The degree *n* of the equation is the order of the surface; and this definition of the order agrees with the geometrical one, that the order of the surface is equal to the number of the inter­sections of the surface by an arbitrary line. Starting from the foregoing point definition of the surface, we might develop the notions of the tangent line and the tangent plane; but it will be more convenient to consider the surface *ab initio* from the more general point of view in its relation to the point, the line and the plane.

2. Mention has been made of the plane curve and the cone; it is proper to recall that the *order of* a plane curve is equal to the

number of its intersections by an arbitrary line (in the plane of the curve), and that its *class* is equal to the number of tangents to the curve which pass through an arbitrary point (in the plane of the curve). The cone is a figure correlative to the plane curve: corresponding to the plane of the curve we have the vertex of the cone, to its tangents the generating lines of the cone, and to its points the tangent planes of the cone. But from a different point of view we may consider the generating lines of the cone as corresponding to the points of the curve and its tangent planes as corresponding to the tangents of the curve. From this point of view we define the order of the cone as equal to the number of its intersections (generating lines) by an arbitrary plane through the vertex, and its class as equal to the number of the tangent planes which pass through an arbitrary line through the vertex. And in the same way that a plane curve has singularities (singular points and singular tangents) so a cone has singularities (singular generating lines and singular tangent planes).

3. Consider now a surface in connexion with an arbitrary line. The line meets the surface in a certain number of points, and, as already mentioned, the *order* of the surface is equal to the number of these intersections. We have through the line a certain number of tangent planes of the surface, and the *class* of the surface is equal to the number of these tangent planes.

But, further, through the line imagine a plane; this meets the surface in a curve the order of which is equal (as is at once seen) to the order of the surface. Again, on the line imagine a point; this is the vertex of a cone circumscribing the surface, and the class of this cone is equal (as is at once seen) to the class of the sur­face. The tangent lines of the surface which lie in the plane are nothing else than the tangents of the plane section, and thus form a singly infinite series of lines; similarly, the tangent lines of the sur­face which pass through the point are nothing else than the generat­ing lines of the circumscribed cone, and thus form a singly infinite series of lines. But, if we consider those tangent lines of the sur­face which are at once in the plane and through the point, we see that they are finite in number; and we define the *rank* of a surface as equal to the number of tangent lines which lie in a given plane and pass through a given point in that plane. It at once follows that the class of the plane section and the order of the circum­scribed cone are each equal to the rank of the surface, and are thus equal to each other. It may be noticed that for a general surface (\**x*, *y,* z, *w*.)*n*,=0, of order *n* without point singularities the rank is *a, = n(n*-1*),* and the class is *n', =n(n-1)2;* this implies (what is in fact the case) that the circumscribed cone has line singularities, for otherwise its class, that is the class of the surface, would be *a(a-1),* which is not = n(n-1)2.

4. The notions of the tangent line and the tangent plane have been assumed as known, but they require to be further explained

in reference to the original point definition of the sur­face. Speaking generally, we may say that the points of the surface consecutive to a given point on it lie in a plane which is the tangent plane at the given point, and conversely the given point is the point of contact of this tangent plane, and that any line through the point of contact and in the tangent plane is a tangent line touching the surface at the point of contact. Hence we see at once that the tangent line is any line meeting the surface in two consecutive points, or—what is the same thing—a line meeting the surface in the point of contact counting as two intersections and in *n-2* other points. But, from the foregoing notion of the tangent plane as a plane containing the point of contact and the consecutive points of the surface, the passage to the true definition of the tangent plane is not equally obvious. A plane in general meets the surface of the order *n* in a curve of that order without double points; but the plane may be such that the curve has a double point, and when this is so the plane is a tangent plane having the double point for its point of contact. The double point is either an acnode (isolated point), then the surface at the point in question is convex towards (that is, concave away from) the tangent plane; or else it is a crunode, and the surface at the point in question is then concavo-convex, that is, it has its two curvatures in opposite senses (see below, par. 16). Observe that in either case any line whatever in the plane and through the point meets the surface in the points in which it meets the plane curve, viz. in the point of contact, which *qua* double point counts as two intersections, and in *n—2* other points; that is, we have the preceding definition of the tangent line.

5. The complete enumeration and discussion of the singularities of a surface is a question of extreme difficulty which has not yet been solved.@@1 A plane curve has point singularities and line singularities; corresponding to these we have for the surface isolated point singularities and isolated plane singularities, but there are besides continuous singularities applying to curves on or torses circumscribed to the surface, and it is among these that we have the non-special singularities which play the most important part in the theory. Thus the plane curve represented by the general equation (\**x*, *y, z)n=0,* of any given order *n,* has the non-special line singularities of inflexions and double tangents; corresponding to this the surface represented by the general equation (\**x*, *y, z,* *w*)n=0, of any given order *n,* has, not the isolated plane singularities, but the continuous singularities of the spinode curve or torse and the node-couple curve or torse. A plane may meet the surface in a curve having (1) a cusp (spinode) or (2) a pair of double points; in each case there is a singly infinite system of such singular tangent planes, and the locus of the points of contact is the curve, the envelope of the tangent planes the torse. The reciprocal singularities to these are the nodal curve and the cuspidal curve: the surface may intersect or touch itself along a curve in such wise that, cutting the surface by an arbitrary plane, the curve of inter­section has at each intersection of the plane with the curve on the surface (1) a double point (node) 01 (2) a cusp. Observe that these are singularities not occurring in the surface represented by the general equation (\**x*, *y,* z, w)*n* =0 of any order; observe further that in the case of both or either of these singularities the definition of the tangent plane must be modified. A tangent plane is a plane such that there is in the plane section a double point in addition to the nodes or cusps at the intersections with the singular lines on the surface.

6. As regards isolated singularities, it will be sufficient to mention the point singularity of the conical point (or cnicnode) and the corresponding plane singularity of the conic of contact (or cnictrope). In the former case we have a point such that the consecutive points, instead of lying in a tangent plane, lie on a quadric cone, having the point for its vertex; in the latter case we have a plane touching the surface along a conic; that is, the complete intersection of the surface by the plane is made up of the conic taken twice and of a residual curve of the order *n—4.*

7. We may, in the general theory of surfaces, consider either a surface and its reciprocal surface, the reciprocal surface being taken to be the surface enveloped by the polar planes (in regard to a given quadric surface) of the points of the original surface; or—what is better—we may consider a given surface in reference to the reciprocal relations of its order, rank, class and singularities. In either case we have a series of unaccented letters and a corre­sponding series of accented letters, and the relations between them are such that we may in any equation interchange the accented and the unaccented letters; in some cases an unaccented letter may be equal to the corresponding accented letter. Thus, let *n, n,* be as before the order and the class of the surface, but, instead of immediately defining the rank, let *a* be used to denote the class of the plane section and *a'* the order of the circumscribed cone; also let *S, S'* be numbers referring to the singularities. The form of the relations is *a=a,* (=rank of surface); *a'* = *n* (*n*-1) — *S; n' = n (n —* ι)2 — 5; *a = n' (n'* — 1) — 5'; *n = n' (n'* — 1)2 — *S'.* In these last equations 5, *S'* are merely written down to denote proper corresponding combinations of the several numbers referring to the singularities collectively denoted by 5, *S'* respectively. The theory, as already mentioned, is a complex and difficult one.

8. A torse or developable corresponds to a curve in space in the same manner as a cone corresponds to a plane curve: although capable of representation by an equation *U =* (\**x*, *y,* z, *w*)n =0, and so of coming under the foregoing point definition of a surface, it is an entirely distinct geo­metrical conception. We may indeed, *qua* surface, regard it as a surface characterized by the property that each of its tangent planes touches it, not at a single point, but along a line; this is equivalent to saying that it is the envelope, not of a doubly infinite series of planes, as is a proper surface, but of a singly infinite system of planes. But it is perhaps easier to regard it as the locus of a singly infinite system of lines, each line meeting the consecutive line, or, what is the same thing, the lines being tan­gent lines of a curve in space. The tangent plane is then the plane through two consecutive lines, or, what is the same thing, an oscu­lating plane of the curve, whence also the tangent plane intersects the surface in the generating line counting twice, and in a residual curve of the order »—2. The curve is said to be the edge of re­gression of the developable, and it is a cuspidal curve thereof; that is to say, any plane section of the developable has at each point of intersection with the edge of regression a cusp. A sheet of paper bent in any manner without crumpling gives a developable;

@@@1 In a plane curve the only singularities which need to be con­sidered are those that present themselves in Plücker's equations, for every higher singularity whatever is equivalent to a certain number of nodes, cusps, inflexions and double tangents. As re­gards a surface, no such reduction of the higher singularities has as yet been made.