of a polygonal figure having *m* geometrical equations of condition, and *x* for the most probable value of the error of any observed angle, we have

= = n for a single figure,

*3 IV mm σ*

= tfh3 for a group of figures,

[wι]

the brackets [] in each case denoting the sum of all the quantities involved, *e*3 usually gives the best value of the theoretical error, then *e*2. As a rule the value by *e*1 is too small ; but to this there are notable exceptions, in which it was found to be much too great. The instrument with which the angles were measured in these instances gave very discrepant results at different settings of the circle ; but this was caused by large periodic errors of graduation which did not affect the “ concluded angles,” because they were eliminated by the systematic changes of setting, so the results were really more precise than was apparent.

When weights were determined for the final simultaneous reduction of triangulations executed by different instru­ments, it became necessary to find a factor *p* to be applied as a modulus to each group of angles measured with the same instrument and under similar conditions, to convert the as yet relative weights into absolute measures of preci­sion. *ρ* was made = *e*1÷*e*3 whenever data were available, if not to *e*1 *÷ e*2 ; then the absolute weight of an observed angle in any group was taken as *wρ*2 and the *e.m.s.* of the angle as 1 *÷ρ√w.* The average values of the *e.m.s.* thus determined for large groups of angles, measured with the 36-inch and the 24-inch theodolites, ranged from ± 0"·24 to ± 0"·67, the smaller values being usually obtained at hill stations, where the atmospheric conditions were most favourable.

13. *Harmonizing Angles of Trigonometrical Figures.—* Every figure, whether a single triangle or a polygonal net­work, was made consistent by the application of corrections to the observed angles to satisfy its geometrical conditions. The three angles of every triangle having been observed, their sum had to be made = 180° + the spherical excess; in networks it was also necessary that the sum of the angles measured round the horizon at any station should be exactly = 360°, that the sum of the parts of an angle measured at different times should equal the whole, and that the ratio of any two sides should be identical, what­ever the route through which it was computed. These are called the *triangular, central, toto-partial,* and *side* condi­tions ; they present *n* geometrical equations, which contain *t* unknown quantities, the errors of the observed angles, *t* being always >n. When these equations are satisfied and the deduced values of errors are applied as corrections to the observed angles, the figure becomes consistent. Primarily the equations were treated by a method of suc­cessive approximations ; but afterwards they were all solved simultaneously by the so-called method of minimum squares, which leads to the most probable of any system of corrections ; it is demonstrated under Earth, Figure of the (vol. vii. p. 599). The following is a general out­line of the process :—

Let *x* be the most probable value of the error and *u* the recipro­cal of the weight of any observed angle *X,* and let *α, b,. .. n* bethe coefficients of *x* in successive geometrical equations of condition whose absolute terms are *ea, eb,. . . en ;* then we have the following group of *n* equations containing *t* unknown quantities to be satis­fied, the significant coefficients of *x* being 1 in the triangular, toto- partial, and central, and ± cot *X* in the side equations :—

*a1x1 + a2x2 + . . . + atxt=ea}*

*b]Xl +b2r2 + . . . ÷ btXt =* ***eb*** I (g)

n1x1 + >½a½ +. · · +ni¾ = en)

The values of *x* will be the most probable when [{{}}}] is a minimum, a condition which introduces *n* indeterminate factors λ*a*, . . . λ*n*,

whose values are obtained by the solution of the following equa­tions :— *[oa.u]λa* + [α⅛4i]λft +. . . + [an. *u]λn = etA*

[aδ. w]λa + [δδ. w]λ4 +. . . + [⅛n,w]λn = *ek* p

[a>ι.n]λlj + [δπ.a]λft +.. . + *[nn.*ιz]λn = *en J* the brackets indicating summations of *t* terms as to left of (3).

Then the value of any, the *p*th, *x* is

*xp —- up* {iq,Xα -f- *bp∖k* -f-. . , + (5)\*

The minimum or [3 is =[eλ] (6).

In the application to a single triangle we have *x1+x2+x3=e,* λ = *e*÷(*u*1+*u*2+*u*3); *x1* = *u*1λ; *x2* = *u*2λ; *x3* = *u*3λ*.*

In the application to a simple polygon, by changing symbols and putting *X* and *Y* for the exterior and *Z* for the central angles, with errors *x, y,* and *z* and weight reciprocals *u, v,* and *w,* *a* for cot *X* and *b* for cot *Y, e* for any triangular error, *ec* and *es* for the central and side errors, λ*c* and λ*s* for the factors for the central and side equations, and *W* for *u + v + w,* the equations for obtaining the factors become

Γ w2-k i-w(αii-δ-r)Ι. *Γwe~] ∖*

Lw-7f> -L—^Jx∙=i∙-W **lm**

-[⅛^⅛]x.+[A+δ⅛-S^⅛]x.=e.-[⅛∑⅛J∫'' λ and the general expressions for the errors of the angles are—

*x=^,{e + {aW - au + bvj∖t- w∖c}* 1

y= - (δ JK+αii-frυ)λ,-λw>c} J- (8).

*w* I

2=^jp∙ {c-(αw - *bvy)∖s + {u* + r)λcJ∙ j

14. *Calculation of Sides of Triangles.—*The angles having been made geometrically consistent *inter se* in each figure, the side-lengths are computed from the base-line onwards by Legendre’s theorem, each angle being dimin­ished by one-third of the spherical excess of the triangle to which it appertains. The theorem is applicable without sensible error to triangles of a much larger size than any that are ever measured.

15. *Calculation of Latitudes and Longitudes of Stations and Azimuths of Sides.—*A station of origin being chosen of which the latitude and longitude are known astronomi­cally, and also the azimuth of one of the surrounding stations, the differences of latitude and longitude and the reverse azimuths are calculated in succession, for all the stations of the triangulation, by Puissant’s formulæ (*Traité de Géodésie,* Paris, 1842, 3d ed.).

*Problem.—*Assuming the earth to be spheroidal, let A and B be two stations on its surface, and let the latitude and longitude of A be known, also the azimuth of B at A, and the distance between A and B at the mean sea-level ; we have to find the latitude and longitude of B and the azimuth of A at B.

The following symbols are employed *α* the major and *b* the minor semi-axis ; *e* the excentricity, = -■ *~~f-~~ ; p* the radius of

tι( 1 - *c%y}*

curvature to the meridian in latitude λ, = —, : *v* the normal

{l-e⅝in2λ}U

to the meridian in latitude λ, = 1 ; λ and *L* the given

{l-e2sin2λ}1 o

latitude and longitude of A ; λ + ∆λ and *L* + Δ*L* the required lati­tude and longitude of B ; *A* the azimuth of B at A ; *B* the azimuth of A at B ; ∆*A=B-(π + A)* ; *c* the distance between A and B. Then, all azimuths being measured from the south, we have

*r c*

- cos *A* cosec 1"

1 2

- - — sin2√4 tan X cosec 1"

δx"H 3 c'2 c2 . r ox

- -r —-ι ⅛ cos2√4 sin 2λ cosec 1"

4 *ρ.v* 1 - *ei*

1

+ - —3 sin3√i cos *A(l* + 3 tan2 λ) cosec 1"

I θ *P∙v* J

*c* sin *A ...* Ί

r- cosec 1

*v* cos λ

1 *e2* sin 2 *A* tan λ ,,,

+ s —, r cosec 1

λ *τ"- i \*v* cθs ^ lorn

~ 1 c3 (1 + 3 tan2λ) sin 2A cos√∕ „ *i '∙ ',*

- 5 ~ ½ cosec 1

o *vj* cos λ

, 1 c3 sin3∠f tan2 λ ,,,

+ s -⅞ ; cosec 1

3 *vi* cos X j