following tests. If *1∖lu\** approaches a limit *l* as *n* is indefinitely increased, ∑wn will converge if *l* is less than unity and will diverge if *l* is greater than unity (Cauchy s test) ; if *uη+ιluη* approaches a limit *l* as *η* is indefinitely increased, ∑κn wi∏ converge if *l* is less than unity and diverge if *l* is greater than unity (D’Alembert’s test). Nothing is settled when the limit Z is unity, except in the case when *l* remains greater than unity as it approaches unity. The series then diverges. It may be remarked that if *un+∖∣un* approaches a limit and V«» approaches a limit, the two limits are the same. The choice of the more useful test to apply to a particular series depends on its form.

In the case in which *un±∖iun* approaches unity remaining con­stantly less than unity, J. L. Raabe and J. M. C. Duhamel have given the following further criterion. Write *u1Jun+ι* = ι+<⅛> where αn is positive and approaches zero as *n* is indefinitely increased. If *nan* approaches a limit *l,* the series converges for *l* > i and diverges for Z< **I.** For *l* = i nothing is settled except for the case where *l* remains constantlyjess than unity as it approaches it; in this case the series diverges.

If ∫(w) is positive and decreases as *n* increases, the series ∑tf(n) is convergent or divergent with the series ∑α"∕(αn) where *a* is any number >2 (Cauchy’s condensation test). By means of this theorem we can show that the series whose general terms are

2\_ i I i

w(Iw)α\* wlw(l2Λ)α\* wlwl2w(l3w)β\* ' \* \*’

where lw denotes Iog w, Pw denotes log-log w, Pn denotes log log log n, and **so** on, are convergent if α> ι ana divergent if **a** ≈=or< I.

By comparison with these series, a sequence of criteria, known as the logarithm criteria, has been established by De Morgan and J. L. Bertrand. A. De Morgan’s form is as follows: writing ‰ = ι∕≠(w), put 2>o = xφ'(x)∕≠(x), *p∖ = (po~*l)l\*, *pt≈(pi*“ i)⅛ ∕>s = (∕⅛-i)bx,-· . where Kx denotes log log log.. .x. If the limit, when *x* is infinite, of the first of the functions *p?, pi, pi,..*., whose limit is not unity, is greater than unity the series is convergent, if less than unity it is divergent.

In Bertrand’s form we take the series of functions

⅛∙⅛1,,∙,s⅛zι,n∙∙∙∙

íf the limit, when *n* is infinite, of the first of these functions, whose limit is not unity, is greater than unity the series is convergent, if less than unity it is divergent. Other forms of these criteria may be found in Chrystal’s *Algebra,* vol. ii.

Though sufficient to test such series as occur in ordinary mathe- matîcs, it is possible to construct series for which they entirely fail, ít follows that in a convergent series not only must we have Lt *un* — o but also Lt *nun* t=o, Lt wlnuw = o, &c. Abel bas, however, shown that no function φ(w) can exist such that the series ∑wn is convergent or divergent as Lt *Φ(n)un* is or is not zero.

ιι. Two or more absolutely convergent series may be added toge ther, th u s *(uι +ui* +... )+(rl+t⅛ +... ) = (in +r1 )+(«2 +t⅛) ÷ ..., that is, the resulting series is absolutely convergent and has for its sum the sum of the sums of the two series. Similarly two or more absolutely convergent series may be multiplied together thus («l+Wî+«î+· ..) (t⅛+t⅛+tfj+...) =WM + (Mjt⅛-l-ttjΓl)-h⅛V3 +

. . 1⅛t⅛÷W1t>ι)÷.. .>

and the resulting scries is absolutely convergent and its sum is the product of the sums of the two series. This was shown by Cauchy, who also showed that the series ∑wn, where wr,≈WjΓn+w2Γn,ι +

... ÷\*w∑⅛, is not necessarily convergent when both series are semî- convergent. A striking instance is furnished’by the series ι -+

.. which is convergent, while its square

H ' —. · · may be shown to be divergent. >F. K. L. Mertens

has shown that a sufficient condition is that one of the two series should be absolutely convergent, and Abel has shown that if ∑wn converges at all, it converges to the product of ÍX and 2rft. But more properly the multiplication of two series gives rise to a double series of which the general term is *umυn.*

12. Before considering a double series we may consider the case of a series extending backwards and forwards to infinity

. ..W-m+. . . + *ì + U~*1 +«o + 4-tt2 + . . . +ttn+. . .

Such a series may be absolutely convergent and the sum is then independent of the order of the terms and is equal to the sums of the two series <⅛+mi-∣-W2-∣-. .. and tt-ι+w-t+..., but, if not absolutely convergent, the expression has no definite meaning until it is explained in what manner the terms are intended to be grouped together; for instance, the expression may be used to denote the foregoing sum of two series, or to denote the series w0-∣-(mi-∣-w-i)+ (ttj+w-2) +. ·. . and the sum may have different values, or there may be no sum, accordingly. Thus, if the series be . ..-⅛-l + **o+i + } +...,** with the former meaning the two series o+{+⅜ + ... and — J -⅛-... are each divergent, and there is no sum; but with the latter meaning the series is **o+o+o-¼...** which has a sum o. So, if the series be taken to denote the limit of (ii0+tt1+... {-ttn)4- («\_i+tí\_4-.. .+w-m), where *n* and *m* are each of them ultimately

infinite, there may be a sum depending on the ratio *n : m,* which sum acquires a determinate value only when this ratio is given, ln the case of the series given above, if this ratio is *k,* the sum of the series is log *k.*

13. Ina singly infinite series we have a general term *un,* where *n* is an integer positive in the case of an ordinary series, and positive or negative in the case of a back-and-forwards series. SimilarIy for a doubly infinite series we have a general term *um,n* where *m, n* arc integers which may be each of them positive, and the form of the series is then

**«0,0, Wθιb Wo,2, - - - I «1,0, «1,1, «1,2». - -**

or they may be each of them çositive or negative. The latter is the more general supposition, and includes the former, since *um,n* may =o, for *m* or *n* each or either of them negative. To attach a definite meaning to the notion of a sum, we may regard *m, n* as the rectangu- lar coordinates of a point in a plane; if *m* and *n* are each positive we attend only to the positive quadrant of the plane, but otherwise to the whole plane. We may imagine a boundary depending on a para- meter T, which for T infinite is at every point thereof at an infinite distance from the boundary; for instance, the boundary may be the circle xs+y2 =T, or the four sides of a rectangle, *χ* = =tαT, *y ≈ =\*=0T.* Suppose the form is given and the value of **T,** and let the sum Sw,∏ be understood to denote the sum of the terms ι⅛,\* within the boundary , then, if as T increases without limit, Sw,w continually approaches a determinate limit (dependent, it may be, on the form of the boundary) *for such form of boundary* the series is said to be convergent, and the sum of the doubly infinite series is the limit of Sm,n. The condition of convergency maybe otherwise stated ; it must be possible to take T so large that the sum Rm,\* for all terms ttm,n which correspond to points outside the boundary shall be as small as we please.

14. lt is easy to see that, if each of the terms *um,n* is positive and the series is convergent for any particular form of boundary, it will be convergent for any other form of boundaιy, and the sum will be the same in each case. Suppose that in the first case the boundary is the curve Ji(x, y)-=T. Draw any other boundary ∫2(x, v)=Ti. Wholly within this we can draw a curve *fl(x, y)≈vι* of the first family, and wholly outside it we can draw a second curve of the first family, *fi(x,* y) «T2. The sum of all the points within ∕2(x, *y)* =Tr lies between the sum of all the points within *f1(x, y}* =T1 and the sum of all the points within *fl(x,* y)=T2. It therefore tends to the common limit to which these two last sums tend. The sum ιs therefore independent of the form of the boundary. Such a series is said to be absolutely convergent, and similarly a doubly infinite series of positive and negative terms is absolutely convergent when the series formed by taking all its terms positively is convergent.

15. It is readily, seen that when the series is not absolutely con- vergent the sum will depend on the form of the boundary. Consider the case in which *m* and *n* are always positive, and the boundary is the rectangle formed by *x∙=m,y≈n,* and the axes. Let the sum within this rectangle be Sm,n. This may have a limit when we first make w infinite and then rn∙, it may have a limit when we first make rn infinite and then *n,* but the limits are not necessariIy the same; or there may be no limit in either of these cases but a limit depending on the ratio of rn to n,that is to say, on the shape of the rectangle..

When the product of two series is arranged as a doubly infinite series, summing for the rectangular boundaryx = αT,y =0T we obtain the product of the sums of the series. When we arrange the double series in the form WιΓι4-(uιΓ2-Hκ2rι) +... we are summing over the triangle bounded by the axes and the straight line x-∣-y\*=T, and the results are not necessarily the same if the terms are not all posi­tive. For full particulars concerning muItiplc series the reader may consult E. Goursat, *Cours d'analyse,* vol. 1.; **G.** Chrystab *Algebra,* vol. ii.; or T. J. ΓA. Bromwich, *The Theory of Infinite Series.*

16. ln the. series so far considered the terms arc actual numbers, or, at least, if the terms are functions of a variable, we have, con­sidered the convergency only when that variable has an assigned value. In the case, however, of a series «i(z)+«j(z) + ..., where Wι(z), «a(z),v . are singIe-valued continuous functions of the general complex variable 2, if the series converges for any value of *z,* in general it converges for all values of *z,* whose representative points lie within a certain area called the “ domain of convergence ” and within this area defines a function which we may call S(z). It might be supposed that S(z) was necessariIy a continuous function of *z,* but this is not the case. **G. G.** Stokes (1847) and P. L. Seidel (1848) independently discovered that in the neighbourhood of a point of discontinuity the convergence is infinitely slow and thence arises the notion ol *uniform* and *non-uniform* convergence.

17. If for any value of *z* the series κι(s)+i⅛(z)÷. . converges it

is possible to find an integer *n* such that ∣ S(z) — Sn(z)∣<e, )S(z}~ Sα+ι(z) I < €,..., where e is any arbitrarily assigned positive quantity however small. For a given e the least value of *n* will vary through­out any region from point to point of that region.. lt may, however be possible to find an integer *v* which is a superior limit to all th< values of *n* in that region, and we thus have, throughout this region I S(z)~Sp(z) I < e,J S (z)-Sμ+j(z) ∣< . .where *z* is any point in th<

region and *v* is a finite integer depending only on e and not on *z*