ûo + *aix + aiχ2 + .. + apxf.* It is clear that we can regard the series *uo+UιX+u2X2A-.*. .as the expansion in powers of *x* of an expression of the form

(δ0+¼x+ ... +0p-ιxw)∕(θo+αι^÷ \*. \* +⅝\*p), and by splitting this expression into partial fractions we can obtain the general term of the series. If we know that a series is a recurring series and know the number of terms in its scale of relation, we can determine this scale if we are given a sufficient number of terms of the series and obtain its general term. It follows that the general term of a recurring series is of the form ∑≠(w)αn, where *φ(n)* is a rational integral algebraic function of «, and α is independent of *n*. The series whose general term is of the form Kαn+<∕>(w), where *φ(n)* is a rational integral algebraic function of degree *r,* is a recurring series whose scale of relation is (1 — *ax)* (1— x)h^1, but the general term of this series may be obtained by another method. Suppose we have a series *u0, u*1*, u*2*,...* From this we can form a series t⅛, ι⅛, v2,... where *υn'=un+ι-un\*,* from **»0, t⅛j t⅛,...** we similarly form another series and so on; we write τ>ft≡ ∆wn, and we suppose E to be an operation such that Eurt≡‰+ι (the notation is that of the calculus of finite differences); the operations E and 1 ÷Δ are equivalent and hence the operations En and (1 +∆)n are equivalent, so that we obtain wn≈=tt0 ÷ *n∆uv +* ---— Δ2mo÷ \* · · This is true whatever the form of wn. When

u∏is of the form Kαn+φ(w), where *φ(n)* is of degree r, ∆r¾o, Δr+¼>,

... form a geometrical progression, of which the common difference is α —I, or vanish if the term Kαn is absent. In either case we readily obtain the expression for *un.*

2. The general problem of finite series is to find the sum of *n* terms of a series of which the law of formation is given. By finding the sum to *n* terms is meant finding some simple function of *n*, or a sum of a finite number of simple functions, the number being independent of *n,* which shall be equal to this sum. Such an ex­pression cannot always be found even in the case of the simplest series. The sum of *n* terms of the arithmetic progression *a*, a+ò, α+2&,... is wα-∣-⅜w(n-ι)δ; the sum of *n* terms of the geometric progression *a, ab,* αά2, ...is *a(ι* — δn)∕(ι — *b)* ; yet we can find no simple expression to represent the sum of *n* terms of the harmonic progression

I+Ì + I+...+“

3. The only type of series that can be summed to *n* terms with complete generality is a recurring series. If we let Sn=2⅛÷Mι%+

... +Mn-ιXn^1t where *u0i..* .is a recurring series with a given scale of relation, for simplicity take it to be 1+∕>x+gx2, we shall have

**Sn(l** *-VpX-Vq%2)* =Wθ4"*(ul+puQ)x + (fiuπ-l+qun-z)xP+qun-lXn+1.* If *x* had a value that made ι-b∕>χ+iξχ2 vanish, this method would fail, but we could find the sum in this case by finding the general term of the series. For particular cases of recurring aeries we may

proceed somewhat differently. If the nth term is *unxn* we have from the equivalence of the operations E and 1 +Δ,

*1 , l , \_ „ Xul —* Xn+1‰+l 1 X2∆tti-Xπ+2ΔWn+ι

UiX-H2X2+ . . . +ttnXn= 1-χ + ~~(i—χ)~~~~2~~

f χs∆¼-Xft+8Δ2Wft+i 1

+ (∏=Tp + · · ·

in general, and for the case of χ = unity we have

» . I I n.w — ia t w.w —i.n —2a9 t

«I+W2+ · · · +Wn=WMι-t ΔMi4 Δ⅛ιT . . .,

which will give the sum of the series very readily when *un* is a polynomial in *n* or a polynomial + a term of the form Kαn.

4. Other types of series, when they can be summed to *n* terms at all, are summed by some special artifice. Summing the series to 3 or 4 terms may suggest the form of the sum to *n* terms which can then be established by induction. Or it may be possible to express ttnin the form wλ+i-*wn,* in which case the sum to *n* terms is ¾1-W1. Thus, if un = a(α+⅛)(α-∣-2⅛) . . . (α+w — *1h)∣c(c+b)(c+2b) . . .* (c+w — *lb),* the relation (r-⅜-w⅛)‰+1 ≈ (α-bn⅛)‰ can be thrown into the form *(c+nb)un+ι-*(c÷n- ιδ)wn = (α-*c+b)unt* whence the sum can be found. Again, if wn = tan *nx* tan (n4^1)χ, the summation can be effected by writing *un* in the form cot *x* (tan w + ix—tan *nx)* — 1. Or a series may be recognized as a coefficient in a product. Thus, if ∕(x)≡M0÷tt1\*÷tt2X2÷..., 2⅛+tt1-)-...+«η is the coefficient of χn in ∕(%)∕(ι-x); in this way the sum of the first *n* coefficients in the expansion of (1 — χ)-\* may be found. The sum of one series may be deduced from that of another by differentiation or integration. For further information the reader may consult G. Chrystal's *Algebra* (vol. ii.).

5. The sum of an infinite series may be deduced from\* the sum to *n* terms, when this is known, by increasing *n* indefinitely and finding the limit, if any, to which it tends, but a series may often be summed to infinity when it cannot be summed to *n* terms; the sum of the infinite series p+^3÷∣2÷∙ · - is the sum to *n* terms cannot be found.

For methods and transformations by means of which the sum to

*n* terms of a series may be found approximately when it cannot be found exactly, the reader may consult G. Boole’s *Treatise on the Calculus of Finite Differences.*

Infinite Series.

6. Let ttι, **«2,** *u3,...un,* be a series of numbers real or complex, and let Sn denote fh⅛÷... ⅛un. We thus form a sequence of numbers Sι,S2,... Sn. This sequence may tend to a definite finite limit S as n increases indefinitely. In this case the series «i÷«2÷... ÷«» is said to be *convergent,* and to converge to a sum S. If by taking *n* sufficiently large ∣Sn∣ can be made to exceed any assignable quantity, however large, the series is said to be *divergent.* If the sequence S∏ S2,... tends to finite but different limits according to the form of *n* the series is said to oscillate, and is also classed under the head of divergent series. The sum of *n* terms of the geometric series 1+χ+χ2+.. .is (1 —χn)∕(ι ~χ). If *x* is less than unity Sn clearly tends to the limit 1/(1—χ), and the series is con- vergent and its sum is l∕(ι — X). If *x* is greater than unity S» clearly can be made greater than any assignable quantity by taking *η* large enough, and the series is divergent. The series **I —1 + 1 —1 + ...,** where Sn is unity or zero, according as *n* is odd or even, is an example of an oscillating series. The condition of convergency may also be presented under the following form. Let j>Rn denote Sn+,>-Sn: let e be any arbitrarily assigned positive quantity as small as we

please; if we can find a number *m* such that for w=or>n, LRn∣<e for all values 1, 2,... of *p,* then the series converges. The least value of the number *m* corresponding to a given value of c, if it can be found, may be regarded as a measure of rapidity of the con- vergency of the series; it may happen that when *uη* involves a variable x, *m* increases indefinitely as *x* approaches some value; in this case the convergence of the series is said to be infinitely slow for this value of *x.*

7. An infinite series may contain both positive and negative terms. The terms may be positive and negative alternately or they may occur in groups which without altering the order of the terms of the series may each be collected into a single term; thus all series may be regarded as belonging to one of two types, *Ui-i-u^-j-us-V* .. .in which the terms are all positive, or «1—w2d-w3-.. .in which the terms are alternately positive and negative.

8. It is dear that if a series is convergent *un* must tend to the limit zero as *n* is increased indefinitely. This condition though necessary is by no means sufficient. If all the terms of a convergent series are positive a series obtained by writing its terms in any other order is convergent and converges to the same sum. For if Sn denotes the sum of w terms of the first series and ∑n denotes the sum of *n* terms of the new series, then, when *n* **is** any large number, we can choose numbers *p* and *q* such that Si>Σn>Sj>j so that ∑n tends to the common limit of Sp and Sff, which is the sum of the original series. If Wι, ti2j «s,... are all positive, and if after some fixed term, say the *pih, un* continually decreases and tends to the limit zero, the series «1—m2÷«3—«4+ ...is convergent. For ISp+2n-Sp∣ lies between ttp+2∣ and ∣ttp+ι-*uρ+2n}* so that, when *n* is

increased indefinitely, **∣Sp+2n∣** remains finite; also ∣Sp+2a+i-Sj,+2n∣ tends to zero, so that the series converges. If *un* tends to a limit *a,* distinct from zero, then the series **t⅛-**t>2-J-t⅛-.. ..where *υn=un-*α, converges and the series «1—tt2+tts... oscillates. As examples we may take the series 1 — ⅜+⅜-i÷ ... and 2—⅞ + **J-**î+ ...; the first of these converges, the second oscillates.

9. The series «i÷ms÷w5÷ ...» tt2÷tt4+t⅛÷... may each of them diverge, though the series «ι—«2+«3—.. .converges. A series such that the series formed by taking all its terms positively is convergent is said to be *absolutely convergent* ; when this is not the case the series is said to be *semi-convergent* or *conditionally con­vergent.* A series **of** complex numbers in which *uη-pηjτiqη,* where ∕>nand çnare real (i being √ — 1), is said to be convergent when the series Λ+∕⅛÷∕>j÷..., îi÷Çî÷çs÷... are separately convergent, and if they converge to P and Q respectively the sum of the series is P-∏Q. Such a series is said to be absolutely convergent when the series of moduli of wn, *i.e.,* Σ(Aι2÷<Zn2)⅛, is convergent; this is sufficient but not necessary for the separate convergence of the *p* and *q* series.

There is an important distinction between absolutely convergent and conditionally convergent series. In an absolutely convergent series the sum is the same whatever the order of the terms; this is not the case with a conditionally convergent series. The two series I—i + ⅜-Id-..., and ι+⅜-⅛ + ⅛4-⅛-i + ..., in which the terms are the same but in different orders, are convergent but not absolutely convergent. If we denote the sum of the first by S and the sum of the second by Σ it can be shown that Σ = JS. G. F. B. Riemann and P. G. L. Dιrichlet have shown that·the terms of a semi- convergent series may be so arranged as to make the series converge to any assigned value or even to diverge.

10. Tests for convergency of series of positive terms are obtained by comparing the series with some series whose convergency or divergency is readily established. If the series of positive terms **«i÷«2÷tta÷.. ∙, t⅛÷t⅛+t⅛-∣-...** are such that ‰∕rn is always finite, then they are convergent or divergent together; if WΛ+ι∕‰<rn+ι∕fn and *∑vn* is convergent, then *^∑un* is convergent; if *un+ι∕un>vn^[∣vn* and ∑fn is divergent, then ∑wn is divergent. By comparison with the ordinary geometric progression we obtain the