in general meet this line; but there is a determinate number of consecutive lines which do meet it; or, attending for the moment to only one of these, say the congruence line is met by a consecutive congruence-line. In particular, each normal is met by a consecutive normal; this again is met by a consecutive normal, and so on. That is, we have a singly infinite system of normals each meeting the consecutive normal, and so forming a *torse* ;. starting from different normals successively, we obtain a singly infinite system of such torses. But each normal is in fact met by two consecutive normals, and, using in the construction first the one and then the other of these, we obtain two singly infinite systems of torses each intersecting the given surface at right angles. In other words, if in place of the normal we consider the point on the surface, we obtain on the surface two singly infinite systems of curves such that for any curve of either system the normals at consecutive points intersect each other; moreover, for each normal the torses of the two systems intersect each other at right angles ; and therefore for each point of the surface the curves of the two systems intersect each other at right angles. The two systems of curves are said to be the curves of curvature of the surface.

The normal is met by the two consecutive normals in two points which are the centres of curvature for the point on the surface; these lie either on the same side of the point or on opposite sides, and the surface has at the point in question like curvatures or opposite curvatures in the two cases respectively (see above, par. 4). .

17. In immediate connexion with the curves of curvature we have the so-called asymptotic curves (Haupt-tangentenlinien). The tangent plane at a point *of* the surface cuts the surface in a curve having at that point a node. Thus we have at the point of the surface two directions of passage to a consecutive point, or, say, two elements of arc; and, passing along one of these to the consecutive point, and thence to a consecutive point, and so on, we obtain on the surface a curve. Starting successively from different points of the surface *we* thus obtain a singly infinite system of curves; or, using first one and then the other of the two directions, we obtain two singly infinite systems of curves, which are the curves above referred to. The two curves at any point are equally inclined to the two curves of curvature at that point, or—what is the same thing—the supplementary angles formed by the two asymptotic lines are bisected by the two curves of curvature. In the case of a quadric surface the asymptotic curves are the two systems of lines on the surface.

*Geodetic Lines.*

18. A geodetic line (or curve) is a shortest curve on a surface; more accurately, the element of arc between two consecutive points of a geodetic line is a shortest arc on the surface. We are thus led to the fundamental property that at each point of the curve the osculating plane of the curve passes through the normal of the surface; in other words, any two consecutive arcs *PP',P'P"* are *in piano* with the normal at *P'.* Starting from a given point *P* on the surface, we have a singly infinite system of geodetics

proceeding along the surface in the direction of the several tangent ines at the point *P* ; and, if the direction *PP,* is given, the property gives a construction by successive elements of arc for the required geodetic line.

Considering the geodetic lines which proceed from a given point *P* of the surface, any particular geodetic line is or is not again intersected by the consecutive generating line; if it is thus inter­sected, the generating line is a shortest line on the surface up to, but not beyond, the point at which it is first intersected by the con­secutive generating line; if it is not intersected, it continues a shortest line for the whole course.

In the analytical theory both of geodetic lines and of the curves of curvature, and in other parts of the theory of surfaces, it is very convenient to consider the rectangular co-ordinates *x,* y, z of a point of the surface as given functions of two independent parameters *p, q;* the form of these functions of course determines the surface, since by the elimination of *p, q* from the three equations we obtain the equation in the co-ordinates *x, y, z.* We have for the geodetic lines a differential equation of the second order between *p* and *q∙,* the general solution contains two arbitrary constants, and is thus capable of representing the geodetic line which can be drawn from a given point in a given direction on the surface. In. the case of a quadric surface the solution involves hyperelliptic integrals of the first kind, depending on the square root of a sextic function.

*Curvilinear Co-ordinates.*

19. The expressions of the co-ordinates *x, y, z* in terms of *p, q* may contain a parameter *r,* and, if this is regarded as a given con­stant, these expressions will as before refer to a point on a given surface. But, if *p, q, r* are regarded as three independent para­meters *x, y, z* will be the co-ordinates of a point in space, deter­mined by means of the three parameters *p, q,* r; these parameters are said to be the curvilinear co-ordinates, or (in a generalized sense of the term) simply the co-ordinates of the point. We arrive other­wise at the notion by taking *p, q, r* each as a given function of *x, y. z;* say we have *p =f1(x, y,* z), *q =f1(x, y, z), r=fi(x, y,* z), which equations of course lead to expressions for *p, q, r* each as a function of *x, y, z.* The first equation determines a singly infinite set of sur­faces: for any given value of *p* we have a surface; and similarly the second and third equations determine each a singly infinite set of surfaces. If, to fix the ideas, f1, f2, *f3* are taken to denote each a rational and integral function of *x, y, z,* then two surfaces of the same set will not intersect each other, and through a given point of space there will pass one surface of each set; that is, the point will be determined as a point of intersection of three surfaces be­longing to the three sets respectively; moreover, the whole of space will be divided by the three sets of surfaces into a triply infinite system of elements, each of them being a parallelepiped.

*Orthotomic Surfaces; Parallel Surfaces.*

*20.* The three sets of surfaces may be such that the three surfaces through any point of space whatever intersect each other at right angles ; and they are in this case said to be orthotomic. The term curvilinear co-ordinates was almost appropriated *by* Lamé, to whom this theory is chiefly due, to the case in question: assuming that the equations *p=f1(x, y, z), q=f2(x, y, z), r=f3(x, y,* z) refer to a system of orthotomic surfaces, we have in the restricted sense *p, q, r* as the curvilinear co-ordinates of the point.

An interesting special case is that of confocal quadric surfaces. The general equation of a surface confocal with the ellipsoid ^2 ∕y2 g2 -yî g2

^ + i≡+c∙≡≈≡ 1 is +-⅛qT +^∑qT = 1; and, if in this

equation we consider *x, y, z* as given, we have for θ a cubic equation with three real roots *p, q, r,* and thus we have through the point three real surfaces, one an ellipsoid, one a hyperboloid of one sheet, and one a hyperboloid of two sheets.

21. The theory is connected with that of curves of curvature by Dupin's theorem. Thus in any system of orthotomic surfaces each surface of any one of the three sets is intersected by the surfaces of the other two sets in its curves of curvature.

22. No one of the three sets, of surfaces is altogether arbitrary: in the equation *p =f*1*(x, y,* z), *p* is not an arbitrary function of *x,* *y*, z, but it must satisfy a certain partial differential equation of the third order. Assuming that *p* has this value, we have *q=f2 (x, y, z)* and *r=f*2*(x, y,* z) determinate functions of *x,y,z* such that the three sets of surfaces form an orthotomic system.

23. Starting from a given surface, it has been seen (par. 16) that the normals along the curves of curvature form two systems of torses intersecting each other, and also the given surface, at right angles. But there are, intersecting the two systems of torses at right angles, not only the given surface, but a singly infinite system of surfaces. If at each point of the given surface we measure off along the normal one and the same distance at pleasure, then the locus of the points thus obtained is a surface cutting all the normals of the given surface at right angles, or, in other words, having the same normals as the given surface; and it is therefore a parallel surface to the given surface. Hence the singly infinite system of parallel surfaces and the two singly infinite systems of torses form together a set of orthotomic surfaces.

*The Minimal Surface.*

**24.** This is the surface of minimum area—more accurately, a surface such that, for any indefinitely small closed curve which can be drawn on it round any point, the area of the surface is less than it is for any other surface whatever through the closed curve. It at once follows that the surface at every point is concavo-convex; for, if at any point this was not the case, we could, by cutting the surface by a plane, describe round the point an indefinitely small closed plane curve, and the plane area within the closed curve would then be less than the area of the element of surface within the same curve. The condition leads to. a partial differential equation of the second order for the determination of the minimal surface: considering z as a function of *x, γ,* and writing as usual *p, q, r, s, t* for the first and the second differential coefficients of z in regard to *x, y* respectively, the equation (as first shown by Lagrange) is (1 + *q*2*)r — 2pqs* + (1 *+ p2)t* =0, or, as this may also be written, ~ . , ? , , + ~ -, $ , „ = 0. The general

*dy* √ ι+p2+g2 *dx* √ι+∕>2+g2

integral contains of course arbitrary functions, and, if we imagine these so determined that the surface may pass through a given closed curve, and if, moreover, there is but one minimal surface passing through that curve, we have the solution of the problem of finding the surface of minimum area within the same curve. The surface continued beyond the closed curve is a minimal surface, but it is not of necessity or in general a surface of minimum area for an arbitrary bounding curve not wholly included within the given closed curve. It is hardly necessary to remark that the plane is a minimal surface, and that, if the. given closed curve is a plane curve, the plane is the proper solution; that is, the plane area within the given closed curve is less than the area for any other surface through the same curve. The given closed curve is not