but we cannot with a single sheet of paper properly exhibit the form in the neighbourhood of the edge of regression: we need two sheets connected along a plane curve, which, when the paper is bent, becomes the edge of regression and appears as a cuspidal curve on the surface.

It may be mentioned that the condition which must be satisfied in order that the equation *U=o* shall represent a developable is *H(U)=o;* that is, the Hessian or functional determinant formed with the second differential coefficients of *U* must vanish in virtue of the equation *U* = o, or—what is the same thing— H(*U*) must contain *U* as a factor. If in cartesian co-ordinates the equation is taken in the form *z—f* (x, y)=o, then the condition is *rt-*s2 = o identically, where *r*, *s*, *t* denote as usual the second differential coefficients of z in regard to *x, y* respectively.

9. A regulus or ruled surface is the locus of a singly infinite system of lines, where the consecutive lines do not intersect ; this

is a true surface, for there is a doubly infinite series of tangent planes—in fact any plane through any one of the lines is a tangent plane of the surface, touching it at a point on the line, and in such wise that, as the tangent plane turns about the line, the point of contact moves along the line. The complete intersection of the surface by the tangent plane is made up of the line counting once and of a residual curve of the order n —1. A quadric surface is a regulus in a two­fold manner, for there are on the surface two systems of lines each of which is a regulus. A cubic surface may be a regulus (see below, par. it).

*Surfaces of the Orders 2,3 and 4.*

10. A surface of the second order or a quadric surface is a surface such that every line meets it in two points, or—what comes to the

same thing—such that every plane section thereof is a conic or quadric curve. Such surfaces have been studied from every point of view. The only singular forms are when there is (1) a conical point (cnicnode), when the sur­face is a cone of the second order or quadricone; (2) a conic of contact (cnictrope), when the surface is this conic; from a different point of view it is a “ surface aplatie ” or flattened surface. Ex­cluding these degenerate forms, the surface is of the order, rank and class each = 2, and it has no singularities. Distinguishing the forms according to *reality,* we have the ellipsoid, the hyper­boloid of two sheets, the hyperboloid of one sheet, the elliptic paraboloid and the hyperbolic paraboloid (see Geometry : § *Ana­lytical).* A particular case of the ellipsoid is the sphere ; in abstract geometry this is a quadric surface passing through a given quadric curve, the circle at infinity. The tangent plane of a quadric surface meets it in a quadric curve having a node, that is, in a pair of lines; hence there are on the surface two singly infinite sets Of lines. Two lines of the same set do not meet, but each line of the one set meets each line of the other set; the surface is thus a reçulus in a two­fold manner. The lines are real for the hyperboloid of one sheet and for the hyperbolic paraboloid; for the other forms of surface they are imaginary.

11. We have next the surface of the third order or cubic surface, which has also been very completely studied. Such a surface

may have isolated point singularities (cnicnodes or points of higher singularity), or it may have a nodal line; we have thus 21+2, =23 cases. In the general case of a surface without any singularities, the order, rank and class are =3, 6, 12 respectively. The surface has upon it 27 lines, lying by threes in 45 planes, which are triple tangent planes. Ob­serve that the tangent plane is a plane meeting the surface in a curve having a node. For a surface of any given order *n* there will be a certain number of planes each meeting the surface in a curve with 3 nodes, that is, triple tangent planes; and, in the particular case where n = 3, the cubic curve with 3 nodes is of course a set of 3 lines; it is found that the number of triple tangent planes is, as just mentioned, =45. This would give 135 lines, but through each line we have 5 such planes, and the number of lines is thus =27. The theory of the 27 lines is an extensive and interesting one; in particular, it may be noticed that we can, in thirty-six ways, select a system of 6×6 lines, or "double sixer,” such that no two lines of the same set intersect each other, but that each line of the one set intersects each line of the other set.

A cubic surface having a nodal line is a ruled surface or regulus; in fact any plane through the nodal line meets the surface in this line counting twice and in a residual line, and there is thus on the surface a singly infinite set of lines. There are two forms.

12. As regards quartic surfaces, only particular forms have been much studied. A quartic surface can have at most 16 conical

points (cnicnodes); an instance of such a surface is Fresnel’s wave surface, which has 4 real cnicnodes in one of the principal planes, 4×2 imaginary ones in the other two principal planes, and 4 imaginary ones at infinity— in all 16 cnicnodes; the same surface has also 4 real+12 imaginary planes each touching the surface along a circle (cnictropes)—in all 16 cnictropes. It was easy by a mere homographic transforma­tion to pass to the more general surface called the tetrahedroid; but this was itself only a particular form of the general surface with 16 cnicnodes and 16 cnictropes first studied by Kummer. Quartic surfaces with a smaller number of cnicnodes have also been considered.

Another very important form is the quartic surface having a nodal conic ; the nodal conic may be the circle at infinity, and we have then the so-called anallagmatic surface, otherwise the cyclide (which includes the particular form called Dupin’s cyclide). These correspond to the bicircular quartic curve of plane geometry. Other forms of quartic surface might be referred to.

*Congruences and Complexes.*

13. A congruence is a doubly infinite system of lines. A line depends on four parameters and can therefore be determined so as to satisfy four conditions; if only two conditions are imposed on the line we have a doubly infinite system of lines or a congruence. For instance, the lines meet­ing each of two given lines form a congruence. It is hardly necessary to remark that, imposing on the line one more condition, we have a ruled surface or regulus; thus we can in an infinity of ways separate the congruence into a singly infinite system of reguli or of torses (see below, par. 16).

Considering in connexion with the congruence two arbitrary lines, there will be in the congruence a determinate number of lines which meet each of these two lines; and the number of lines thus meeting the two lines is said to be the *order-class* of the congruence. If the two arbitrary lines are taken to intersect each other, the congruence lines which meet each of the two lines separate them­selves into two sets—those which lie in the plane of the two lines and those which pass through their intersection. There will be in the former set a determinate number of congruence lines which is the *order* of the congruence, and in the latter set a determinate number of congruence lines which is the *class* of the congruence. In other words, the order of the congruence is equal to the number of congruence lines lying in an arbitrary plane, and its class to the number of congruence lines passing through an arbitrary point.

The following systems of lines form each of them a congruence: (A) lines meeting each of two given curves; (B) lines meeting a given curve twice; (C) lines meeting a given curve and touching a given surface; (D) lines touching each of two given surfaces; (E) lines touching a given surface twice, or, say, the bitangents of a given surface.

The last case is the most general one; and conversely for a given congruence there will be in general a surface having the congruence lines for bitangents. This surface is said to be the *focal surface* of the congruence; the general surface with 16 cnicnodes first pre­sented itself in this manner as the focal surface of a congruence. But the focal surface may degenerate into the forms belonging to the other cases A, B, C, D.

14. A complex is a triply infinite system of lines—for instance, the tangent lines of a surface. Considering an arbitrary point in connexion with the complex, the complex lines which pass through the point form a cone; considering a plane in connexion with it, the complex lines which lie in the plane envelop a curve. It is easy to see that the class of the curve is equal to the order of the cone ; in fact each of these numbers is equal to the number of complex lines which lie in an arbitrary plane and pass through an arbitrary point of that plane; and we then say *order* of complex = order of curve; *rank* of complex=class of curve =order of cone; *class* of complex=class of cone. It is to be observed that, while for a congruence there is in general a surface having the congruence lines for bitangents, for a complex there is not in general any surface having the complex lines for tangents; the tangent lines of a surface are thus only a special form of complex. The theory of complexes first presented itself in the researches of Malus on systems of rays of light in connexion with double refraction.

15. The analytical theory as well of congruences as of complexes is most easily carried out by means of the six co-ordinates of a line ; viz. there are co-ordinates *(a, b, c, f,* g, *h)* connected by the equation *af+bg+ch=O,* and therefore such that the ratios *a:b:c:f:g·.h* constitute a system of four arbitrary parameters. We have thus a congruence of the order *n* represented by a single homogeneous equation of that order (\**a*, *b, c, f, g, h)n=o* between the six co-ordinates; two such relations determine a congruence. But we have in regard to congruences the same difficulty as that which presents itself in regard to curves in space: it is not every congru­ence which can be represented completely and precisely by *two* such equations (see Geometry: *§ Line).*

The linear equation (\**a*, *b, c, f,* g, *h)* =0 represents a congruence of the first order or linear congruence; such congruences are inter­esting both in geometry and in connexion with the theory of forces acting on a rigid body.

*Curves of Curvature; Asymptotic Lines.*

16. The normals of a surface form a congruence. In any con­gruence the lines consecutive to a given congruence line do not