences the same difficulty as that which presents itself in regard to curves in space : it is not every congruence which can be represented com­pletely and precisely by *two* such equations.

The linear equation (\*)(*a*, *b, c, f, g, h)* = 0 represents a congruence of the first order or linear congruence ; such congruences are interesting both in geometry and in con­nexion with the theory of forces acting on a rigid body.

*Curves of Curvature ; Asymptotic Lines.*

16. The normals of a surface form a congruence. In any congruence the lines consecutive to a given congruence line do not in general meet this line ; but there is a deter­minate number of consecutive lines which do meet it ; or, attending for the moment to only one of these, say the congruence line is met by a consecutive congruence line. In particular, each normal is met by a consecutive normal ; this again is met by a consecutive normal, and so on. That is, we have a singly infinite system of normals each meeting the consecutive normal, and so forming a torse ; starting from different normals successively, we obtain a singly infinite system of such torses. But each normal is in fact met by two consecutive normals, and, using in the construction first the one and then the other of these, we obtain two singly infinite systems of torses each intersecting the given surface at right angles. In other words, if in place of the normal we consider the point on the surface, we obtain on the surface two singly infinite systems of curves such that for any curve of either system the normals at consecutive points intersect each other ; moreover, for each normal the torses of the two systems intersect each other at right angles ; and therefore for each point of the surface the curves of the two systems intersect each other at right angles. The two systems of curves are said to be the curves of curvature of the surface.

The normal is met by the two consecutive normals in two points which are the centres of curvature for the point on the surface ; these lie either on the same side of the point or on opposite sides, and the surface has at the point in question like curvatures or opposite curvatures in the two cases respectively (see *supra,* No. 4).

17. In immediate connexion w’ith the curves of curvature we have the so-called asymptotic curves (Haupt-tangenten- linien). The tangent plane at a point of the surface cuts the surface in a curve having at that point a node. Thus we have at the point of the surface two directions of passage to a consecutive point, or, say, two elements of arc ; and, passing along one 'of these to the consecutive point, and thence to a consecutive point, and so on, we obtain on the surface a curve. Starting successively from different points of the surface we thus obtain a singly infinite system of curves ; or, using first one and then the other of the two directions, we obtain two singly infinite systems of curves, w’hich are the curves above referred to. The two

curves at any point are equally inclined to the two curves of curvature at that point, or—what is the same thing— the supplementary angles formed by the two asymptotic lines are bisected by the two curves of curvature. In the case of a quadric surface the asymptotic curves are the two systems of lines on the surface.

*Geodesic Lines.*

18. A geodesic line (or curve) is a shortest curve on a surface ; more accurately, the element of arc between two consecutive points of a geodesic line is a shortest arc on the surface. We are thus led to the fundamental property that at each point of the curve the osculating plane of the curve passes through the normal of the surface; in other words, any two consecutive arcs *PP', PP'* are *in plano* with the normal at *P.* Starting from a given point *P* on the surface, we have a singly infinite system of geodesics proceeding along the surface in the direction of the several tangent lines at the point P; and, if the direction *PP* is given, the property gives a construction by successive elements of arc for the required geodesic line.

Considering the geodesic lines which proceed from a given point *P* of the surface, any particular geodesic line is or is not again intersected by the consecutive generating line ; if it is thus intersected, the generating line is a shortest line on the surface up to, but not beyond, the point at w’hich it is first intersected by the consecutive generating line ; if it is not intersected, it continues a shortest line for the whole course.

In the analytical theory both of geodesic lines and of the curves of curvature, and in other parts of the theory of surfaces, it is very convenient to consider the rectangular coordinates *x, y, z* of a point of the surface as given functions of two independent parameters *p, q* ; the form of these functions of course determines the surface, since by the elimination of *p, q* from the three equations we obtain the equation in the coordinates *x, y, z.* We have for the geodesic lines a differential equation of the second order between *p* and *q;* the general solution contains two arbi­trary constants, and is thus capable of representing the geodesic line which can be drawm from a given point in a given direction on the surface. In the case of a quadric surface the solution involves hyperelliptic integrals of the first kind, depending on the square root of a sextic function.

*Curvilinear Coordinates.*

19. The expressions of the coordinates *x, y, z* in terms of *p, q* may contain a parameter *r,* and, if this is regarded as a given constant, these expressions will as before refer to a point on a given surface. But, if *p, q, r* are regarded as three independent parameters, *x, y, z* will be the co­ordinates of a point in space, determined by means of the three parameters *p,q,r ',* these parameters are said to be the curvilinear coordinates, or (in a generalized sense of the term) simply the coordinates of the point. We arrive otherwise at the notion by taking *p, q, r* each as a given function of *x, y, z;* say we have *p=f*1*(x,y,z), q=f*2*(x,y,z), r=f*3*(x,y,z),* which equations of course lead to expressions for *p, q, r* each as a function of *x, y, z.* The first equation determines a singly infinite set of surfaces : for any given value of *p* we have a surface ; and similarly the second and third equations determine each a singly infinite set of surfaces. If, to fix the ideas, *f*1, *f*2, *f*3 are taken to denote each a rational and integral function of *x, y, z,* then two surfaces of the same set will not intersect each other, and through a given point of space there will pass one surface of each set ; that is, the point will be determined as a point of intersection of three surfaces belonging to the three sets respectively; moreover, the whole of space will be divided by the three sets of surfaces into a triply infinite system of elements, each of them being a parallelepiped.