The Hypercubic Manifold in Homotopy Type Theory

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Intro (1) : HoTT

Homotopy Type Theory

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- Vladimir Voevodsky : proposed the "Univalence Axiom"
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- "Univalent Foundations" as a foundational system for maths

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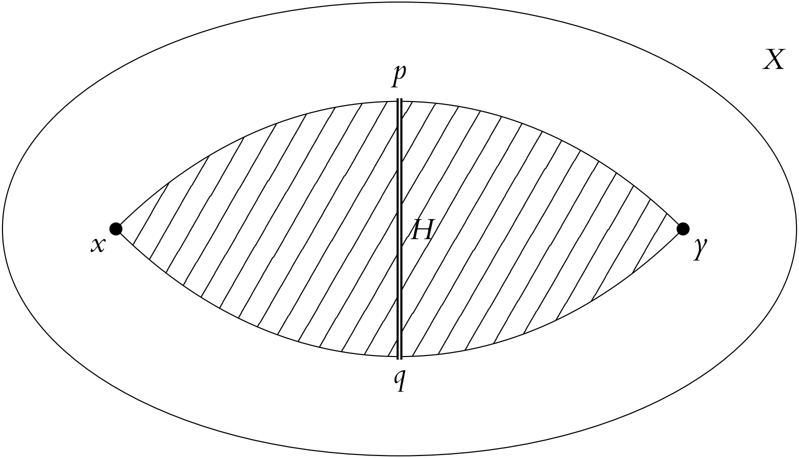
- Loops: a loop at point x is a path p such that $p(0) = p(1) = x$.

Intro (3) : Homotopy theory



Intro (4) : Homotopy theory

Paths operations

- inversion : given $p : x \to y$ there is a path $p^{-1} : y \to x$
- concatenation : given $p : x \to y$ and $q : y \to z$ there is a path $p \cdot q : x \to z$

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Fundamental group $\pi_1(X, x)$

Take a point x of X and consider loops at point x.

- Equality: homotopy (of loops)
- Composition : path concatenation (associative)
- Inverse element : inverse path
- Unit element : constant path at x

Two (path connected) homeormorphical spaces (more precisely, homotopically equivalent) have the same fundametal groups !

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- Given a finite group G is there a (path-connected) space X such that $\pi_1(X) = G$?

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- Is there a (path-connected) space such that $\pi_1(X) = G$ and $\pi_n(X) = 1$ for any $n \geq 2$? Answer : yes, the **Eilenberg Mac-Lane** space of G

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We manipulate types A, B and terms $a : A$ of some certain type.

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We distinguish between **definitional equality** $\equiv$ used when defining objects and **propositional equality** which is a type theoretical concept.

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Propositional Equality

Given a type A and terms $a, b : A$ there is a type $a =_A b$. We say that elements a and b are (propositionally) equal when there exists some element :

$$p : a =_A b$$

Intro (7) : What's the connection with Type Theory ?

Types	Topology
A	a space A
$a : A$	a is a point of A
$p : x =_A y$	p is a path from x to y in the space A

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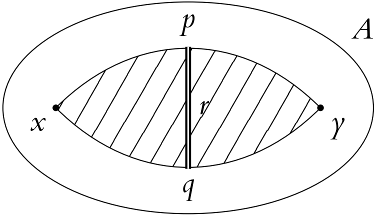

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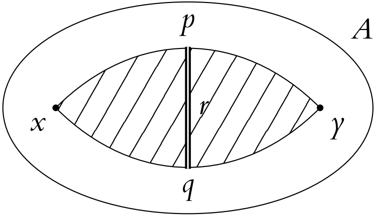

Synthetic homotopy

This connection is at the core of synthetic homotopy theory which allows us to define every object we previously talked about in the synthetic framework of type theory.

Intro (8) : The Hypercubic Manifold $\mathbb{HM}^1$

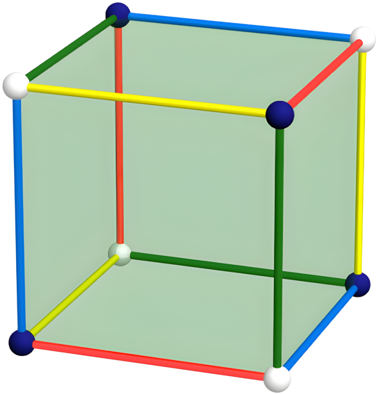

Figure: The hypercubic manifold $\mathbb{HM}^1$ (analysis-situs)

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- We know how to define Eilenberg Mac-Lane spaces for finite groups in HoTT but the construction is not well understood
- We would like to achieve a "nicer" construction that would yield a handy induction principle
- This would open up to define "Higher Group" actions and computer checked cohomology calculations.
- It turns out that $\pi_1(\mathbb{HM}^1) = Q$ and that by computing the "homotopy fiber" of a certain map my advisors have a way to provide such a "nice" construction in the case of the group Q.

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Some constructions on types

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Coproduct types

Give types A, B one has a coproduct type $A + B$ given by the constructors :

- $\mathrm{inl} : A \to A + B$
- $\mathrm{inr} : B \to A + B$

Manipulating types

Introduction rules

They encapsulate how to build an element of a certain type.

- To build an element of $f : A \to B$ one needs an expression $\phi(x)$ such that $a : A \vdash \phi(a) : B$ and to set $f :\equiv \lambda x.\phi(x)$
- To build an element of $A \times B$ one needs elements $a : A$ and $b : B$ to form $(a, b) : A \times B$

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Induction principles

They encapsulate how to build **dependent** functions from a source type A.

- To build a function of type $f : \prod_{x:A\times B} P(x)$ one only needs to give its value on pairs (a, b).

A working example : product types (1)

Projections

We can define projections $\mathrm{pr}_1 : A \times B \to A$ and $\mathrm{pr}_2 : A \times B \to B$ by the **induction** principle for product types by setting $\mathrm{pr}_1((a, b)) :\equiv a$ and $\mathrm{pr}_2((a, b)) :\equiv b$.

Identity types (1)

Definition

Given a type A and elements $a, b : A$ there exists a type $a =_A b$ called the identity type of a and b. We say that "a is equal to b" if there exists some element $p : a =_A b$

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Reflexivity

For any type A and point $x : A$ there is an element $\text{refl}_x : x =_A x$

Identity types (2)

Path induction

To prove a predicate that depends on $x, y : A$ and $p : x =_A y$ one only needs to prove it in the case where $x \equiv y$ and $p \equiv \mathrm{refl}$.

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Example (Functions preserve equals)

For any $f : A \to B$, $x, y : A$ and $p : a =_A b$ there is an element

$$f(p) : f(a) =_B f(b)$$

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- We then have to provide an element of $f(x) =_B f(x)$
- We conclude by setting $f(\mathrm{refl}_x) :\equiv \mathrm{refl}_{f(x)}$

$\square$

HoTT : The homotopical interpretation of Type Theory (1)

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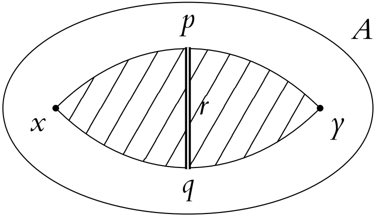

Figure: two points $x, y : A$, two paths $p, q : x =_A y$ and a homotopy $r : p =_{x =_A y} q$

HoTT : The homotopical interpretation of Type Theory (2)

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reflexivity	constant path
symmetry	path inversion
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- **Symmetry :** Given a path $p : x =_A y$, one has an inverse path $p^{-1} : y =_A x$ such that for any x, $\mathsf{refl}_x^{-1} \equiv \mathsf{refl}_x$.
- **Transitivity :** Given paths $p : x =_A y$ and $q : y =_A z$ one has a concatenated path $p \cdot q : x =_A z$ such that for any x, $\mathsf{refl}_x \cdot \mathsf{refl}_x \equiv \mathsf{refl}_x$.

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There are additional properties that one can prove such as

$$p \cdot p^{-1} = \mathrm{refl}$$

that justify this whole interpretation.

Transports

Equalities and dependent functions

Take $f : \prod_{x:A} P(x)$, $x, y : A$ and $p : x = y$.
Note that we can't have a path between $f(x) : P(x)$ and $f(y) : P(y)$.

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Transport

By path induction, for any x, y and $p : x = y$ one can define :

$$\mathrm{transport}^P : \prod_{a,b:A} \prod_{p:a=_A b} P(a) \to P(b)$$

We may write $\mathrm{transport}^P(a, b, p)$ as p_* if the context is clear.

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Example

The map p_* encapsulates how are transported elements of the "fiber" $P(x)$ to the fiber $P(y)$ by going along the path $p : x = y$.

Fibrations (1)

Fibration

Take a continuous map $p : E \to B$. It is called a fibration if it has a certain path lifting property from the **base space** B to the **total space** E.

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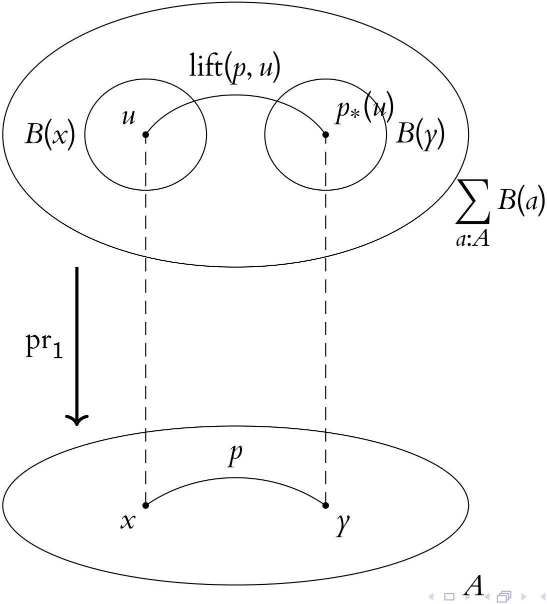
Then, the type $\sum_{x:A} B(x)$ equipped with its first projection pr_1 is a fibration.
More precisely :
Given $x : A$ and $u : P(x)$, for any $y : A$ and $p : x = y$, one has a path

$$\mathrm{lift}(p, u) : (x, u) = (y, p_*(u))$$

such that $\mathrm{pr}_1(\mathrm{lift}(p, u)) = p$.

Fibrations (2)



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Dependent functions yield such paths :

$$\mathrm{apd}_f : \prod_{p:x=_A y} p_*(f(x)) = f(y)$$

Equivalences and Univalence

Equivalences

We can build them from quasi-equivalences. Two types are quasi-equivalent if there exists maps $f : A \to B$ and $g : B \to A$ such that :

$$\prod_{x:A} g(f(x)) = x \quad \text{and} \quad \prod_{y:B} f(g(y)) = y$$

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Univalence

One has :

$$\left(A \simeq B\right) \simeq \left(A = B\right)$$

Higher Inductive Types

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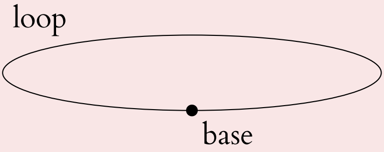
When dealing with HITs we allow ourselves to use paths constructors, paths between paths constructors and so on.

The circle $\mathbb{S}^1$

The circle $\mathbb{S}^1$ is defined by the following HIT :

- base : $\mathbb{S}^1$
- loop : base $=_{\mathbb{S}^1}$ base



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The case of $\mathbb{S}^1$

To define a map $f : \prod_{x:\mathbb{S}^1} P(x)$, one needs to specify its value $f(\text{base}) : P(\text{base})$. Then, what should you send loop to ?

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The case of $\mathbb{S}^1$

To define a map $f : \prod_{x:\mathbb{S}^1} P(x)$, one needs to specify its value $f(\text{base}) : P(\text{base})$. Then, what should you send loop to ?
Recall that by transport :

$$P(\text{base}) \overset{\text{loop}_*}{\simeq} P(\text{base})$$

Since f is a dependent function, it should send loop to a dependent path between $\text{loop}_*(f(\text{base}))$ and $f(\text{base})$.

Some other HITs (1)

Coequalizers

Given types B, A and maps $f, g : B \to A$ the (homotopy) **coequalizer** type $\mathrm{CoEq(f, g)}$ is given by the HIT:

- $c : A \to \mathrm{CoEq(f, g)}$
- $p : \prod_{b:B} c(f(b)) = c(g(b))$

$$\begin{tikzcd} B \arrow[r, shift left, "g"] \arrow[r, shift right, "f"'] & A \arrow[r, "c"] & \mathrm{CoEq}(f, g) \end{tikzcd}$$

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The circle as a coequalizer

$$\begin{tikzcd} \mathbf{1} \arrow[r, shift left, "1"] \arrow[r, shift right, "1"'] & \mathbf{1} \arrow[r, "c"] & \mathrm{CoEq}(1, 1) \simeq \mathbb{S}^1 \end{tikzcd}$$

Cubical Type Theory : Completing Squares

Filling property

Given an incomplete square of paths

$$\begin{tikzcd} a \arrow[r, "p"] \arrow[d, "q"'] & b \arrow[d, "r"] \\ c & d \end{tikzcd}$$

Then there is a unique path s from c to d that completes the square by making it commute, that is :

$$q \cdot s = p \cdot r$$

Why do we complete squares?

$$
\begin{tikzcd}
a_{0,1} \arrow[rr, "s"] & {} & a_{1,1} \\
a_{0,0} \arrow[u, "p \equiv c\ i_0"] \arrow[rr, "r"'] & {} \arrow[u, "c\ i"] & a_{1,0} \arrow[u, "q \equiv c\ i_1"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
i_0 \arrow[rr, bend right, "i"] & & i_1
\end{tikzcd}
$$

Some other HITs (2) : The torus $\mathbb{T}^2$

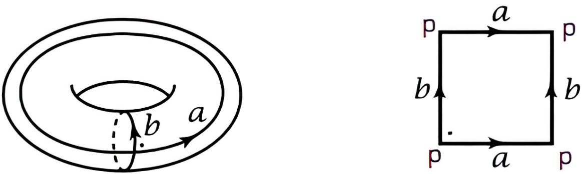

Figure: The torus $\mathbb{T}^2$ and its presentation as a cell complex (Hatcher)

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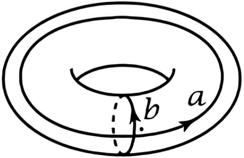

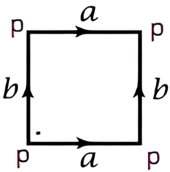

Figure: The torus $\mathbb{T}^2$ and its presentation as a cell complex (Hatcher)

This leads to the following HIT definition of $\mathbb{T}^2$:

- $p : \mathbb{T}^2$
- $a : p = p$
- $b : p = p$
- $\text{fill} : a \cdot b = b \cdot a$

Some other HITs (3) : The Torus $\mathbb{T}^2$

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```
data Torus2 : Type where
p : Torus2
a : p ≡ p
b : p ≡ p
fill : Square b b a a
```

Somer other HITs : Fundamental groups

Fundamental group presentation

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Given a presentation of G we can build a space H with $\pi_1(H) = G$.

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The Hypercubic Manifold $\mathbb{HM}^1$

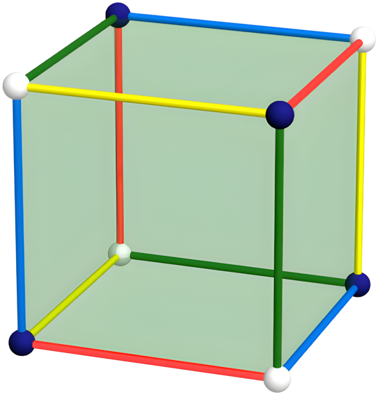

Figure: The hypercubic manifold $\mathbb{HM}^1$ (analysis-situs)

Defining $\mathbb{HM}^1$ (1)

The HIT/CW-complex correspondence

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Challenges

- Identifying opposite sides under a quater of a turn rotation
- Filling the cube

The Hypercubic Manifold $\mathbb{HM}^1$

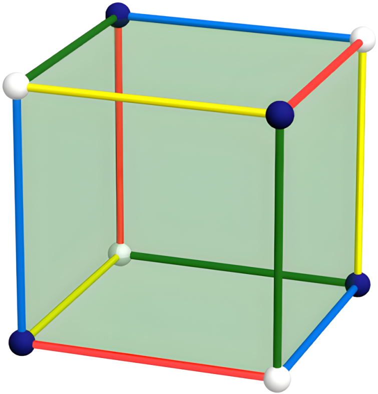

Figure: The hypercubic manifold $\mathbb{HM}^1$ (analysis-situs)

Defining $\mathbb{HM}^1$ (2) : Rotation

Begin with a filled square, that is a commutative diagram of paths :

$$\begin{tikzcd} \cdot \arrow[r, "s"] & \cdot \\ \cdot \arrow[u, "p"] \arrow[r, "r"'] & \cdot \arrow[u, "q"'] \end{tikzcd}$$

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We have just defined a map :

$$\boxed{\mathrm{rot} : \mathrm{Square}\ p\ q\ r\ s \to \mathrm{Square}\ \bar{r}\ \bar{s}\ q\ p}$$

Defining $\mathbb{HM}^1$ (3) : Result

```
data Hypercubic : Type where
    blueV : Hypercubic
    whiteV : Hypercubic
    yellowE : whiteV ≡ blueV
    greenE : whiteV ≡ blueV
    redE : whiteV ≡ blueV
    blueE : whiteV ≡ blueV
    f1 : Square (sym yellowE) greenE (sym blueE) redE
    f3 : Square (sym yellowE) blueE (sym redE) greenE
    f5 : Square (sym blueE) greenE (sym redE) yellowE
    3-cell : Cube f1 (rot f1) f3 (anti-rot f3) f5 (rot f5)
```

Figure: Synthetic description of the hypercubic manifold as a cubical HIT in cubical agda

Homotopy fiber

Given a map $f : A \to B$ and $b : B$ the fiber of f over b is given by :

$$\mathrm{fib}_f(b) :\equiv \sum_{x:A} f(x) =_B b$$

Homotopy fibers

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The final goal of the internship was to compute such a fiber, it has not yet been computed but we will look at an instructive example instead.

A fiber calculation (1)

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Let now m be a nonnegative integer and $r \in \mathbb{Z}_m^\times$ (the integers mod m).
Then we have a short exact sequence of groups :

$$\mathbb{Z} \xrightarrow{\times m} \mathbb{Z} \xrightarrow{1 \mapsto \bar{r}} \mathbb{Z}_m$$

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Which, for topological reasons, yields a fibration of Eilenberg Mac-Lane spaces:

$$\mathbb{S}^1 \xrightarrow{\text{base} \mapsto \text{base}, \quad \text{loop} \mapsto \text{loop}^m} \mathbb{S}^1 \xrightarrow{g} B\mathbb{Z}_m$$

This is in fact a fibration whose base space is $B\mathbb{Z}_m$, of total space $\mathbb{S}^1$, and for which the fiber of g over the canonical element $*$ of $B\mathbb{Z}_m$ should be $\mathbb{S}^1$.

A fiber calculation (2)

By definition, the fundamental group at $*$ in $\mathrm{B}\mathbb{Z}_m$ is $\mathbb{Z}_m$, we choose a presentation of it as the powers of s_m the succesor mod m. Then the map g is given by :

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We can now begin !

A fiber calculation (3) : diagrams

If $P : \mathrm{B}\mathbb{Z}_m \to \mathcal{U}$ is defined by $P :\equiv x \mapsto x = s_m$ then we have a fibration over $\mathbb{S}^1$ given by $P \circ g$ whose total space is by definition:

$$\sum_{x:\mathbb{S}^1} P \circ g(x) :\equiv \sum_{x:\mathbb{S}^1} (g(x) = s_m) :\equiv \mathrm{fib}_g(s_m)$$

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Recalling that $\mathbb{S}^1$ is obtained as a coequalizer type, we obtain the following commutative diagram :

$$\begin{tikzcd}
1 \arrow[r, shift left] \arrow[r, shift right] & 1 \arrow[r, "\mathrm{base}"] \arrow[dr, "s_m = s_m"'] & \mathbb{S}^1 \arrow[r, "g"] \arrow[d, dashed, "P \circ g"] & \mathrm{B}\mathbb{Z}_m \arrow[dl, "P"] \\
 & & \mathcal{U} &
\end{tikzcd}$$

A fiber calculation (4) : The flattening lemma

Flattening Lemma

Suppose given a fibration over a coequalizer type:

$$B \underset{g}{\overset{f}{\rightrightarrows}} A \xrightarrow{c} W \xrightarrow{P} \mathcal{U}$$

Then, one has a coequalizer diagram between the total spaces:

$$\sum_{b:B} P \circ c \circ f(a) \underset{(b,x)\mapsto(f(b),x)}{\overset{(b,x)\mapsto(g(b),p_b^*(x))}{\rightrightarrows}} \sum_{a:A} P \circ c(a) \longrightarrow \sum_{w:W} P(w)$$

A fiber calculation (5)

In our case, the lemma yields the following coequalizer diagram :

$$
\begin{tikzcd}
s_m = s_m \arrow[r, shift left, "\mathrm{Id}"] \arrow[r, shift right, "p \mapsto p^r"'] & s_m = s_m \arrow[r] & \mathrm{fib}_g(s_m)
\end{tikzcd}
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We get that $\mathrm{fib}_g(s_m)$ is equivalent to the HIT W defined by the following constructors :

- $c : \mathrm{Fin}\ m \mapsto W$
- $p : \prod_{x:\mathrm{Fin}\ m}(c(x) = c(s_m^r(x)))$

A fiber calculation (6)

Since r is coprime to m, it turns out that s_m^r is of order m so the orbit of $c(0)$ under the action of the group $\langle s_m^r \rangle$ is the whole type W.

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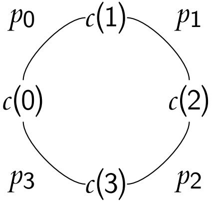

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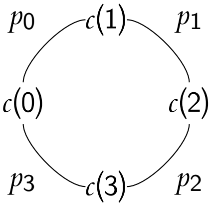

Hence :

$$\boxed{\mathrm{fib}_g(s_m) \simeq \mathbb{S}^1}$$

A fiber calculation (7) : sketch of proof

The idea is that the general case should come from the case $r = 1$.

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For $r = 1$, we have that :

$$p_0 \cdot p_1 \cdot \ldots \cdot p_m : c(0) = c(0)$$

and we can successively contract each path :

$$p_i : c(i) \longrightarrow c(\overline{i + 1}_r)$$

to end up with a type with a constructor $c(0)$ and a path $p_0 : c(0) = c(0)$, which is the circle.

A fiber calculation (8) : An extension

Let's consider the case where r and m are not coprime.
In that case, let δ be the gcd of r and m, then s_m^r is now only of order $\frac{m}{\delta}$.

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We then naturally obtain :

$$\boxed{\mathrm{fib}_g(s_m) = \coprod_{i=1}^{\delta} \mathbb{S}^1}$$

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Towards $\mathbb{HM}^1$ as ${}^{\mathbb{S}^3}\!/_{Q \curvearrowright \mathbb{S}^3}$

The final bit of the work would consist in an other fiber calculation that expresses $\mathbb{HM}^1$ as a homotopy quotient of $\mathbb{S}^3$ under an action of Q.

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