4.7 Hypercubic manifold

The *quaternion group* is

$$Q = \langle a, b \mid a^4 = 1, a^2 = b^2, aba = b \rangle$$

By Knuth-Bendix completion, we obtain the rewriting system with 7 rules

```
a,b |
A : aaaa -> "", B : bb -> aa, C : aba -> b, D : baaa -> ab,
E : baa -> aab, F : aaab -> ba, G : bab -> a
```

$$Q = \langle e, i, j, k \mid i^2 = e, j^2 = e, k^2 = e, ijk = e, e^2 = 1 \rangle$$

By Knuth-Bendix completion, we obtain the rewriting system with 24 rules

```
i,j,k,e |
A : ii -> e, B : jj -> e, C : kk -> e, D : ijk -> e, E : ee -> "",
F : ije -> ek, G : ke -> ek, H : je -> ej, I : ejk -> ie, J : ie -> ei,
K : jk -> i, L : eik -> j, M : eji -> k, N : ekjk -> eki, O : ij -> k,
P : kj -> ei, Q : ekik -> kj, R : ji -> ek, S : ekj -> i, T : ki -> j,
U : ik -> ej, V : eki -> ik, W : ekji -> e, X : ekik -> eif
```

We can deduce the relation $iji = i$ by

$$iji \to ki \to j$$

The correspondence between generators should be

$$\begin{array}{r|l} a & i \\ b & j \\ ab & k \\ abab & e \end{array}$$

We have a CW-complex called the *hypercubic manifold*[2] H^1 such that $\pi_0(H^1) = 0$, $\pi_1(H^1) = Q$ and $\pi_2(H^1) = 0$, but higher $\pi_n(H^1)$ are not right ($\pi_n(H^1) = \pi_n(S^3)$ for $n > 1$).

From the picture in analysis situs, we get the presentation

$$Q = \langle r, g, b, y \mid \bar{b}y\bar{g}r = 1, \bar{b}g\bar{r}y = 1, \bar{y}g\bar{b}r = 1 \rangle$$

(orienting the 1-generators as white $\to$ blue)

And thus

$$\begin{aligned} Q &= \langle r, g, b, y \mid by\bar{g}r = 1, bg\bar{r}y = 1, \bar{y}gbr = 1 \rangle && \text{rename } \bar{b} \to b \\ &= \langle r, g, b, y \mid by\bar{g}r = 1, bg\bar{r}y = 1, gbr = y \rangle \\ &= \langle r, g, b, y \mid by\overline{gr} = 1, bgry = 1, gb\bar{r} = y \rangle && \text{rename } \bar{r} \to r \\ &= \langle r, g, b, y \mid by = rg, bgry = 1, gb = yr \rangle \end{aligned}$$

By KB, we obtain the rewriting system with 13 rules

[2] https://analysis-situs.math.cnrs.fr/La-variete-hypercubique.html

```
r,g,b,y |
A : rg -> by, B : bgry -> "", C : yr -> gb, D : yby -> gbg, E : bgbyb -> r,
F : bgbgbg -> ry, G : ybgbg -> gbgby, H : ybgb -> gbgr, I : gbybg -> y,
J : ybgr -> gryb, K : gbgbgby -> ybg, L : rybg -> "", M : gbgbgr -> yb
```

We propose the correspondence

$$\begin{array}{c|l} i & bg \\ j & by \\ k & bgby \\ e & bgbg \end{array} \qquad \begin{array}{c|l} r & ek \\ g & kj \\ b & e \\ y & ik \end{array}$$

Writing Q_1, Q_2 and Q_3 for the three presentations. The above gives morphisms $f : Q_2 \to Q_3$ and $g : Q_3 \to Q_2$ with $g \circ f$ being the identity on Q_2 but the other way round is problematic:

$$f(g(b)) = f(e) = bgbg \neq b$$

This is not correct!! We should contract one of the two 0-cells using a 1-cell. We remove y and obtain

$$Q = \langle r, g, b \mid b = rg, bgr = 1, gb = r \rangle$$

```
r,g,b |
A : rg -> b, B : bgr -> "", C : gb -> r, D : br -> g, E : rr -> bb,
F : rbb -> bg, G : bbg -> rb, H : bbb -> gr, I : gg -> bb, J : grb -> "",
K : grbg -> g, L : rbg -> grb, M : rbg -> ""
```

and the correspondence should be

$$\begin{array}{c|l} i & bbb \\ j & g \\ k & r \\ e & bb \end{array} \qquad \begin{array}{c|l} r & k \\ g & j \\ b & ei \end{array}$$