The Hypercubic Manifold in Homotopy Type Theory

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Propositional Equality

Given a type A and terms a, b : A there is a type $a =_A b$. We say that elements a and b are (propositionally) equal when $a =_A b$ is inhabited.

Some constructions on types

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Coproduct type

Give types A, B one has a coproduct type A + B given by the constructors :

- inl : $A \rightarrow A + B$
- inr : $B \rightarrow A + B$

Manipulating types

Introduction rules

They encapsulate how to build an element of a certain type.

- To build an element of $f: A \to B$ one needs an expression $\phi(x)$ such that $a: A \vdash \phi(a): B$ and to set $f:= \lambda x.\phi(x)$
- To build an element of $A \times B$ one needs to take elements a : A and $b: B \text{ to form } (a, b): A \times B$

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- To build an element of $f: A \to B$ one needs an expression $\phi(x)$ such that $a: A \vdash \phi(a): B$ and to set $f: \equiv \lambda x.\phi(x)$
- To build an element of A × B one needs to take elements a: A and
 b: B to form (a, b): A × B

Induction principles

They encapsulate how to build **dependent** functions from a source type A.

• To build a function of type $f: \prod_{x:A\times B} P(x)$ one only needs to give its value on pairs (a, b).

A working example : product types (1)

Projections

We can define projections $\operatorname{pr}_1: A \times B \to A$ and $\operatorname{pr}_2: A \times B \to B$ by the **induction** principle for product types by setting $\operatorname{pr}_1((a,b)) :\equiv a$ and $\operatorname{pr}_2((a,b)) :\equiv b$.

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Propositional uniqueness for product types

We have a type theoretic statement that expresses that the type $A \times B$ is the type of "pairs of elements of A and B":

$$\operatorname{uniq}_{A \times B} : \prod_{x: A \times B} \left(x =_{A \times B} \left(\operatorname{pr}_{1}(x), \operatorname{pr}_{2}(x) \right) \right)$$

A working example : product types (2)

Sketch of proof:

• By induction, we only need to build an element :

$$\operatorname{uniq}_{A \times B}((a, b)) : (a, b) = (\operatorname{pr}_1(a, b), \operatorname{pr}_2(a, b))$$

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• We now have an element refl : $(a, b) =_{A \times B} (a, b)$