Physics 718: White dwarfs, neutron stars, black holes and related topics in relativistic astrophysics

Lecture I Preliminaries

Course Description:

Texts:
Black Holes, White Dwarfs, and Neutron Stars: The Physics of Compact Objects,
by Shapiro and Teukolsky [S-T]
We will also use a set of class notes and
The Physics of Stars, by A. C. Phillips.

Additional books that might be useful:

Accretion Power in Astrophysics, by Frank, King, & Raine

High Energy Astrophysics, by Malcolm S. Longair

An Introduction to Modern Astrophysics by Carroll & Ostlie

Rotating Relativistic Stars by Friedman & Stergioulas

Gravitation (Misner, Thorne & Wheeler), *General Relativity* (Wald), *A First Course in General Relativity* (Schutz), or *Gravitation & Cosmology* (Weinberg)

In addition, much material will be related to the modern astrophysical literature. You will have to read papers. The normal place to start is ADS:
`http://adsabs.harvard.edu/abstract_service.html`.
You can search by author, title, keyword, object, etc.

After you find a paper you can click on the links to read it. Note that some papers are published in journals that you cannot read from home. Most modern papers, though, are also listed at
`http://arxiv.org`
where you can read them for free. [Note that most papers will use cgs units. Be careful!]

Evaluation will be:

- Biweekly problem sets (70%), with the lowest grade dropped
- Final Project (30%). The final project will entail a 1-page project proposal that is due on March 11 that must include expected sources, an in-class oral presentation (20 minutes), and a final written report.

I.1 Precision

We often do not know things very precisely. So we use $\sim$ and $\approx$ and related symbols. $\sim$ is for when we know something to *an order of magnitude*. So we if we know that $x \sim 5$, we know that x is between $5/3$ and $5 * 3$, where 3 is roughly $\sqrt{10}$. This means that the possible range for x is in total a factor of 10. We will also sometimes use $\sim$ to mean *scales as*. For example, if you were to estimate the height of a person as a function of their weight (for a wide range of people), you might expect that as you double the weight, the height changes by $2^{1/3}$. We could write height$\sim$ weight$^{1/3}$. There will be a lot of variation, but this is roughly correct.

$\approx$ means more precision. It doesn't necessarily have an exact definition. But generally, if we say $x \approx 5$, that means that 4 is probably OK but 2 is probably not.

Finally, we have $\propto$, which means *proportional to*. This is more precise that the *scales as* use of $\sim$. So while for a person height$\sim$ weight$^{1/3}$ is OK, for a sphere (where we know that volume is $4\pi/3r^3$) we could write volume$\propto r^3$: we take this as correct, but leave off the constants ($4\pi/3$ in this case).

I.2 Small Angles

For small angles θ, $\sin\theta \approx \tan\theta \approx \theta$ and $\cos\theta \approx 1$. We need θ to be in radians. But we also often deal with fractions of a circle. A circle has $360°$. We break each degree into 60 minute (or *arcminutes*): $1° = 60'$. And each arcminute into 60 seconds (or *arcseconds*): $1' = 60''$, so $1° = 3600''$. But we also know that 2π radians is $360°$, so we can convert between radians and arcsec. This will come up frequently: $1'' = 360 \times 3600/2\pi \approx 1/206265$ radians.

I.3 Units and Celestial Constants

Astronomy emphasizes *natural* units ($\odot$ is for the Sun, $\oplus$ is for the Earth). We also use cgs units.

- $M_\odot = 2 \times 10^{33}$ g (solar mass)
- $R_\odot = 7 \times 10^{10}$ cm (solar radius)
- $M_\oplus = 6 \times 10^{27}\,\mathrm{g} \approx 3 \times 10^{-6}\,M_\odot$(earth mass)
- $M_\mathrm{J} = 2 \times 10^{30}\,\mathrm{g} \approx 10^{-3}\,M_\odot$(Jupiter)
- $L_\odot = 4 \times 10^{33}$ erg/s (solar luminosity or power)
- light year $= 10^{18}$ cm: the *distance* light travels in one year (moving at $c = 3 \times 10^{10}\,\mathrm{cm\,s^{-1}}$)
- Astronomical Unit = AU = 1.5×10^{13} cm (distance between earth and sun)
- parsec = parallax second (we will understand this later) = pc = $3 \times 10^{18}\,\mathrm{cm} = 206,265\,\mathrm{AU}$

- energies: eV=electron volt=1.6×10^{-12} erg (typical chemical reaction is eV; typical nuclear reaction is MeV)
- temperatures: often express as $k_B T$, where $k_B = 1.4 \times 10^{-16}$ erg/K is Boltzmann's constant. $k_B T$ is an energy, can express in eV; 10^6 K is 86 eV
- Masses often expressed as energies (also in eV) via $E = mc^2$, so:
  - $m_e = 511\,\mathrm{keV}$ (electron)
  - $m_n \approx m_p \approx 1\,\mathrm{GeV}$ (neutron or proton)
  - $m_\gamma = 0$ (photon — rest mass)

And then we use usual metric-style prefixes to get things like kpc, Mpc, etc.

Google/Wolfram Alpha/astropy can be very helpful when checking unit conversions.

Lecture II Basic Concepts

Phillips Chapter 1. These are (in many cases) things we will go back over later.

II.2 Big Bang Nucleosynthesis

What ingredients do we have to make a star? Universe is mostly H, then He. Rest is details. How did it get that way?

It started out very hot (we know this since the Universe is expanding and we see the left-over radiation at 3 K now). Was a soup of interacting sub-atomic particles (electrons, positrons, neutrinos, quarks). Eventually (after 10^{-4} s) free quarks got bound up into neutrons, protons, ... Neutrons and protons in particular were in equilibrium:

$$\nu_e + n \to e^- + p$$

and

$$\bar{\nu}_e + p \to e^+ + n$$

at the same time. However, n is slightly more massive than p. At a temperature T, the ratio of these is given by the mass (energy) difference:

$$\frac{N_n}{N_p} = e^{-\Delta mc^2/k_B T}$$

This is a **Boltzmann factor** (will come back). Δmc^2 is energy difference, 1.3 MeV (remember that $m_p c^2 \approx 1$ GeV, so difference is 0.1%).

As T goes down and universe expands, the reactions go more slowly and we get more protons wrt neutrons. Finally it effectively stopped, and the ratio was frozen. This happened at $T \sim 10^{10}$ K, with $N_n/N_p \approx 1/5$. Then went down a little more (to 1/7) through natural decay of n.

At 10^9 K, could make deuteron:

$$n + p \to d + \gamma$$

From these could make ^{3}He, then ^{4}He. ^{4}He is very stable, so a lot of things got stuck there, except for a little ^{7}Li. But there were still a lot of protons left over. How much He?

$N_n/N_p \approx 1/7$. So take 2 neutrons, 14 protons (16 particles total, or a mass of ≈ 16 amu). Make a single ^{4}He nucleus, then 12 protons left. So out of 16 amu, 4 amu are in ^{4}He, or mass of He is $\approx 25\%$ total mass. This is pretty close to what we see.

II.3 Gravitational Contraction

Stars are one big fight against gravity. Temporary relief from thermonuclear fusion. But what are they fighting against?

Spherical system with M, R. Only have pressure, gravity. density is $\rho(r)$, pressure is $P(r)$.

Start at the center. How much mass out to r?

$$m(r) = \int_0^r dr' \rho(r') 4\pi r'^2$$

($dr'\rho(r')4\pi r'^2$ is the mass of a shell at r'). Gravity only cares about enclosed mass, so:

$$g(r) = \frac{Gm(r)}{r^2}$$

What about pressure? Pressure on a parcel between r and $r + \Delta r$ (area=ΔA, volume=$\Delta r \Delta A$). If pressure at the top is the same as the bottom, no net force. But what if it is not the same?

$$P(r + \Delta r) \approx P(r) + \frac{dP}{dr}\Delta r$$

so difference (top $-$ bottom, or inward) in force (pressure times area) is:

$$\left[P(r) + \frac{dP}{dr}\Delta r - P(r)\right]\Delta A = \frac{dP}{dr}\Delta r \Delta A$$

But acceleration is force / mass, and mass is volume times density ($\Delta M = \rho(r)\Delta r \Delta A$). So acceleration from pressure is:

$$\frac{dP}{dr}\frac{1}{\rho(r)}$$

The total acceleration is then:

$$\frac{d^2 r}{dt^2} = -g(r) - \frac{1}{\rho(r)}\frac{dP}{dr}$$

So if the star isn't moving, then P must increase toward the center ($dP/dr < 0$).

II.3.1 Free Fall

What if $P = 0$? Deal with energies, not acceleration. Convert potential energy to kinetic. Start at r_0, mass enclosed m_0. Initial $K = 0$, $U = -Gm_0^2/r_0$. $K + U$ is always the same, and $K = m_0 v^2/2 = m_0(dr/dt)^2/2$. So:

$$\frac{1}{2}\left(\frac{dr}{dt}\right)^2 - \frac{Gm_0}{r} = \frac{-Gm_0}{r_0}$$

Can get the time to go all the way to the center ($r = 0$):

$$t_{\mathrm{FF}} = \int_{r_0}^{0} dr \frac{dt}{dr} = -\int_{r_0}^{0} dr \left[\frac{2Gm_0}{r} - \frac{2Gm_0}{r_0}\right]^{-1/2}$$

The integral is a little messy, but you can show that the free-fall time $t_{\mathrm{FF}}$is just:

$$\frac{\pi}{2}\left(\frac{r_0^3}{2Gm_0}\right)^{1/2}$$

Only depends on m_0/r_0^3. What has these units? $\rho = m_0/(4\pi r_0^3/3)$! So

$$t_{\mathrm{FF}} = \sqrt{\frac{3\pi}{32G\rho}}$$

For the Sun, $1/2$ hour. But for most things, eventually Pressure will stop collapse.

II.3.2 Hydrostatic Equilibrium

Assume 0 acceleration. Then:

$$\frac{dP}{dr} = -\frac{Gm(r)\rho(r)}{r^2} = -\rho(r)g(r)$$

This is a very important result: the equation of **hydrostatic equilibrium** (HSE). Applies to any stable system (atmospheres, stars, etc).

If the whole thing is in equilibrium at all r, then this will be true everywhere. Can then look at total potential energy:

$$\int_0^R dr\, 4\pi r^3 \frac{dP}{dr} = -\int_0^R dr\, \frac{Gm(r)\rho(r)4\pi r^3}{r^2}$$

where we multiplied both sides by $4\pi r^3$ and integrated. RHS is:

$$U_{\mathrm{G}} = -\int_0^M dm\, \frac{Gm(r)}{r}$$

where $dm = 4\pi r^2 \rho(r) dr$. Integrate LHS by parts:

$$P(r)4\pi r^3|_0^R - 3\int_0^R dr\, 4\pi r^2 P(r)$$

The first term is 0 ($P(R) = 0$). Second is average P times V: $\langle P \rangle V$. So:

$$\langle P \rangle = -\frac{U_G}{3V}$$

This is a very important result — one way of expressing the **virial theorem**. Can work for lots of things. What about particles in a box?

II.3.3 Kinetic Origin of Pressure

Box with side L has N particles. Particle hits top/bottom at a rate $v_z/2L$ (collisions/s) and imparts $2p_z$ (redirects with equal velocity). So momentum per time per area is $2p_z v_z/2L/L^2 = p_z v_z/L^3$. But momentum per time is force, and force per area is pressure. Total of N particles:

$$P = \frac{N}{L^3}\langle p_z v_z \rangle$$

Assume all directions are the same, so:

$$\langle p_x v_x \rangle = \langle p_y v_y \rangle = \langle p_z v_z \rangle = \langle \vec{p} \cdot \vec{v} \rangle/3 = \frac{n}{3}\langle \vec{p} \cdot \vec{v} \rangle/3$$

n is N/V or number density.

Can then do this for different types of particles. Total energy of a particle $\epsilon^2 = p^2c^2 + m^2c^4$ (kinetic + rest-mass). NR: $p \ll mc$, so $\epsilon = mc^2 + p^2/2m$. UR: $p \gg mc$ so $\epsilon = pc$. Can show:

$$P_{NR} = \frac{2}{3}\frac{K}{V}$$

$$P_{UR} = \frac{1}{3}\frac{K}{V}$$

So if NR:

$$\langle P \rangle = \frac{2}{3}\frac{K}{V} = -\frac{U_G}{3V}$$

or $U_G = -2K$. Other ways to write this are ($E = U + K$): $E = -K$, $E = U/2$. Overall $E < 0$ so the system is bound.

Strange consequence: add a little energy slowly. Add 1% of total energy, U goes down by 2%, K goes up by 1%. So for a contracting cloud converting gravitational energy to radiation (early model for the Sun) will get hotter (K goes up) as it contracts.

What if energy comes from nuclear reactions at the center? E goes up, so K goes down. Therefore it cools down! Adding energy makes it cooler?

II.4 Ideal Gas Law

You may know from chemistry:

$$PV = n_{\text{mol}}RT$$

with n_{mol} the # of moles, and R the ideal gas constant. But we are physicists. So the total number of molecules is $N_{\text{molec}} = N_A n_{\text{mol}}$, and we can write:

$$PV = N_{\text{molec}}\frac{R}{N_A}T$$

Divide both sides by V:

$$P = \frac{N_{\text{molec}}}{V}\frac{R}{N_A}T$$

Have $k_B = R/N_A$ is Boltzmann's constant. And:

$$n \equiv \frac{N_{\text{molec}}}{V}$$

is the **number density**: the number of particles per volume (units are cm^{-3}, since number doesn't have a unit).

$$P = nk_BT$$

Can also write in terms of **mass density** ρ ($\text{g}\,\text{cm}^{-3}$):

$$\rho \equiv m_{\text{molec}}n$$

so

$$P = \frac{\rho}{m_{\text{molec}}}k_BT$$

II.5 Star Formation

Cloud collapses under influence of gravity. Details complicated. But basic conditions must be satisfied. Gravity must be stronger than pressure (kinetic energy).

$$U = -f\frac{GM^2}{R}$$

(f depends on density distribution, $f \sim 1$).

$$K = \frac{3}{2}Nk_BT$$

Need $|U| > K$ for collapse. Can write this as:

$$M > M_J = \frac{3k_BT}{2G\bar{m}}R$$

where $\bar{m}$ is average mass of particle ($M = N\bar{m}$). Or

$$\rho > \rho_J = \frac{3}{4\pi M^2}\left(\frac{3k_BT}{2G\bar{m}}\right)^3$$

These are the **Jeans mass and density**.

So want a big cloud to collapse. But does a big cloud make a big star? Generally it breaks up along the way (*fragmentation*).

$T = 20\,\mathrm{K}$, $M = 10^3 M_\odot$, needs $\rho = 10^{-25}\,\mathrm{g/cm^3}$ ($n = 0.1\,cm^{-3}$) to collapse (not too bad). But for $1M_\odot$ density needs to be 10^6 times higher.

II.6 The Sun

$M = 1M_\odot$, $R = 1R_\odot$. So average density is $1.4 \times 10^0\,\mathrm{g\,cm^{-3}}$. $t_{\mathrm{FF}} = 30\,\mathrm{min}$, which isn't happening, so there must be pressure.

$$\langle P\rangle = \frac{-U}{3V} \approx \frac{1}{3}\frac{GM_\odot^2}{R_\odot}\frac{3}{4\pi R_\odot^3} = \frac{GM_\odot^2}{4\pi R_\odot^4} \approx 10^{14}\,\mathrm{Pa}$$

Can also say $\langle P\rangle = \langle\rho\rangle k_BT/\bar{m}$ (ideal gas law). $\bar{m} \approx 0.5\,\mathrm{amu}$ (ionized H). So

$$k_BT \approx \frac{GM_\odot\bar{m}}{3R_\odot} \approx 0.5\,\mathrm{keV}$$

or $T \approx 6 \times 10^6\,\mathrm{K}$. Hotter (and denser etc.) toward center.

II.6.1 What Powers the Sun and How Long Will It Last?

We take the Solar luminosity to be 4×10^{33} erg/s, and try to find a way to get that amount of energy out over a long time.

The first estimate was due to Lord Kelvin (1862, in Macmillan's Magazine). This estimate (known now at the Kelvin-Helmholtz time, $t_{\rm KH}$) was shown to be $< 100\,$Myr. But Darwin said (at the time) that fossils were at least 300 Myr old. So something weird was going on. Kelvin's estimate may have been wrong by a bit, but it couldn't be that bad. So there had to be some unknown energy source.

The lifespan of the Sun could be due to:

1. Chemical energy
2. Gravitational energy
3. Thermal energy (could it have just been a lot hotter in the past?)
4. Fission?

The answers for all of these are no. Kelvin's estimate concerned specifically gravitational. Chemical energy isn't enough, since we know about how much chemical energy a given reaction can release for a given amount of stuff. Same with fission.

II.6.2 Gravito-Thermal Collapse, or the Kelvin-Helmholtz Timescale

This ascribes the luminosity to the change in total energy: L is change in $E = K + U$.

If you do this you get a timescale of $t_{\rm KH} \sim 10^7$ yr, which is $\gg t_{\rm ff}$:

$$t_{\rm KH} \sim \frac{E}{L}$$

But $E \sim GM_\odot^2/R_\odot \sim 10^{48}$ erg.

That is because as collapse occurs, $|U|$ increases so K increases too. That heats up the star, which slows down the collapse.

We can use the Virial theorem to get the central temperature T_c of the Sun. We assume that the center (the hottest/densest bit) dominates K:

$$K \sim \frac{3}{2} k_B T_c \frac{M}{\bar{m}}$$

with $\bar{m} \approx m_p$ the average particle mass. And $K = -U/2$, with $U \sim -GM_\odot^2/R_\odot$. So we find $T_c \sim GM_\odot m_H/k_B R_\odot \sim 10^7$ K. This is pretty good (the real number is about 1.6×10^7 K).

II.6.3 Stellar Radiation

Stars are not quite blackbodies. But we describe them by their effective (surface) temperature $T_{\rm eff}$ and defined by

$$L = 4\pi R^2 \sigma T_{\rm eff}^2. \tag{II.1}$$

where R is the radius and L the luminosity. We also infer the luminosity from the flux F with

$$L = 4\pi d^2 F \tag{II.2}$$

(with d =distance). Note, though, that instead of fluxes and luminosities we often use magnitudes, with

$$m = -2.5 \log_{10} \frac{F}{F_0} \tag{II.3}$$

, comparing the observed flux against some reference flux F_0. Apparent magnitudes relate to fluxes, and absolute magnitudes to lumonisities (defined by apparent magnitude at a distance of 10 pc).

We can contrast "bolometric" quantities (integrated over all wavelengths) and actual observed quantities, which are defined over finite bandpasses. Then instead of measuring temperature directly we use "color" as a proxy:

$$B - V = m_B - m_V = -2.5 \log_{10} F_B/F_V \tag{II.4}$$

is the magnitude difference (flux ratio) between the B(lue)=4000 Å and V(isual)=5500 Å bands. So a hotter star would be bluer and would have $B - V < 0$. We put these together on a "color-magnitude" or Hertzprung-Russell (HR) diagram. Eventually people recognized different parts of the diagram: main sequence (where stars spend most of their lives burning H→He) with $L \sim M^{3.5}$, followed by giant branc(es) and white dwarfs.

For the Sun, $T_{\rm Eff} \approx 6000$ K. This means the blackbody peaks in the visible portion of the spectrum. And this is much cooler than the interior.

How does it get from very hot interior to cool exterior?

Center of the Sun: nuclear reaction releases energy in the form of neutrinos (which escape) and photons (gamma-rays). How long to get out? A naive answer is $\sim R_\odot/c = 2$ s. But not for photons.

It actually takes $\sim 10^7$ yrs. Why? Because a star is a very crowded place, and photons (even though they move fast) cannot move very far before they wack into something else and end up going in another direction. They easily bounce (scatter) off of ions, electrons, and atoms, and even other photons.

Each bounce tends to make the photon lose energy, but more photons are then produced, conserving energy. In the center the photons start out as X-ray photons, but by the time they get to the surface of the star they are optical photons. They get there via a *random walk*.

Assume that a photon will move (on average) a distance $l_{\rm mfp}$ before it hits something and changes direction. That distance is the *mean free path*. It travels a distance d after N collisions. We can determine what $d(N)$ is. Assume each one moves $\vec{l}_i$ for $i = 1 \dots N$, with $|\vec{l}_i| = l_{\text{mfp}}$. So the total distance is the vector sum:

$$\vec{d} = \sum^{N} \vec{l}_i$$

We want the magnitude of this, $|\vec{d}| = \sqrt{\vec{d} \cdot \vec{d}}$. But

$$\vec{d} \cdot \vec{d} = \sum_i^N \vec{l}_i \cdot \vec{l}_i + \sum_{i \neq j} \vec{l}_i \vec{l}_j$$

The second term there will go to 0 on average, since the directions are different. So $|\vec{d}|^2 = N|\vec{l}| = Nl_{\text{mfp}}$, or $d = \sqrt{N} l_{\text{mfp}}$. This is in fact a general result with applicability to a wide range of areas.

From this we can determine how long does it take for a photon to diffuse out of the star. To go a distance d, it takes:

$$\begin{array}{rl} \frac{d}{c} & l_{\text{mfp}} > d \\ N\frac{l_{\text{mfp}}}{c} = \frac{d^2}{l_{\text{mfp}} c} & l_{\text{mfp}} < d \end{array}$$

This is also often referred to as a "drunkard's walk". So to go $R_\odot$ it takes:

$$\frac{R_\odot^2}{lc}$$

which is a factor of $R_\odot/l$ longer than basic escape. So luminosity (energy per time) also changes by that factor. Naive luminosity for central temperature is:

$$L = 4\pi R_\odot^2 \sigma T_I^4$$

but in reality it is $T_{\text{Eff}} = 6000\,\text{K} \ll T_I = 6{,}000{,}000\,\text{K}$. So:

$$T_{\text{Eff}} \approx T_I \left(\frac{l}{R_\odot}\right)^{1/4}$$

which would give $l \sim 1\,\text{mm}$ (very small!). Which would give about 50,000 yr to diffuse (too small, but not horrible).

II.7 Stellar Life Cycles

Big Bang: mostly H and He. Stars make the rest. T at the center of a star is pretty close to constant, set by fusion (hotter $\to$ faster $\to$ bigger $\to$ cooler). So $M/R \sim$ constant.

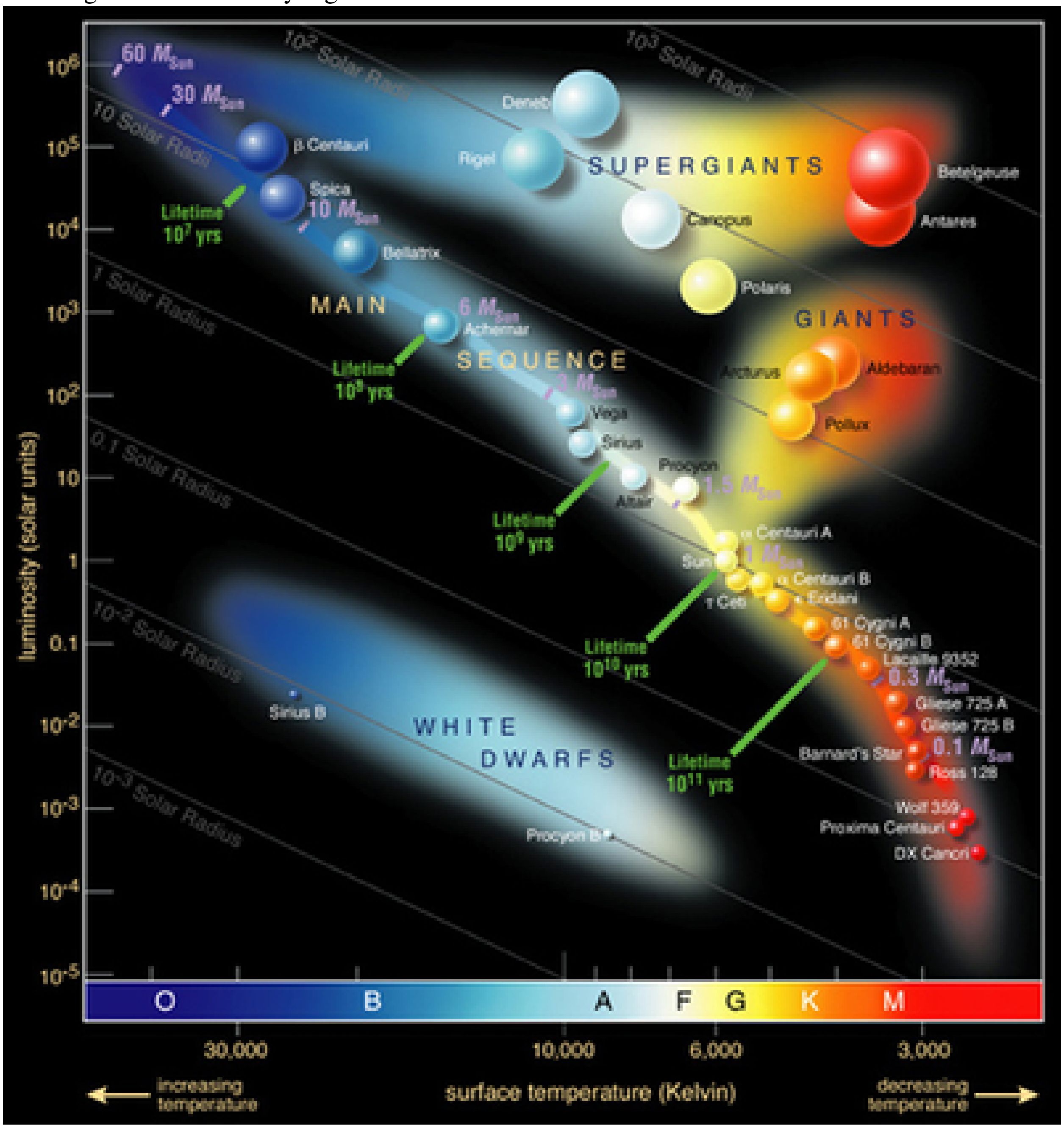
Energy escape determines luminosity. Since $L \sim R^2 T_I^4 (l/R) \sim R^2 (M/R)^4 (l/R) \sim \rho M^3$.

Since $L \sim M^3$ (roughly), determined by how fast energy can escape. So lifetime is $\sim M/L \sim M^{-2}$: bigger stars use up their fuel much faster. About 10^{10} yr for the Sun.

II.8 Color-Magnitude Diagram

Plot T_{Eff} increasing to the left, L increasing up. Hotter is the same as bluer, so often plot color (blue to the left) on the x-axis. We can directly observe color. And instead of L plot magnitude, where $m = m_\odot - 2.5\log_{10}(L/L_\odot)$. So it decreases going up, but that still means brighter.

Most of the stars define the **Main Sequence**. This turns out to be where normal H fusion is occuring. Can also identify regions for **Red Giants** and **White Dwarfs**.



Lecture III Fluids

(See, e.g., *The Feynman Lectures in Physics*, vol. II, Chap. 40, http://www.feynmanlectures.caltech.edu; and the Thorne-Blandford giant *Modern Classical Physics*, Chapter 13, http://www.pmaweb.caltech.edu/Courses/ph136/yr2012/)

In physics, the word *fluid* refers both to liquids and gases. More generally, what distinguishes a fluid from a solid is that a fluid cannot maintain a shear stress.[1] The stress is normal to any surface that is at rest relative to the fluid, and its magnitude is independent of the orientation of the surface. Suppose that inside a fluid one makes a small cut, say the vertical cut on shown in the figure below. The cut separates the matter on one side of a small plane from the matter on the other side. What force $\boldsymbol{F}$ is needed to keep the matter on the left side of that plane in the state it would have been in had there been no cut?

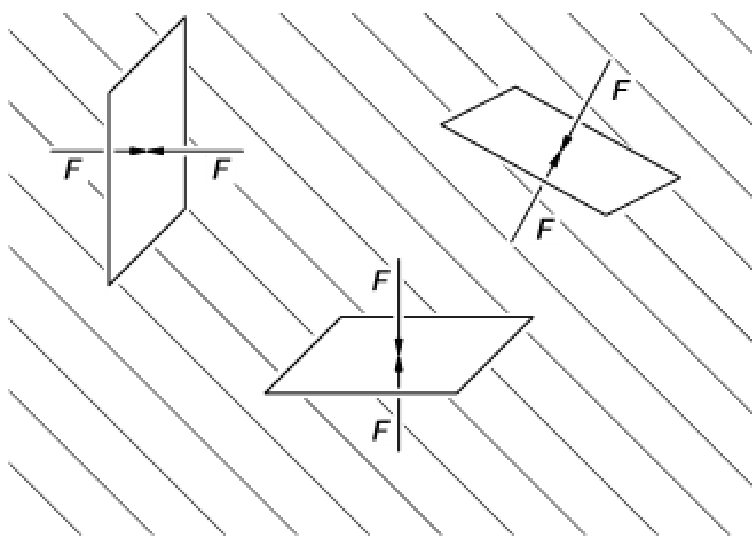


Figure 1: (This is Fig. 40-1 of the *Feynman Lectures*, vol. II)

This $\boldsymbol{F}$ is, of course, the same force that the right side was exerting on the left side just before the cut was made (and is equal and opposite to the force exerted by the left side side on the right side). Let $\boldsymbol{A} = \boldsymbol{n}A$, where A is the area of the cut and $\boldsymbol{n}$ is a unit normal pointing to the right. Then for small cuts, $\boldsymbol{F} \propto \boldsymbol{A}$ and its magnitude is independent of the orientation of the surface: $\boldsymbol{F} = P\boldsymbol{A}$, with P the pressure.

Solids, on the other hand, maintain shear stresses: The force needed to keep the matter on one side of a cut in place is not, in general, normal to the cut, and it depends on the orientation: For small cuts, the force is linear in the area, but the general linear map from an element of area to a force (from a vector to a vector) is a tensor,

$$F^a = T^a{}_b A^b. \tag{III.5}$$

[1]An *imperfect fluid*, a fluid with viscosity, is intermediate between a perfect fluid and a solid. We are assuming here that viscosity is negligible.

The map $T^a{}_b$ from areas to forces is is called the *stress tensor*. For a fluid, $T^a{}_b$ is independent of orientation – invariant under rotations, and the only 2-index tensors invariant under rotations are multiples of the identity:

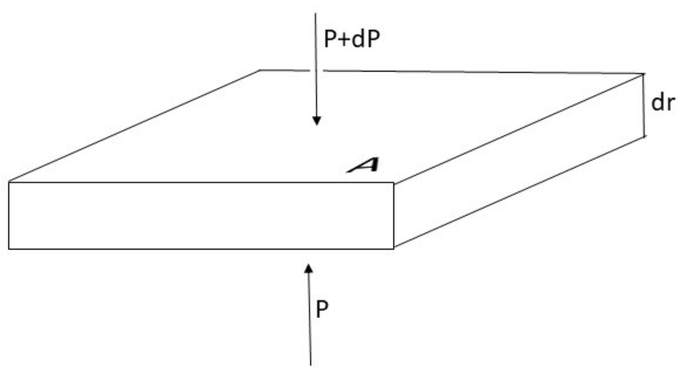
$$T^a{}_b = P\delta^a_b, \tag{III.6}$$

implying $F^a = PA^a$.

III.1 Hydrostatic Equilibrium of a Newtonian Star

A star is pulled together by gravity and supported against collapse by pressure: The pressure below a column of fluid supports the weight of the fluid above it in a spherical (nonrotating) star. This means that the pressure $P(r)$ at radius r is equal to the weight of a column of unit area extending from r to the surface of the star.

The structure of a star is determined by the balance of pressure and gravity on a slab of fluid (*fluid element*) of area A and thickness dr shown in the figure. Let M, R be the total mass and radius of the star, $\rho(r)$ the density of the fluid at radius r and $m(r)$ the mass inside the radius r. The pressure P at the bottom of the fluid element is larger than the pressure $P + dP$ at the top ($dP < 0$), and the difference between the force on the bottom and the force on the top is equal to the weight of the fluid.



The pressure P at radius r exerts a force $F_r = PA$ on the bottom of the fluid element; the smaller pressure $P + dP$ at radius $r + dr$ exerts a force $F_r = -(P + dP)A$ on the top. The buoyant force, the net force on the fluid element due to the pressure difference, is then

$$PA - (P + dP)A = -AdP.$$

The gravitational force on the mass dm of the fluid element cares only about the enclosed mass $m(r)$: $F_r = -g\,dm = -\dfrac{Gm(r)dm}{r^2}$.

The balance between the pressure gradient and gravity is then given by

$$\begin{aligned} -AdP = \frac{Gm(r)dm}{r^2} &= \frac{Gm(r)}{r^2}\rho A dr \\ \frac{dP}{dr} &= -\frac{Gm(r)}{r^2}\rho. \end{aligned} \tag{III.7}$$

This is the equation of hydrostatic equilibrium.

Summary:
Hydrostatic equilibrium of a star is governed by the equations (III.7),

$$m(r) = \int_0^r dr' \rho(r') 4\pi r'^2, \tag{III.8}$$

$$\frac{d\Phi}{dr} = -\frac{1}{\rho}\frac{dP}{dr}, \quad r \le R \tag{III.9}$$

and

$$\nabla^2\Phi = 4\pi G\rho, \quad \Phi \to 0 \text{ as } r \to \infty. \tag{III.10}$$

The last two equations, (III.9) and (III.10), are redundant inside the star.

If we know the equation of state, the relation $P = P(\rho)$ between pressure and density, we can simultaneously integrate Eqs. (III.7) and (III.8) to find the quantities $P(r)$, $\rho(r)$ and $m(r)$ that describe the structure of the star: For each central density, there will be one star, obtained by integrating the equations outward until reaching the radius at which the pressure vanishes – the surface of the star.

This is how one models white dwarfs and neutron stars (with the relativistic equation of hydrostatic equilibrium replacing the Newtonian approximation for neutron stars). For living stars, the pressure depends on temperature as well as density $P = P(\rho, T)$ and one must adjoin equations determining the temperature. These could include heat flow (radiative transfer), reaction rates, and densities of each nuclear species.

III.2 Hydrodynamics A: $\boldsymbol{F} = m\boldsymbol{a}$ for radial motion

If a spherical star is not static – if it is contracting, expanding or spherically oscillating, the sum of the forces due to pressure and gravity is mass× acceleration $= dm\, a_r = \rho\, Adr\, \dfrac{d^2r}{dt^2}$:

$$-\frac{dP}{dr}Adr - \frac{Gm(r)}{r^2}\rho\, Adr = \rho\, Adr\, \frac{d^2r}{dt^2}.$$

Then the motion is governed by the equations

$$\frac{d^2r}{dt^2} = -\frac{Gm(r)}{r^2} - \frac{1}{\rho(r)}\frac{dP}{dr}, \qquad m(r) = \int_0^r dr' \rho(r') 4\pi r'^2. \tag{III.11}$$

Free Fall

What if $P = 0$? For free fall from rest, the matter initially inside a shell at r_0 stays inside the contracting shell. This is not obvious, but we'll start by assuming it and then show that the resulting solution has that property.[2] For a particle initially at r_0, the mass enclosed is then always its intial

[2] This is enough to verify the assumption, because solutions to the dynamical equations with specified initial conditions are unique: That is, we find a solution, we are guaranteed that it is the unique solution, and it satifies the assumption. The assumption is therefore true.

value, $m(r(t)) = m(r_0) \equiv m_0$. This allows us to write conservation of energy E for a freely falling particle of mass m and initial radius r_0 in the form $E = K + U = \frac{1}{2}\mathrm{m}\dot{r}^2 - \frac{Gm_0\mathrm{m}}{r^2}$. At $t = 0$, we have $\dot{r} = 0$ and $r = r_0$, implying $E_0 = -Gm_0\mathrm{m}/r_0^2$. Then conservation of energy, $E/\mathrm{m} = E_0/\mathrm{m}$, has the form

$$\frac{1}{2}\left(\frac{dr}{dt}\right)^2 - \frac{Gm_0}{r} = -\frac{Gm_0}{r_0} \tag{III.12}$$

with solution

$$t = \int_{r_0}^{r} dr \frac{dt}{dr} = -\int_{r_0}^{r} dr \left[\frac{2Gm_0}{r} - \frac{2Gm_0}{r_0}\right]^{-1/2}. \tag{III.13}$$

This is simpler when written in terms of $x = r/r_0$, the fraction of the initial radius:

$$t = \sqrt{\frac{r_0^3}{2Gm_0}} \int_x^1 dx' \left[\frac{1}{x'} - 1\right]^{-1/2} \equiv f(x). \tag{III.14}$$

The time to reach the center, $r = 0$, is the free-fall time t_{FF}. Using $\displaystyle\int_0^1 dx \left[\frac{1}{x} - 1\right]^{-1/2} = \frac{\pi}{2}$, we have

$$t_{\mathrm{FF}} = \frac{\pi}{2}\sqrt{\frac{r_0^3}{2Gm_0}}, \tag{III.15}$$

a time that depends only on m_0/r_0^3. What is this? The average density inside r_0 at $t = 0$ is $\rho_0 = m_0/(4\pi r_0^3/3)$. So the free-fall time depends only on the initial density: With negligible pressure, a large or a small cloud with the same initial density takes the same time to collapse,

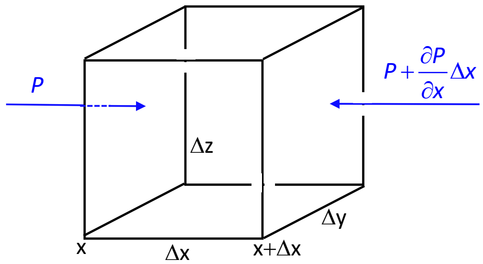
$$t_{\mathrm{FF}} = \sqrt{\frac{3\pi}{32G\rho_0}}. \tag{III.16}$$

Note that the relation $t \propto \rho^{-1/2}$ follows from dimensional analysis: $\frac{1}{\sqrt{G\rho}}$ is the only combination of G, m and r whose dimension is time. Similarly, the period of a particle in circular orbit near the surface of a star or planet, the period of radial oscillations, and the time a sound wave takes to cross a star are each of order $\frac{1}{\sqrt{G\rho}}$. For the Sun, this characteristic time is 1/2 hour.

Because the free-fall time for a shell decreases as the average density inside the shell increases, and the density is higher at smaller radii, our initial assumption is right: Matter inside a collapsing shell stays inside the shell. (Explictly, the integral (III.14) is $t = f(x) = \sqrt{\frac{3}{8\pi G\rho_0}}\left[\cos^{-1}\sqrt{x} + \sqrt{x(1-x)}\right]$ and can be found by the substitution $x = \cos^2\theta$.)

III.2.1 The Euler Equation for general motion

The Euler equation is the equation of motion, $\boldsymbol{F} = m\boldsymbol{a}$ for a fluid element. For radial motion, it is Eq. (III.11), and we'll now derive the 3-dimensional version. Again consider a fluid element, shown here as a small box of fluid with a density ρ and velocity $\mathbf{v}$.



The pressure on the left face is $P(x)$; the pressure on the right face is $P(x+\Delta x) = P(x)+\frac{\partial P}{\partial x}\Delta x$. With $A = \Delta y\Delta z$ the area of the left and right faces of the box, the net force in the x-direction is

$$\begin{aligned} F_x &= P(x)A - P(x+\Delta x)A \\ &= -\frac{\partial P}{\partial x}V. \end{aligned}$$

where $V = \Delta x\Delta y\Delta z$ is the volume of the fluid element. Replacing the index x by y and z, we have

$$\boldsymbol{F} = -\boldsymbol{\nabla} P\ V.$$

We want to write $\boldsymbol{F} = m\boldsymbol{a}$ or

$$-\boldsymbol{\nabla} P\ V = \rho V\ \boldsymbol{a},$$

and we need to find $\boldsymbol{a}$ in terms of the velocity field $\mathbf{v}(x,t)$. The vector field $\mathbf{v}(\mathbf{x},t)$ has the meaning that at time t the fluid element at $\mathbf{x}$ has velocity $\mathbf{v}(\mathbf{x},t)$. Thus at time $t+\Delta t$ that same fluid element is at $\mathbf{x}+\mathbf{v}(\mathbf{x},t)\Delta t$ and has velocity $\mathbf{v}(\mathbf{x}+\mathbf{v}(\mathbf{x},t)\Delta t, t+\Delta t)$. The fluid element has changed its velocity by

$$\begin{aligned} \Delta\mathbf{v} &= \mathbf{v}(\mathbf{x}+\mathbf{v}(\mathbf{x},t)\Delta t, t+\Delta t) - \mathbf{v}(\mathbf{x},t) \\ &= \left(\frac{\partial\mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v}\right)\Delta t \end{aligned}$$

in time Δt, and its acceleration is therefore

$$\boldsymbol{a} = (\partial_t + \mathbf{v}\cdot\nabla)\,\mathbf{v}. \tag{III.17}$$

In this way we obtain Euler's equation of motion ("Principes généraux du mouvement des fluides," Mémoires de l'Académie des Sciences de Berlin, 1757)

$$\rho\,(\partial_t + \mathbf{v}\cdot\boldsymbol{\nabla})\,\mathbf{v} = -\boldsymbol{\nabla} P\ . \tag{III.18}$$

In the presence of a gravitational field, with potential Φ satisfying $\nabla^2\Phi = 4\pi G\rho$, there is an additional force $-\rho V\ \boldsymbol{\nabla}\Phi$ on each fluid element; and the Euler equation becomes

$$\rho\,(\partial_t + \mathbf{v}\cdot\boldsymbol{\nabla})\,\mathbf{v} = -\boldsymbol{\nabla} P - \rho\boldsymbol{\nabla}\Phi. \tag{III.19}$$

III.2.2 Conservation of mass: The continuity equation

As a fluid element moves its volume changes. Because its mass is conserved (in our present Newtonian approximation) a fractional increase $\Delta V/V$ in its volume is equal to the fractional decrease $\Delta\rho/\rho$ in its density.

$$\rho V = \text{ constant } \Longrightarrow \frac{\Delta\rho}{\rho} = -\frac{\Delta V}{V} \quad \text{or} \quad \frac{d\rho/dt}{\rho} = -\frac{dV/dt}{V}, \tag{III.20}$$

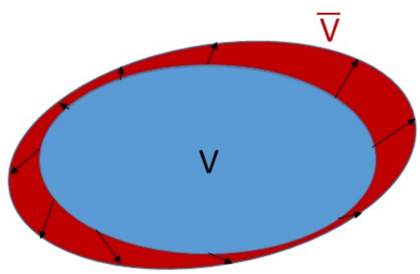
where $\rho = \rho(t, \mathbf{x}(t))$, $V = V(t, \mathbf{x}(t))$.

We'll begin with the change in the volume of the fluid element as it moves. It is helpful first to recall or notice the geometrical meaning of the divergence of a vector field. If each point in a volume V moves by a small amount $\boldsymbol{\xi}(x)$, from an initial position $\mathbf{x}$ to a final position $\bar{\mathbf{x}} = \mathbf{x} + \boldsymbol{\xi}$, the volume of the box changes by $\Delta V = \overline{V} - V = \nabla\cdot\boldsymbol{\xi}\, V$: That is, $\boldsymbol{\nabla}\cdot\xi$ is the fractional change in volume.

This is really Gauss's theorem: As illustrated by the figure below, moving V to $\overline{V}$ moves each point of the surface S of a volume V along $\boldsymbol{\xi}$, changing the the volume of the box by

$$\Delta V = \int_S \boldsymbol{\xi}\cdot d\boldsymbol{S}\,,$$

to lowest order in $\boldsymbol{\xi}$.



Gauss's theorem now implies

$$\Delta V = \int_V \boldsymbol{\nabla}\cdot\boldsymbol{\xi}\, dV. \tag{III.21}$$

For a small volume V the volume then changes by $\Delta V = V\boldsymbol{\nabla}\cdot\boldsymbol{\xi}$, or

$$\frac{\Delta V}{V} = \boldsymbol{\nabla}\cdot\boldsymbol{\xi}, \tag{III.22}$$

to lowest order in V and $\boldsymbol{\xi}$. That is, as claimed, $\nabla\cdot\boldsymbol{\xi}$ is the fractional change in volume.

Go back now to a fluid with a velocity field $\mathbf{v}(x,t)$. In a time Δt the fluid at $\mathbf{x}$ moves to $\mathbf{x}+\boldsymbol{\xi}$, where $\boldsymbol{\xi} = \mathbf{v}\Delta t$. Then, writing $\nabla\cdot\boldsymbol{\xi} = \nabla\cdot(\mathbf{v}\,\Delta t)$, we have

$$\frac{dV/dt}{V} = \boldsymbol{\nabla}\cdot\mathbf{v}. \tag{III.23}$$

The change in density is given by

$$\frac{d}{dt}\rho(t,\mathbf{x}(t)) = \partial_t\rho + \partial_i\rho\frac{dx_i}{dt} = (\partial_t + \mathbf{v}\cdot\boldsymbol{\nabla})\rho. \tag{III.24}$$

Finally, with these expressions for dV/dt and $d\rho/dt$, conservation of mass (III.20) is

$$\frac{1}{\rho}(\partial_t + \mathbf{v}\cdot\boldsymbol{\nabla})\rho = -\boldsymbol{\nabla}\cdot\mathbf{v},$$

or

$$\partial_t\rho + \nabla\cdot(\rho\mathbf{v}) = 0. \tag{III.25}$$

This is commonly called the *continuity equation.*

To summarize: A fluid is characterized by its pressure P, density ρ and 3-velocity $\mathbf{v}$. Its motion is governed by the equations

$$\nabla^2\Phi = 4\pi G\rho, \qquad \lim_{r\to\infty}\Phi = 0, \tag{III.26}$$

$$\partial_t\,\rho + \nabla\cdot(\rho\mathbf{v}) = 0,$$

$$\rho\,(\partial_t + \mathbf{v}\cdot\boldsymbol{\nabla})\,\mathbf{v} = -\boldsymbol{\nabla}P - \rho\boldsymbol{\nabla}\Phi. \tag{III.27}$$

The Euler equation in this form neglects viscosity and magnetic fields, and this is appropriate in computing the structure and oscillations of white dwarfs and neutron stars. Magnetic fields and effective viscosity from neutrino loss are important in collapse and mergers, and viscosity can also be important for stability of rapidly rotating neutron stars. For neutron stars, the Newtonian approximation is off by 10-15%, and we will use the relativistic version of the Euler equation to compute their structure.

Finally, the Newtonian potential Φ satisfies

III.3 Newtonian energy conservation

We'll begin with conservation of energy for a fluid element with no gravitational field. Including the field is simple for a time-independent field but needs a little discussion when the field is time-dependent, and we'll save that for the end.

The energy of a fluid element of mass M is a sum, $E = \frac{1}{2}Mv^2 + U$, of its kinetic energy and internal energy. For a fluid element of volume V, we have seen that conservation of mass written in terms of $\rho = M/V$ has the form (III.25). Denote by ε the internal energy density, $\varepsilon = U/V$ (we'll use u for the energy per unit mass, $u = E/M = \varepsilon/\rho$). Conservation of energy does not have the form $\partial_t\varepsilon + \nabla\cdot(\varepsilon\mathbf{v}) = 0$, because pressure does work as the fluid element moves. The change in internal energy per unit volume, $\partial_t\varepsilon + \nabla\cdot(\varepsilon\mathbf{v})$ is equal to the work done per unit volume per unit time:

$$\partial_t\varepsilon + \nabla\cdot(\varepsilon\mathbf{v}) = -P\boldsymbol{\nabla}\cdot\mathbf{v}, \tag{III.28}$$

where from Eq. (III.23), $\boldsymbol{\nabla}\cdot\mathbf{v}$ is the change in volume per unit volume per unit time. We are assuming that heat flow is negligible, that the entropy of each fluid element is conserved, and this relation is equivalent to the first law of thermodynamics, $dU = TdS - PdV$, when $dS = 0$. That is, dividing by V and writing $\dfrac{d}{dt} = (\partial_t + \mathbf{v}\cdot\boldsymbol{\nabla})$ for the change per unit time along the flow gives the first law in the form

$$\frac{1}{V}(\partial_t + \mathbf{v}\cdot\boldsymbol{\nabla})U = -\frac{P}{V}(\partial_t + \mathbf{v}\cdot\boldsymbol{\nabla})V, \tag{III.29}$$

and Eq. (III.23) then implies (III.28).

The second contribution to the change in the energy of a fluid element is the change in its kinetic energy. Recall that energy conservation for a mass m is obtained by manipulating $(\boldsymbol{F} = m\boldsymbol{a})\cdot\mathbf{v}$, using $m\boldsymbol{a}\cdot\mathbf{v} = \partial_t(\frac{1}{2}mv^2)$. Conservation of energy has the form $\partial_t(\frac{1}{2}mv^2) = \boldsymbol{F}\cdot\mathbf{v}$, equating the rate of change of kinetic energy to the power, the rate at which forces are doing work. So we multiply the Euler equation (III.19) by $\mathbf{v}$:

$$\begin{aligned} 0 &= v^a\rho\,(\partial_t + \mathbf{v}\cdot\boldsymbol{\nabla})\,v_a + v^a\nabla_a P \\ \text{1st term} &= \rho\,(\partial_t + \mathbf{v}\cdot\boldsymbol{\nabla})\frac{1}{2}v^2 = \partial_t(\frac{1}{2}\rho v^2) + \boldsymbol{\nabla}\cdot(\mathbf{v}\frac{1}{2}\rho v^2) \end{aligned} \tag{III.30}$$

where we have used conservation of mass, $\partial_t\rho + \boldsymbol{\nabla}\cdot(\rho\mathbf{v}) = 0$. With no pressure, no internal energy, and no gravitational field, we have energy conservation for dust:

$$\partial_t(\frac{1}{2}\rho v^2) + \boldsymbol{\nabla}\cdot(\mathbf{v}\frac{1}{2}\rho v^2) = 0.$$

In our case, the change in the sum of two contributions to energy satisfies

$$\partial_t\left(\frac{1}{2}\rho v^2 + \varepsilon\right) + \boldsymbol{\nabla}\cdot\left(\mathbf{v}\frac{1}{2}\rho v^2 + \mathbf{v}\varepsilon\right) = -P\boldsymbol{\nabla}\cdot\mathbf{v} - \mathbf{v}\cdot\boldsymbol{\nabla}P$$

or

$$\partial_t\left(\frac{1}{2}\rho v^2 + \varepsilon\right) + \boldsymbol{\nabla}\cdot\left[\rho\mathbf{v}\left(\frac{1}{2}v^2 + \frac{\varepsilon + P}{\rho}\right)\right] = 0.$$

The quantity $\dfrac{\varepsilon + P}{\rho}$ appearing in the energy flux term is the enthalpy per unit mass, written

$$h = \frac{\varepsilon + P}{\rho}, \tag{III.31}$$

and the conservation law is commonly written in the form

$$\partial_t\left[\rho\left(\frac{1}{2}v^2 + u\right)\right] + \boldsymbol{\nabla}\cdot\left[\rho\mathbf{v}\left(\frac{1}{2}v^2 + h\right)\right] = 0, \tag{III.32}$$

where, as mentioned above, $u = \varepsilon/\rho$ is the internal energy per unit mass.

When a time-independent gravitational field is present (e.g. for matter orbiting or falling onto a star or black hole, where one can ignore the field of that matter), the additional energy density of the fluid element is $\rho\Phi$, the additional term in the the Euler equation is $\rho\boldsymbol{\nabla}\Phi$, and our energy conservation equation becomes

$$\partial_t\left[\rho\left(\frac{1}{2}v^2 + u + \Phi\right)\right] + \boldsymbol{\nabla}\cdot\left[\rho\mathbf{v}\left(\frac{1}{2}v^2 + h + \Phi\right)\right] = 0. \tag{III.33}$$

This has the general form of a conserved current,

$$\partial_t \mathcal{E} + \boldsymbol{\nabla} \cdot \mathbf{f} = 0, \tag{III.34}$$

with the energy density $\mathcal{E}$ and the energy flux $\mathbf{f}$ playing the roles of charge density and current density.

When the gravitational field is time dependent, one needs to include the energy flux of the field itself and that exercise will be one of the assigned problems. In GR, however, there is no well-defined local energy density of a time-dependent gravitational field. The reason is that conservation of energy is associated with time-translation invariance of the geometry. When the geometry is time dependent, the invariance is gone. What remains is the total energy of the spacetime, associated with the asymptotic time-translation invariance of an asymptotically flat spacetime. In the Newtonian approximation, gravity is a field on flat space (the metric linearized about flat space), and this allows one to define a local energy.

III.4 Lie derivatives, Gauss's Theorem, and Stokes's Theorem

Lie derivatives

Lie derivatives arise naturally in the context of fluid flow and are a tool that can simplify calculations and aid one's understanding of relativistic fluids.

Begin, for simplicity, in a Newtonian context, with a stationary fluid flow with 3-velocity $\mathbf{v}(\mathbf{r})$. A function f is said to be *dragged along* by the fluid flow, or *Lie-derived* by the vector field $\mathbf{v}$ that generates the flow, if the value of f is constant on a fluid element, that is, constant along a fluid trajectory $\mathbf{r}(t)$:

$$\frac{d}{dt} f[\mathbf{r}(t)] = \mathbf{v} \cdot \nabla f = 0. \tag{III.35}$$

The *Lie derivative* of a function f, defined by

$$\mathcal{L}_{\mathbf{v}} f = \mathbf{v} \cdot \nabla f, \tag{III.36}$$

is the directional derivative of f along $\mathbf{v}$, the rate of change of f measured by a comoving observer.

Consider next a vector that joins two nearby fluid elements, two points $\mathbf{r}(t)$ and $\bar{\mathbf{r}}(t)$ that move with the fluid: Call the connecting vector $\lambda\mathbf{w}$, so that for small λ the fluid elements are nearby: $\lambda\mathbf{w} = \bar{\mathbf{r}}(t) - \mathbf{r}(t)$. Then $\lambda\mathbf{w}$ is said to be *dragged along* by the fluid flow, as shown in Fig. (2). In the figure, the endpoints of $\mathbf{r}(t_i)$ and $\bar{\mathbf{r}}(t_i)$ are labeled $\mathrm{r_i}$ and $\bar{\mathrm{r}}_\mathrm{i}$.

A vector field $\mathbf{w}$ is *Lie-derived* by $\mathbf{v}$ if, for small λ, $\lambda\mathbf{w}$ is dragged along by the fluid flow. To make this precise, we are requiring that the equation

$$\mathbf{r}(t) + \lambda\mathbf{w}(\mathbf{r}(t)) = \bar{\mathbf{r}}(t) \tag{III.37}$$

be satisfied to $O(\lambda)$. Taking the derivative of both sides of the equation with respect to t at $t = 0$, we have

$$\begin{aligned} \mathbf{v}(\mathbf{r}) + \lambda\mathbf{v} \cdot \nabla\mathbf{w}(\mathbf{r}) &= \mathbf{v}(\bar{\mathbf{r}}) = \mathbf{v}[\mathbf{r} + \lambda\mathbf{w}(\mathbf{r})] \\ &= \mathbf{v}(\mathbf{r}) + \lambda\mathbf{w} \cdot \nabla\mathbf{v}(\mathbf{r}) + O(\lambda^2), \end{aligned} \tag{III.38}$$

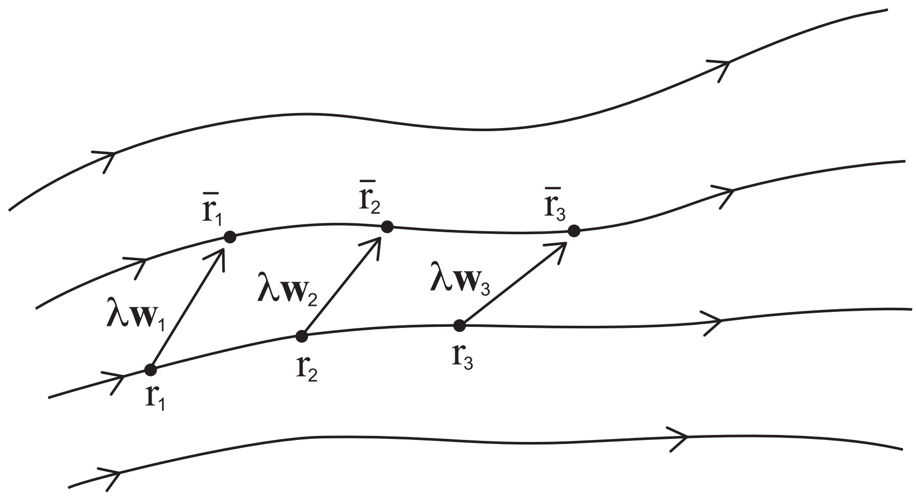


Figure 2: Two nearby fluid elements move along the flow lines, their successive positions labeled $\mathbf{r}_i$ and $\bar{\mathbf{r}}_i$. A vector field $\lambda\mathbf{w}$ is said to be dragged along by the flow when, as shown here, it connects successive positions of two nearby fluid elements.

which holds if and only if

$$[\mathbf{v},\mathbf{w}] \equiv \mathbf{v}\cdot\nabla\mathbf{w} - \mathbf{w}\cdot\nabla\mathbf{v} = 0. \tag{III.39}$$

The commutator $[\mathbf{v},\mathbf{w}]$ is the *Lie derivative* of $\mathbf{w}$ with respect to $\mathbf{v}$, written

$$\mathcal{L}_{\mathbf{v}}\mathbf{w} = [\mathbf{v},\mathbf{w}]. \tag{III.40}$$

Then $\mathbf{w}$ is Lie-derived by $\mathbf{v}$ when $\mathcal{L}_{\mathbf{v}}\mathbf{w} = \mathbf{0}$. The Lie derivative $\mathcal{L}_{\mathbf{v}}\mathbf{w}$ compares the change in the vector field $\mathbf{w}$ in the direction of $\mathbf{v}$ to the change that would occur if $\mathbf{w}$ were dragged along by the flow generated by $\mathbf{v}$.

In a curved spacetime the Lie derivative of a function f is again its directional derivative,

$$\mathcal{L}_{\mathbf{u}}f = u^\alpha\nabla_\alpha f. \tag{III.41}$$

If u^α is the 4-velocity of a fluid, generating the fluid trajectories in spacetime, $\mathcal{L}_{\mathbf{u}}f$ is commonly termed the convective derivative of f. The Newtonian limit of u^α is the 4-vector $\boldsymbol{\partial}_t + \mathbf{v}$, and $\mathcal{L}_{\mathbf{u}}f$ has as its limit the Newtonian convective derivative $(\partial_t + \mathbf{v}\cdot\nabla)f$, again the rate of change of f measured by a comoving observer. (Now the flow is arbitrary, not the stationary flow of our earlier Newtonian discussion.)

A connecting vector is naturally a contravariant vector, the tangent to a curve joining nearby points in a flow; and in a curved spacetime, the Lie derivative of a contravariant vector field is again defined by Eq. (III.40),

$$\mathcal{L}_{\mathbf{u}}w^\alpha = u^\beta\nabla_\beta w^\alpha - w^\beta\nabla_\beta u^\alpha. \tag{III.42}$$

We have used a fluid flow generated by a 4-velocity u^α to motivate a definition of Lie derivative; the definition, of course, is the same in any dimension and for any vector fields:

$$\mathcal{L}_{\mathbf{u}}w^a = u^b\nabla_b w^a - w^b\nabla_b u^a. \tag{III.43}$$

Although the covariant derivative operator ∇ appears in the above expression, the Lie derivative is in fact independent of the choice of derivative operator. This is immediate from the symmetry $\Gamma^i_{jk} = \Gamma^i_{(jk)}$, which implies that the components have in any chart the form

$$\mathcal{L}_{\mathbf{u}} w^i = u^j \partial_j w^i - w^j \partial_j u^i. \tag{III.44}$$

We now extend the definition of Lie derivative to arbitrary tensors using the Leibnitz rule, the requirement that, for any vector w^a,

$$\mathcal{L}_{\mathbf{u}}(\sigma_a w^a) = (\mathcal{L}_{\mathbf{u}}\sigma_a) w^a + \sigma_a \mathcal{L}_{\mathbf{u}} w^a. \tag{III.45}$$

Using this and the action (III.41) of the Lie derivative on the scalar $\sigma_a u^a$, we have

$$\mathcal{L}_{\mathbf{u}}\sigma_a = u^b \nabla_b \sigma_a + \sigma_b \nabla_a u^b. \tag{III.46}$$

Because $\mathcal{L}_{\mathbf{u}}(\sigma_a w^a)$ and $\mathcal{L}_{\mathbf{u}} w^a$ in Eq. (III.45) are independent of the choice of derivative operator, definition (III.46) is independent of the choice of derivative operator, and it is easy to check that the components in any chart are given by

$$\mathcal{L}_{\mathbf{u}}\sigma_i = u^j \partial_j \sigma_i + \sigma_j \partial_i u^j. \tag{III.47}$$

Finally, the Lie derivative of an arbitrary tensor $T^{a_1\cdots a_m}{}_{b_1\cdots b_n}$ again follows from the Leibnitz rule, using its form for contravariant and covariant vectors to compute $\mathcal{L}_{\mathbf{u}}(T^{c\cdots d}{}_{a\cdots b} v^a \cdots w^b \sigma_c \cdots \tau_d)$ for arbitrary vectors $v^a, \ldots, w^a$ and covectors $\sigma_a, \ldots, \tau_a$:

$$\begin{aligned} \mathcal{L}_{\mathbf{u}} T^{a\cdots n}{}_{c\cdots d} &= u^e \nabla_e T^{a\cdots b}{}_{c\cdots d} \\ &\quad - T^{e\cdots b}{}_{c\cdots d} \nabla_e u^a - \cdots - T^{a\cdots e}{}_{c\cdots d} \nabla_e u^b \\ &\quad + T^{a\cdots b}{}_{e\cdots d} \nabla_c u^e + \cdots + T^{a\cdots b}{}_{c\cdots e} \nabla_d u^e, \end{aligned} \tag{III.48}$$

independent of the derivative operator, and with components in a chart again given by replacing ∇ by ∂. [3]

Killing vectors

Spacetimes that are symmetric under continuous symmetries like rotations about an axis or time translations have metrics that are invariant under these symmetries. Let's start with rotations of flat space with metric

$$ds^2 = dx^2 + dy^2 + dz^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2. \tag{III.49}$$

The group of rotations about the z-axis maps a point P with coordinates (x, y, z) to the path $\phi \to P(\varphi) = R_\varphi(P)$, with

$$x(\varphi) = x\cos\varphi - y\sin\varphi, \quad y(\varphi) = y\cos\varphi + x\sin\varphi, \quad z(\varphi) = z. \tag{III.50}$$

[3] If, instead of a coordinate basis, an observer uses a frame that is dragged along by $\mathbf{u}$, then a tensor is dragged along by $\mathbf{u}$ (i.e., $\mathcal{L}_{\mathbf{u}}\mathbf{T} = 0$), if its components are constant along a fluid trajectory. It follows that the components of the Lie derivative of any tensor are just $u^m \partial_m T^{i\cdots j}{}_{k\cdots l}$ in a frame dragged along by $\mathbf{u}$.

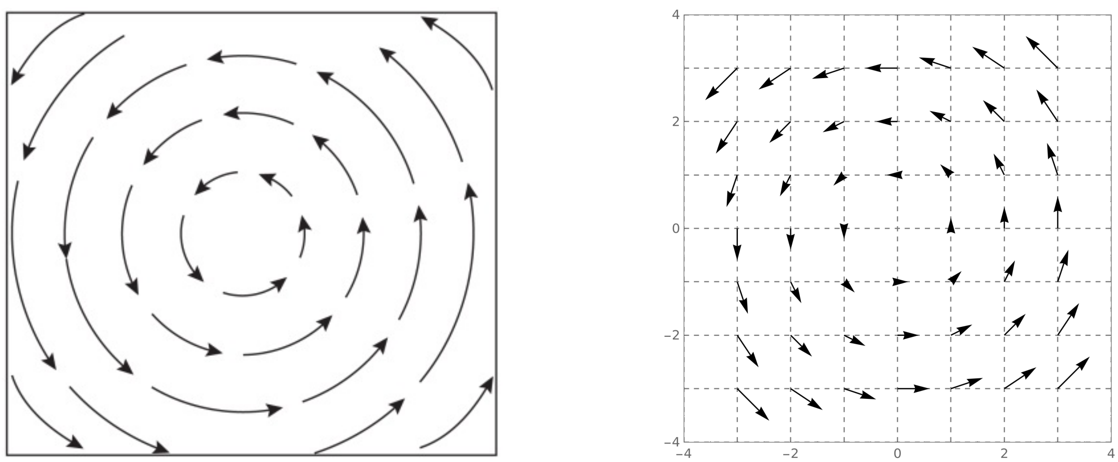


Figure 3: Rotation paths and the vector field $\boldsymbol{\xi}$ tangent to the paths.

The tangent to the path through P is the vector $\boldsymbol{\xi} = \frac{d}{d\phi} R_\phi(P)|_{\varphi=0} = x\boldsymbol{j} - y\boldsymbol{i} \equiv x\boldsymbol{\partial}_y - y\boldsymbol{\partial}_x$, or

$$\xi^x = y, \tag{III.51}$$
$$\xi^y = -x, \tag{III.52}$$
$$\xi^z = 0. \tag{III.53}$$

A fluid rotating with angular velocity Ω has velocity $\mathbf{v} = \Omega\boldsymbol{\xi}$. A vector field $\mathbf{w}$ is rotationally invariant if it is Lie-dragged by this flow, if $\mathcal{L}_{\boldsymbol{\xi}}\mathbf{w} = 0$; for example, the radial vector field $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$, is rotationally invariant.

The metric is rotationally invariant if $\mathcal{L}_{\boldsymbol{\xi}} g_{ab} = 0$. This is equivalent to saying that rotations preserve dot products, that the lengths of vectors and the angles between them are invariant under rotations. In other words, if $\boldsymbol{A}$ and $\boldsymbol{B}$ are Lie-derived by $\boldsymbol{\xi}$, then $\boldsymbol{A}\cdot\boldsymbol{B} = g_{ab}A^aB^b$ is Lie derived by $\boldsymbol{\xi}$.
A check that invariance of dot products of means $\mathcal{L}_{\boldsymbol{\xi}} g_{ab} = 0$:

$$\begin{aligned} 0 &= \mathcal{L}_{\boldsymbol{\xi}}(g_{ab}A^aB^b) = (\mathcal{L}_{\boldsymbol{\xi}} g_{ab})A^aB^b + g_{ab}(\mathcal{L}_{\boldsymbol{\xi}} A^a)B^b + g_{ab}A^a\mathcal{L}_{\boldsymbol{\xi}} B^b = (\mathcal{L}_{\boldsymbol{\xi}} g_{ab})A^aB^b \\ \Rightarrow\ & \mathcal{L}_{\boldsymbol{\xi}} g_{ab} = 0. \end{aligned} \tag{III.54}$$

Definition. A vector field $\boldsymbol{\xi}$ is a Killing vector of a metric g_{ab} if

$$\mathcal{L}_{\boldsymbol{\xi}} g_{ab} = 0. \tag{III.55}$$

Recalling our definition of Lie derivative, we have

$$\begin{aligned} \pounds_{\boldsymbol{\xi}} g_{\alpha\beta} &= \xi^\gamma\nabla_\gamma g_{\alpha\beta} + g_{\alpha\gamma}\nabla_\beta\xi^\gamma + +g_{\gamma\beta}\nabla_\alpha\xi^\gamma. \\ &= \nabla_\alpha\xi_\beta + \nabla_\beta\xi_\alpha \end{aligned} \tag{III.56}$$

and, in any coordinate system $\{x^i\}$,

$$\pounds_{\boldsymbol{\xi}} g_{ij} = \xi^k\partial_k g_{ij} + g_{kj}\partial_i\xi^k + g_{ik}\partial_j\xi^k. \tag{III.57}$$

Note that, if we choose as a coordinate the parameter distance ϕ along the rotation paths e.g., choosing coordinates r, θ, ϕ, then the rotation path is $(r, \theta\phi) \to (r, \theta, \phi + \varphi)$, and its tangent $\boldsymbol{\xi}$ has components $\xi^i = \delta^i_\phi$. Then, by Eq. (III.57), invariance of the metric under rotations in the x-y plane is equivalent to requiring

$$\partial_\phi g_{ij} = 0 \textit{ when } \phi \textit{ is one of the coordinates.} \tag{III.58}$$

More generally, any nonzero vector field $\boldsymbol{\xi}$ is tangent to a family of paths $x^i(\lambda)$: the paths are the solutions to $\dfrac{dx^i}{d\lambda} = \xi^i$. You can think of the paths $x^i(\lambda)$ as the trajectories of fluid elements whose velocity field is $\boldsymbol{\xi}$, and the paths are called the flow of the vector field. Then

$$\mathcal{L}_{\boldsymbol{\xi}} g_{ij} = \partial_\lambda g_{ij} \textit{ in a coordinate system with } \lambda \textit{ as one of the coordinates.} \tag{III.59}$$

Thus, if there are coordinates t, r, θ, ϕ for which the metric is independent of t, then the vector field t^α with components $t^\mu = \delta^\mu_t$ is a Killing vector.

Gauss's Theorem

Gauss's theorem and Stokes's theorem relate differential to integral conservation laws. One proves them in flat space by dividing a volume into a set of coordinate cubes and proving the theorem on each cube. We can quickly see that the flat-space proofs go through in curved space without change.

Because the divergence of a vector has the form $\nabla_a A^a = \dfrac{1}{\sqrt{|g|}}\partial_i(\sqrt{|g|A^i})$ the integral $\displaystyle\int \nabla_a A^a dV \equiv \int \nabla_a A^a \sqrt{|g|} d^n x$ has the form $\displaystyle\int \partial_i(\sqrt{|g|}A^i) d^n x$, involving only the partial derivative of a vector density, $\mathcal{A}^i := A^i\sqrt{|g|}$.[4] As a result, the flat-space proof of Gauss's Theorem, based on the Fundamental Theorem of calculus, $\int_a^b f'(x)dx = f(b) - f(a)$, holds in curved space as well.

The flat-space proof of Gauss's theorem follows from an integration over a coordinate cube V using the fundamental theorem of calculus for the integral over each coordinate.

$$\int_V \partial_i \mathcal{A}^i dx^1\, dx^2\, dx^3 = \int_S (\mathcal{A}^1 dx^2 dx^3 + \mathcal{A}^2 dx^1 dx^3 + \mathcal{A}^3 dx^1 dx^2);$$

and any volume is approximated by an arbitrarily fine division into coordinate cubes.

The integration over a coordinate cube has the identical form in curved space. If V is an n-

[4] Another way to say this is that the divergence of a vector density $\mathcal{A}^a$, here $A^a\sqrt{|g|}$, is defined without introducing a covariant derivative: In any coordinate system $\nabla_a \mathcal{A}^a := \partial_i A^i$.

dimensional coordinate cube

$$\begin{aligned}
\int_V \partial_i \mathcal{A}^i d^n x &= \int_V \partial_1 \mathcal{A}^1 dx^1 dx^2 \cdots dx^n + \cdots + \int_V \partial_n \mathcal{A}^n dx^n dx^1 \cdots dx^{n-1} \\
&= \int_{\partial_{1+}V} \mathcal{A}^1 dx^2 \cdots dx^n - \int_{\partial_{1-}V} \mathcal{A}^1 dx^2 \cdots dx^n + \cdots \\
&\quad + \int_{\partial_{n+}V} \mathcal{A}^n dx^1 \cdots dx^{n-1} - \int_{\partial_{n-}V} \mathcal{A}^n dx^1 \cdots dx^{n-1} \\
&= \int_{\partial V} \mathcal{A}^i dS_i \quad (\partial V \text{ means the boundary of } V) \qquad \text{(III.60)}
\end{aligned}$$

where $dS_i = \pm \epsilon_{ij\cdots k} dx^j \cdots dx^k \dfrac{1}{(n-1)!}$, with

$dS_1 = +dx^2 \cdots dx^n$ for x^1 increasing outward,

$dS_1 = -dx^2 \cdots dx^n$, for x^1 increasing inward.

This form is correct for a region in a space with a metric, independent of the signature of the metric. When $\partial\Omega$ has a unit outward normal n_a (along the gradient of a scalar that increases outward), one can write dS_a in the form $dS_a = n_a dS$. In this case,

$$\int \nabla_a A^a d^n V = \int A^a n_a dS. \qquad \text{(III.61)}$$

Example: As we discuss below, the differential form of baryon conservation is $\nabla_\alpha(nu^\alpha) = 0$. The corresponding integral form is

$$\begin{aligned}
0 &= \int_\Omega \nabla_\alpha(nu^\alpha) d^4V = \int_{\partial\Omega} nu^\alpha dS_\alpha \\
&= \int_{V_2} nu^\alpha dS_\alpha - \left| \int_{V_1} nu^\alpha dS_\alpha \right| .
\end{aligned}$$

Here the fluid is taken to have finite spatial extent, and the spacetime region Ω is bounded by the initial and final spacelike hypersurfaces V_1 and V_2. In a coordinate system for which V_1 and V_2 are surfaces of constant t, with t increasing to the future, we have $dS_\mu = \nabla_\mu t \sqrt{|g|} d^3x = \delta^t_\mu \sqrt{|g|} d^3x$ on V_2, $dS_\mu = -\delta^t_\mu \sqrt{|g|} d^3x$ on V_1, and

$$\int_\Omega \nabla_\alpha(nu^\alpha) d^4V = \int_{V_2} nu^t \sqrt{|g|} d^3x - \int_{V_1} nu^t \sqrt{|g|} d^3x. \qquad \text{(III.62)}$$

If, on a slicing of spacetime one chooses on each hypersurface V a surface element dS_α along $+\nabla_\alpha t$, the conservation law is then

$$N = \int_V nu^\alpha dS_\alpha = \text{ constant}, \qquad \text{(III.63)}$$

with N the total number of baryons.

Note that the fact that one can write the conserved quantity associated with a current j^α in the form,

$$\int_V j^\alpha dS_\alpha = \int_V j^t \sqrt{|g|}\, d^3x,$$

means that there is no need to introduce n_α and $\sqrt{^3g}$ to evaluate the integral. This fact is *essential* if one is evaluating an integral $\int j^\alpha dS_\alpha$ on a null surface, because there is no unit normal. The flux of energy or of baryons across the horizon of a Schwarzschild black hole, for example, can be computed in Eddington-Finkelstein or Kruskal coordinates: In ingoing Eddington-Finkelstein coordinates v, r, θ, ϕ, the horizon is a surface of constant r, and we have

$$\int j^\alpha dS_\alpha = \int j^r \sqrt{|g|}\, dv d\theta d\phi.$$

Stokes's Theorem

The simplest version of Stokes's theorem is its 2-dimensional form, namely Green's theorem:

$$\int_S (\partial_x A_y - \partial_y A_x) dx\, dy = \int_c (A_x dx + A_y dy),$$

where c is a curve bounding the 2-surface S. The theorem involves the integral over a 2-surface of the antisymmetric tensor $\nabla_a A_b - \nabla_b A_a$. In three dimensions, the tensor is dual to the curl of $\mathbf{A}$: $(\nabla \times \mathbf{A})^a = \epsilon^{abc}\nabla_b A_c$; and Stokes's generalization of Green's theorem can be written in either the form

$$\int_S (\boldsymbol{\nabla} \times \boldsymbol{A}) \cdot d\mathbf{S} = \int_c \mathbf{A} \cdot \mathbf{dl}$$

or in terms of the antisymmetric tensor $\nabla_a A_b - \nabla_b A_a$

$$\int_S (\nabla_a A_b - \nabla_b A_a) dS^{ab} = \int_c A_a dl^a, \tag{III.64}$$

where, for an antisymmetric tensor F_{ab}, $F_{ab}dS^{ab}$ means $F_{12}dx^1dx^2 + F_{23}dx^2dx^3 + F_{31}dx^3dx^1$. Written in this form, the theorem is already correct in a curved spacetime. The reason is again that the antisymmetric derivative $\nabla_a A_b - \nabla_b A_a$ has in curved space the same form it has in flat space: Its components in any coordinate system are just $\partial_i A_j - \partial_j A_i$. The antisymmetric derivative $\nabla_a A_b - \nabla_b A_a$ is called the exterior derivative, written dA, and it does not need a covariant derivative for its definition.

Here is the three-line proof for a coordinate square S in a surface of constant coordinates t and z:

$$\begin{aligned}
\int_S (\partial_x A_y - \partial_y A_x) dx dy &= \int_{y_1}^{y_2} dy \int_{x_1}^{x_2} dx (\partial_x A_y - \partial_y A_x) \\
&= \int_{y_1}^{y_2} dy (A_y|_{x_2} - A_y|_{x_1}) + \int_{x_1}^{x_2} dx (A_x|_{y_2} - A_x|_{y_1}) \\
&= \int_c A_i dx^i,
\end{aligned} \tag{III.65}$$

with the boundary c of the square traversed counterclockwise as seen from above the square.

In the more mathematical part of the gravitational physics literature and in differential geometry texts, the theorem is written $\int_S dA = \int_{\partial S} A$, where ∂S is the boundary of S.

III.5 Relativistic Euler equation and the TOV equation

In this section units are chosen to make $c = G = 1$. These are called *gravitational units* or *geometrized units*. With $c = 1$, length and time have the same dimension; if time is measured in seconds, length is measured in light-seconds, where 1 light-second = 3×10^{10} cm, the distance light travels in 1 second. MTW (Misner-Thorne-Wheeler) gives distance in cm, and the unit of time is then $1/(3 \times 10^{10})$ s. With $G = 1$, mass, length and time all have the same dimension, which MTW takes to be length with unit 1 cm. Then the unit of mass is (1 cm)$\times c^2/G = 1.3468 \times 10^{28}$ g. Here is the MTW conversion table:

Box 1.8 GEOMETRIZED UNITS

Throughout this book, we use "geometrized units," in which the speed of light c, Newton's gravitational constant G, and Boltzman's constant k are all equal to unity. The following alternative ways to express the number 1.0 are of great value:

$$1.0 = c = 2.997930 \cdots \times 10^{10}\ \mathrm{cm/sec}$$
$$1.0 = G/c^2 = 0.7425 \times 10^{-28}\ \mathrm{cm/g};$$
$$1.0 = G/c^4 = 0.826 \times 10^{-49}\ \mathrm{cm/erg};$$
$$1.0 = Gk/c^4 = 1.140 \times 10^{-65}\ \mathrm{cm/K};$$
$$1.0 = c^2/G^{1/2} = 3.48 \times 10^{24}\ \mathrm{cm/gauss^{-1}}.$$

One can multiply a factor of unity, expressed in any one of these ways, into any term in any equation without affecting the validity of the equation. Thereby one can convert one's units of measure from grams to centimeters to seconds to ergs to . . . . For example:

$$\begin{aligned} \text{Mass of sun} = M_\odot &= 1.989 \times 10^{33}\ \mathrm{g} \\ &= (1.989 \times 10^{33}\ g) \times (G/c^2) \\ &= 1.477 \times 10^5\ \mathrm{cm} \\ &= (1.989 \times 10^{33}\ g) \times (c^2) \\ &= 1.788 \times 10^{54}\ \mathrm{ergs}. \end{aligned}$$

The standard unit, in terms of which everything is measured in this book, is centimeters. However, occasionally conventional units are used; in such cases a subscript "conv" is sometimes, but not always, appended to the quantity measured:

$$M_{\odot\mathrm{conv}} = 1.989 \times 10^{33}\ \mathrm{g}.$$

Relativistic fluids

First a little more discussion of what is meant by a fluid: A perfect fluid is a model for a large assembly of particles in which a continuous energy density $\epsilon = \rho c^2$, pressure P, and 4-velocity u^α can reasonably describe the macroscopic matter. We will also be assuming that the microscopic particles collide frequently enough that collisions enforce a local thermodynamic equilibrium. In particular, one assumes that a mean velocity field u^α and a mean stress-energy tensor $T^{\alpha\beta}$ (AKA the energy-momentum tensor) can be defined in boxes – fluid elements – small compared to the macroscopic length scale but large compared to the mean free path, and on scales large compared to the size of the fluid elements, the 4-velocity and thermodynamic quantities can be accurately described by continuous fields. Finally, the statement that a fluid cannot maintain a shear stress again means that the stress-energy tensor has no preferred orientation for an observer moving with the average velocity u^α of the fluid:

Denote by

$$q^{\alpha\beta} = g^{\alpha\beta} + u^\alpha u^\beta \tag{III.66}$$

the projection operator orthogonal to u^α. The lack of a preferred orientation implies that the stress-energy tensor has the form

$$T^{\alpha\beta} = \rho u^\alpha u^\beta + P q^{\alpha\beta} = (\rho + P) u^\alpha u^\beta + P g^{\alpha\beta}. \tag{III.67}$$

That is, the only nonzero parts of $T^{\alpha\beta}$ are the rotational scalars [5]

$$\rho \equiv T^{\alpha\beta} u_\alpha u_\beta \tag{III.68}$$

and

$$P \equiv \frac{1}{3} q_{\gamma\delta} T^{\gamma\delta}. \tag{III.69}$$

In an orthonormal frame with $\mathbf{e_0}$ along $\boldsymbol{u}$, $T^{\alpha\beta}$ has components

$$\|T^{\mu\nu}\| = \left\| \begin{matrix} \rho & & & \\ & P & & \\ & & P & \\ & & & P \end{matrix} \right\|. \tag{III.70}$$

Conservation of baryons

Within the standard model of particle physics and to within the accuracy of observations, baryon number is conserved. (The lower limit on the lifetime of the proton against decay to a positron or positive muon is 10^{34} yr.) Let n be the number density of baryons, measured in the rest frame of a fluid. Conservation of baryons has the form

$$\nabla_\alpha (n u^\alpha) = 0, \tag{III.71}$$

and it can be derived heuristically by essentially following our derivation of conservation of mass in the Newtonian approximation.

The proper volume of a fluid element is the volume V of a piece of fluid orthogonal to u^α through the history of the fluid element. With N the number of baryons in the fluid element, conservation

[5] Because the momentum current

$$q^\alpha_\gamma T^{\gamma\beta} u_\beta$$

is a vector in the 3-dimensional subspace orthogonal to u^α, it is invariant under rotations of that subspace only if it vanishes. Similarly, the symmetric tracefree tensor ${}^3T^{\alpha\beta} - \frac{1}{3} q^{\alpha\beta}\, {}^3T \equiv q^\alpha_\gamma q^\beta_\delta T^{\gamma\delta} - \frac{1}{3} q^{\alpha\beta} q_{\gamma\delta} T^{\gamma\delta}$
transforms as a $j = 2$ representation of the rotation group and can be invariant only if it vanishes. More concretely, in an orthonormal frame with $\mathbf{e_0}$ along $\boldsymbol{u}$, T^{0i} and $T^{ij} - \frac{1}{3}\delta^{ij} T^k_k$ must vanish, implying that $T^{\alpha\beta}$ has the form (III.70)

of baryons means $0 = \Delta N = \Delta(nV)$. The fractional change in V in a proper time $\Delta\tau$ is given by the 3-dimensional divergence of the velocity, in the subspace orthogonal to u^α:

$$\frac{\Delta V}{V} = q^{\alpha\beta}\nabla_\alpha u_\beta \Delta\tau. \tag{III.72}$$

Because $u^\beta u_\beta = -1$, we have $u^\beta \nabla_\alpha u_\beta = \frac{1}{2}\nabla_\alpha(u_\beta u^\beta) = 0$, implying

$$q^{\alpha\beta}\nabla_\alpha u_\beta = \nabla_\beta u^\beta. \tag{III.73}$$

Noting that $u^\alpha \nabla_\alpha n = \dfrac{d}{d\tau}n$, we can write the conservation law in the form

$$0 = \frac{\Delta(nV)}{V} = \Delta n + n\frac{\Delta V}{V} = (u^\alpha \nabla_\alpha n + n\nabla_\alpha u^\alpha)\Delta\tau, \tag{III.74}$$

or

$$\nabla_\alpha(nu^\alpha) = 0. \tag{III.75}$$

The conservation law is often written as conservation of rest mass or baryon mass. To do this one has to define a rest mass per baryon: For example, one could imagine dispersing to infinity all the nucleons and electrons in a neutron star, letting the neutrons decay, and taking the rest mass of the star to be the rest mass of the resulting collection of protons, electrons and neutrinos. Because the electron rest mass is about 1/2000 that of a proton, and the neutrino rest mass is about a millionth that of an electron, the result, to within 1 part in 2,000, is to assign the mass of the proton as the rest mass per baryon: $m_p = 1.67 \times 10^{-24}\text{g} = 938$ MeV. The rest mass density is then $\rho_0 = m_p n$, and conservation of baryons is

$$\nabla_\alpha(\rho_0 u^\alpha) = 0, \tag{III.76}$$

with integral form

$$M_0 = \int_V \rho_0 u^\alpha dS_\alpha = \text{ constant}, \tag{III.77}$$

equivalent to Eq. (III.63). Its Newtonian limit is the mass conservation equation (III.25).

Conservation of energy

For a two-parameter equation of state, five variables determine the state of a perfect fluid; they can be taken to be ρ, P and three independent components of u^α. The dynamical evolution of the fluid is governed by the vanishing divergence of the stress-energy tensor,

$$\nabla_\beta T^{\alpha\beta} = 0, \tag{III.78}$$

and by conservation of baryons,

$$\nabla_\alpha(nu^\alpha) = 0. \tag{III.79}$$

The projection of the equation $\nabla_\beta T^{\alpha\beta} = 0$ along u^α yields an energy conservation law, while the projection orthogonal to u^α is the relativistic Euler equation.

The projection $u_\alpha \nabla_\beta T^{\alpha\beta} = 0$ similarly expresses energy conservation for a fluid element:

$$\begin{aligned} 0 &= u_\alpha \nabla_\beta T^{\alpha\beta} = u_\alpha \nabla_\beta [\rho u^\alpha u^\beta + P q^{\alpha\beta}] \\ &= -\nabla_\beta(\rho u^\beta) + P u_\alpha \nabla_\beta (g^{\alpha\beta} + u^\alpha u^\beta) \\ &= -\nabla_\beta(\rho u^\beta) - P \nabla_\beta u^\beta, \end{aligned}$$

$$\Rightarrow \quad \nabla_\beta(\rho u^\beta) = -P \nabla_\beta u^\beta. \tag{III.80}$$

The equation means that the total energy of a fluid element decreases by the work,

$$P\,dV = PV\ \nabla_\beta u^\beta\, d\tau, \tag{III.81}$$

it does on its surroundings in proper time $d\tau$.

The mass (energy) density is

$$\rho = \rho_0(1+u), \quad \text{or, with } c \text{ restored,} \quad \epsilon \equiv \rho c^2 = \rho_0 c^2 + \rho_0 u, \tag{III.82}$$

with u the internal energy per baryon. Because $\rho_0 c^2$ is the dominant contribution to the energy density, energy conservation is just baryon conservation at leading order in c. Newtonian energy conservation is the order c^0 correction.

Relativistic Euler equation. The projection of the conservation of the stress-energy tensor orthogonal to u^α is

$$q^\alpha{}_\gamma \nabla_\beta T^{\beta\gamma} = 0, \tag{III.83}$$

so that

$$\begin{aligned} 0 &= q^\alpha{}_\gamma \nabla_\beta [\rho u^\beta u^\gamma + P q^{\beta\gamma}] \\ &= q^\alpha{}_\gamma \rho u^\beta \nabla_\beta u^\gamma + q^{\alpha\beta}\nabla_\beta P + q^\alpha{}_\gamma p \nabla_\beta (u^\beta u^\gamma) \\ &= \rho u^\beta \nabla_\beta u^\alpha + q^{\alpha\beta} \nabla_\beta P + P u^\beta \nabla_\beta u^\alpha, \end{aligned}$$

$$\Rightarrow \quad (\rho + P) u^\beta \nabla_\beta u^\alpha = -q^{\alpha\beta} \nabla_\beta P. \tag{III.84}$$

The TOV equation

First a summary, then a detailed derivation from the field equations:

For a spherical star in Schwarzschild coordinates (t, r, θ, ϕ), the metric takes the form

$$ds^2 = -e^{2\Phi(r)}dt^2 + \left[1 - \frac{2m(r)}{r}\right]^{-1} dr^2 + r^2(d\theta^2 + r^2 \sin^2\theta d\phi^2), \tag{III.85}$$

where $\Phi(r)$ and $m(r)$, as well as the pressure $P(r)$, are determined by the Tolman-Oppenheimer-Volkoff (TOV) equations

$$\frac{dm}{dr} = 4\pi r^2 \rho, \tag{III.86}$$

$$\frac{d\Phi}{dr} = -\frac{1}{\rho + P}\frac{dP}{dr}, \tag{III.87}$$

$$\frac{dP}{dr} = -\frac{(\rho + P)(m + 4\pi r^3 P)}{r(r - 2m)}, \tag{III.88}$$

by integrating the above system from the center to the surface, $r = R_{\text{sph}}$, with conditions $m(0) = 0$, $P(0) = P_c$ and $\Phi(0)$ arbitrary. Here, P_c is the chosen value of central pressure. The arbitrariness in the initial value for $\Phi(r)$ is removed by matching the solution at the surface of the star to the analytic exterior solution

$$e^{2\Phi(r)} = 1 - \frac{2M}{r}. \tag{III.89}$$

where $M := m(R)$ is the total mass.

Derivation

We begin with the general spherically symmetric metric,

$$ds^2 = -e^{2\Phi}dt^2 + e^{2\lambda}dr^2 + r^2 d\Omega^2 \tag{III.90}$$

(see, for example, Sect. 5.6, p. 122, of Shapiro-Teukolsky). In this section we look at equilibrium models, and Φ and λ are then functions of r only. This means that the vector $\boldsymbol{t} = \boldsymbol{\partial}_t$, with components

$$t^\mu = \delta^\mu_0 \tag{III.91}$$

is a Killing vector. For collapsing or oscillating stars, Φ and λ depend on r and t.

For our time-independent metric, the Einstein tensor has components

$$G^t{}_t = e^{-2\lambda}\left(\frac{1}{r^2} - \frac{2}{r}\lambda'\right) - \frac{1}{r^2} \tag{III.92}$$

$$= -\frac{1}{r^2}\frac{d}{dr}\left[r(1 - e^{-2\lambda})\right]$$

$$G^r{}_r = e^{-2\lambda}\left(\frac{1}{r^2} + \frac{2}{r}\Phi'\right) - \frac{1}{r^2} \tag{III.93}$$

$$G^\theta{}_\theta = G^\phi{}_\phi = e^{-2\lambda}[\Phi'' + (\Phi')^2 + \frac{1}{r}(\Phi' - \lambda') - \Phi'\lambda']. \tag{III.94}$$

All other components $G^\mu{}_\nu$ vanish.

Spacetime outside a spherical star

In a vacuum, the field equations are

$$G^\mu{}_\nu = 0.$$

The G^t_t-equation,

$$G^t{}_t = -\frac{1}{r^2}\frac{d}{dr}[r(1-e^{-2\lambda})] = 0, \tag{III.95}$$

has the first integral

$$r(1-e^{-2\lambda}) = 2M, \text{ for some constant } M, \tag{III.96}$$

implying

$$e^{2\lambda} = \left(1-\frac{2M}{r}\right)^{-1}. \tag{III.97}$$

From the combination,

$$G^r{}_r - G^t{}_t = \frac{2}{r}\, e^{-2\lambda}(\Phi' + \lambda') = 0, \tag{III.98}$$

we have

$$\Phi' = -\lambda' \tag{III.99}$$

or

$$\Phi = -\lambda + k \qquad e^{2\Phi} = k\left(1-\frac{2M}{r}\right).$$

Reparametrizing the time by writing $\tilde{t} = \frac{1}{\sqrt{k}}t$, and changing the name of $\tilde{t}$ back to t, gives

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2 + r^2 d\Omega^2, \tag{III.100}$$

the *exterior Schwarzschild metric.* The geometry is asymptotically flat: For large r, the metric is the Minkowski metric:

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2. \tag{III.101}$$

For $r = 2M$ the components $g_{\mu\nu}$ are singular, so the form (III.100) provides a metric for a spacetime with a hole in it: $\infty > r > 2M, \ -\infty < t < \infty$. When we discuss black holes, we'll see that this is a coordinate singularity, like the poles in spherical coordinates. Changing to coordinates that are smooth at $r = 2M$ reveals what you know: The surface $r = 2M$ is an event horizon.

For large r, the metric (III.100) takes the post-Newtonian form

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1+\frac{2M}{r}\right)dr^2 + r^2 d\Omega^2. \tag{III.102}$$

For nearly Newtonian stars, g_{tt} determines the effect of the gravitational field on matter, and $-\dfrac{M}{r}$ is the Newtonian potential Φ. Because the trajectories of particles at large r are those of Newtonian particles about a mass M, one calls M the mass of the spacetime.

The equations, $G^\theta{}_\theta = 0,\ \ G^\phi{}_\phi = 0$, are automatically satisfied once $G^t{}_t = 0$ and $G^r{}_r = 0$ (not in general, just in this spherically symmetric case).

Stellar Interior

Equilibrium configurations of stars are accurately modeled as perfect fluids. For static, spherical stars, ρ and P depend only on r, while the 4-velocity u^α is along the Killing vector t^α :

$$u^\alpha = kt^\alpha.$$

But $t^\alpha t_\alpha = g_{tt} = -e^{2\Phi}$, so $u^\alpha u_\alpha = -1$ implies

$$u^\alpha = e^{-\Phi} t^\alpha.$$

Then, from

$$T^{\alpha\beta} = \rho u^\alpha u^\beta + P q^{\alpha\beta},$$

we have

$$T^t{}_t = -\rho \qquad T^r{}_r = P.$$

The field equation components are
$G^t{}_t = 8\pi T^t{}_t$:

$$\begin{aligned} \frac{1}{r^2}\frac{d}{dr}[r(1-e^{-2\lambda})] &= 8\pi\rho \\ r(1-e^{-2\lambda}) &= 2\int_0^r 4\pi\rho r^2 dr =: 2m(r) && \text{(III.103)} \\ e^{2\lambda} &= \left(1-\frac{2m(r)}{r}\right)^{-1}. && \text{(III.104)} \end{aligned}$$

Here $m(r)$ is a kind of mass within a radius r, and $m(r) = M$, the mass measured at infinity, for $r \geq R$.
$G^r{}_r = 8\pi T^r{}_r$:

$$\begin{aligned} e^{-2\lambda}\left(\frac{2\Phi'}{r}+\frac{1}{r^2}\right)-\frac{1}{r^2} &= 8\pi P \\ (1-\frac{2m}{r})(\frac{2\Phi'}{r}+\frac{1}{r^2}) &= 8\pi P + \frac{1}{r^2} = \frac{8\pi P r^2+1}{r^2} \\ \frac{2\Phi'}{r}+\frac{1}{r^2} &= \frac{8\pi P r^2+1}{r(r-2m)} \\ 2\Phi' &= \frac{8\pi P r^2+1}{r-2m}-\frac{1}{r} = \frac{8\pi P r^3+2m}{r(r-2m)} \\ \Phi' &= \frac{m+4\pi P r^3}{r(r-2m)} && \text{(III.105)} \end{aligned}$$

Note that the Newtonian limit $(P \ll \rho,\ \ r \ll m)$ of (III.105) is

$$\Phi' = \frac{m}{r^2}\ ,$$

so that Φ is again the Newtonian potential.

The remaining field equation components, $G^\theta{}_\theta = 8\pi T^\theta{}_\theta,\ G^\phi{}_\phi = 8\pi T^\phi{}_\phi$, are identical to one another and are implied by (III.104), (III.105) and the equation of hydrostatic equilibrium $q^\alpha{}_\gamma \nabla_\beta T^{\gamma\beta}$, which we now obtain: Recall that $q^\alpha{}_\gamma \nabla_\beta T^{\gamma\beta} = 0$ has the form

$$u^\beta \nabla_\beta u_\alpha = -\frac{1}{\rho + P}\, q^\beta_\alpha \nabla_\beta P. \tag{III.106}$$

On the RHS, $q^\beta_\alpha \nabla_\beta P = (\delta_\alpha{}^\beta + u_\alpha u^\beta)\nabla_\beta P$. The second term vanishes because $\partial_t P = 0$. Then

$$q_\alpha{}^\beta \nabla_\beta P = \nabla_\alpha P.$$

Next, we have

$$\begin{aligned}
u^\beta \nabla_\beta\, u_\alpha &= e^{-\Phi} t^\beta \nabla_\beta (e^{-\Phi} t_\alpha) \\
&= e^{-2\Phi} t^\beta \nabla_\beta t_\alpha \ (\text{using } \mathrm{t}^\beta \nabla_\beta \Phi = 0) \\
&= -e^{-2\Phi} t^\beta \nabla_\alpha t_\beta \quad (\nabla_\alpha t_\beta + \nabla_\beta t_\alpha = 0 \ - \text{ Killing vector eq.}) \\
&= -\frac{1}{2} e^{-2\Phi} \nabla_\alpha (t^\beta t_\beta) \\
&= \frac{1}{2} e^{-2\Phi} \nabla_\alpha (e^{2\Phi}) \\
&= \nabla_\alpha \Phi \\
\nabla_\alpha \Phi = &- \frac{1}{\rho + P}\, \nabla_\alpha P
\end{aligned}$$

or

$$\Phi' = -\frac{1}{\rho + P}\, P' \ . \tag{III.107}$$

Eqs. (III.105) and (III.107) imply the equation of hydrostatic equilibrium - the TOV (Tolman-Oppenheimer-Volkov) equation:

$$\frac{dP}{dr} = -(\rho + P)\frac{m + 4\pi r^3 P}{r(r - 2m)} \ . \tag{III.108}$$

A spherical relativistic star is a solution to equations (III.104), (III.105), and (III.108) together with an equation of state; the numerical models that have been constructed usually involve equations of state of the simplest form

$$P = P(\rho)\ , \tag{III.109}$$

which are reasonably accurate for neutron stars. A general equation of state has the form $P = P(\rho, s, z_1, \ldots, z_n)$, with s the entropy per baryon and z_i the concentration of the ith particle species. Neutron stars and dwarfs are cold enough (KT $\ll$ Fermi energy) that they are nearly isentropic (s = constant), and their nuclear reactions have proceeded to completion, so each z_i is itself a function of ρ. That's why $P = P(\rho)$ is a good approximation. (At absolute zero of course, s is constant.)

One obtains a star by integrating eqs. (III.108) and (III.109) together with the defining equation for m. That is, one integrates the system

$$m(r) = \int_0^r \rho\, 4\pi r^2 dr, \quad \frac{dP}{dr} = -(\rho + P)\frac{m + 4\pi r^3 P}{r(r-2m)}\ , \quad P = P(\rho)\ . \tag{III.110}$$

One begins with a central density ρ_c and integrates up to the radius R at which P drops to zero (P is a decreasing function of r). This is the boundary of the star. The metric inside the star is then given by

$$e^{2\lambda} = \left(1 - \frac{2m}{r}\right)^{-1} \tag{III.111}$$

$$\Phi = \Phi(R) + \int_r^R \frac{1}{\rho + P}\frac{dP}{dr} \tag{III.112}$$

and outside by

$$e^{2\Phi} = e^{-2\lambda} = 1 - \frac{2M}{r}. \tag{III.113}$$

To restore G and c to these equations, one can multiply by the (unique) factors of G and c that allow each quantity have the desired conventional units. For example, in Eq. (III.111), both sides are dimensionless. To allow m and r to have dimensions of mass and length, multiply by the unique factor built from G and c that makes m/r dimensionless, namely G/c^2. That is, $\dfrac{G}{c^2}\dfrac{m}{r}$ is dimensionless, so the right side of (III.111) becomes $\left(1 - \dfrac{2Gm}{c^2 r}\right)^{-1}$.

In Eq. (III.113), Φ and λ are dimensionless. If they each are to have the conventional units L^2/T^2 of the Newtonian potential, the dimensionless forms are Φ/c^2 and λ/c^2. With G and c restored, Eqs. (III.112), (III.113) and (III.108), become

$$\Phi = \Phi(R) + \int_r^R \frac{1}{\rho + P/c^2}\frac{dP}{dr}, \quad \text{inside} \tag{III.114}$$

$$e^{2\Phi/c^2} = e^{-2\lambda/c^2} = 1 - \frac{2GM}{c^2 r}, \quad \text{outside} \tag{III.115}$$

$$\frac{dP}{dr} = -G(\rho + P/c^2)\frac{m + 4\pi r^3 P/c^2}{r(r - 2Gm/c^2)}\ . \tag{III.116}$$

Notice that, in the TOV equation, (III.116), each factor has its Newtonian form in the Newtonian hydrostatic equilibrium equation, $dP/dr = -G\rho m/r^2$, when terms involving $1/c^2$ are neglected.

Lecture IV Stellar Structure

IV.1 Properties of Stellar Equilibria and the Virial Theorem

Our equation (III.7) of hydrostatic equilibrium (HSE)

$$\frac{dP}{dr} = -\frac{Gm(r)\rho(r)}{r^2} = -\rho(r)g(r)$$

describes any static, spherically symmetric system (atmospheres, stars, planets). When a particle falls, the loss of potential energy is converted to increased kinetic energy of the particle. For a contracting star, some of the gravitational potential energy again goes into kinetic energy – into heating up the star – but some is radiated. An average of the HSE governs that distribution.

First, recall for a simpler system, a particle or a ring in circular orbit in a $1/r$ potential, that the kinetic energy is half as large as | potential energy|: $K = |U|/2$. The particle's total energy is then $E = K + U = -K$. When orbiting particles radiate energy in light or in gravitational waves, E becomes increasingly negative; because $K = |U|/2$, half of the loss of gravitational potential energy goes into increasing the kinetic energy of the particle and half is radiated: Energy radiated = total energy lost by the system = $|\Delta E| = |\Delta U|/2$. The result is also true for elliptical orbits when averaged over time – over the period T of the orbit:[6] $\langle K\rangle_T = |\langle U\rangle_T|$.

To generalize this to stars, we use HSE to relate the gravitational potential energy of the star to an integral over the pressure.

$$\int_0^R dr\, 4\pi r^3 \frac{dP}{dr} = -\int_0^R dr\, \frac{Gm(r)\rho(r)4\pi r^3}{r^2} \tag{IV.117}$$

where we multiplied both sides by $4\pi r^3$ and integrated. The RHS is

$$U_{\rm G} = -\int_0^M dm\, \frac{Gm(r)}{r} \tag{IV.118}$$

where $dm = 4\pi r^2 \rho(r) dr$. Integrate the LHS by parts:

$$P(r)4\pi r^3|_0^R - 3\int_0^R dr\, 4\pi r^2 P(r) \tag{IV.119}$$

The first term vanishes because $P(R) = 0$. The second term is the volume average of P times V: $\langle P\rangle V$. So:

$$\langle P\rangle = -\frac{U_G}{3V} \tag{IV.120}$$

This is a very important result — one way of expressing the **virial theorem**.

[6] $2\langle K\rangle = \int_0^T dt\, \dot{\mathbf{r}}^2 = \underbrace{\mathbf{r}\cdot\dot{\mathbf{r}}|_0^T}_{0} - \int_0^T dt\, \mathbf{r}\cdot\ddot{\mathbf{r}} = \int_0^T dt\, \mathbf{r}\cdot\hat{\mathbf{r}}\frac{GM}{r^2} = \int_0^T dt\, \frac{GM}{r} = -\langle U\rangle$

We now use the virial theorem to generalize the $K = -U/2$ relation that governs a single particle in a bound orbit. For a non-degenerate gas at temperature T, the energy per particle associated with each degree of freedom is $\frac{1}{2}k_B T$. For n_f degrees of freedom, the total kinetic energy of a gas with N particles is then

$$K = \frac{1}{2} n_f N k_B T.$$

The number of degrees of freedom is associated with γ (adiabatic index, ratio of the specific heats): $\gamma = 1 + 2/n_f$, so the kinetic energy density is:

$$\frac{K}{V} = \frac{n k_B T}{\gamma - 1} = \frac{P}{\gamma - 1}, \tag{IV.121}$$

where $n = N/V$ is the number density of particles. So our Virial theorem becomes

$$(\gamma - 1)\frac{K}{V} = -\frac{U}{3V}. \tag{IV.122}$$

With $\gamma = 5/3$ for a monatomic gas we have $K = -U/2$, agreeing with the relation for a single orbiting particle to a self-gravitating system. When the gas particles have internal degrees of freedom or are relativistic (e.g. photons or electrons in a white dwarf near its upper mass limit or in a neutron star modeled with Newtonian gravity), $\gamma \neq 5/3$, and the generalized relation is Eq. (IV.121). What might happen for different values of γ? Clearly for $\gamma = 1$ this equation looks bad. But even $\gamma = 4/3$ is problematic, since that gives us $K = -U$. That means the total energy approaches 0, and the star becomes unbound. We will see that this happens for relativistic systems.

IV.2 Simple Stellar Models

Want to put all of these together. Make into 4 coupled first-order ODEs $P(r)$, $m(r)$, $T(r)$, $L(r)$. Need boundary conditions. Some are easy: $m(0) = L(0) = 0$ (no mass inside that). At the outside, $P(R)$ and $T(R)$ need to merge into the photosphere which is complicated. We will ignore that for the moment, assume $T(R) = P(R) = 0$.

Combine HSE and mass into:

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi G\rho$$

Second order in $P(r)$, $\rho(r)$. Assume a simple relation between these:

$$P = K\rho^{\gamma} = K\rho^{(n+1)/n}$$

This is a **polytrope** with index n, with $\gamma = (n+1)/n$. So we get:

$$\frac{1}{r^2}\frac{d}{dr}\left[\frac{r^2}{\rho}\frac{d}{dr}\left(K\rho^{(n+1)/n}\right)\right] = -4\pi G\rho$$

We can now make our additional boundary conditions $\rho(0) = \rho_c$, $d\rho/dr(0) = 0$. This sets the central density, and says that there is not a cusp of material. The outer boundary comes from having ρ go to 0, or $\rho(R) = 0$ and $m(R) = M$.

These models are overly simple, but can still be useful. Especially before computers. Let us work a bit on the math.

$$\left(\frac{n+1}{n}\right)\frac{K}{r^2}\frac{d}{dr}\left(r^2\rho^{(1-n)/n}\frac{d\rho}{dr}\right) = -4\pi G\rho$$

Let us simplify the units. $\rho(r) = \rho_c(D_n(r))^n$, where $D_n(r)$ is a function that goes between 0 and 1. So:

$$\left[(n+1)\frac{K\rho_c^{(1-n)/n}}{4\pi G}\right]\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dD_n}{dr}\right) = -D_n^n$$

The bit out in front has units of distance squared. So:

$$\lambda_n \equiv \left[(n+1)\frac{K\rho_c^{(1-n)/n}}{4\pi G}\right]^{1/2}$$

and normalize:

$$\xi \equiv \frac{r}{\lambda_n}$$

So we get:

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{dD_n}{d\xi}\right) = -D_n^n$$

This is the **Lane-Emden** equation for a polytrope. We have written it in terms of dimensionless variables $D_n(\xi)$ to make a physics problem into a math problem, but we must be careful to put the units back in before we give physics results.

Boundary conditions as before, but also stop the integration where $D_n(\xi) = 0$. This is the first 0 of the function, and defines the outer edge at $\xi = \xi_1$.

To compute the mass:

$$M = 4\pi \int_0^R dr\, \rho r^2 = 4\pi \int_0^{\xi_1} d(\lambda_n \xi)\, (\lambda_n \xi)^2 \rho_c D_n^n = 4\pi \lambda_n^3 \rho_c \int_0^{\xi_1} d\xi\, \xi^2 D_n^n$$

We don't necessarily have to solve for D_n and integrate to get this, since we can recognize that $\xi^2 D_n^n = -d/d\xi(\xi^2 dD_n/d\xi)$, so

$$M = -4\pi \lambda_n^3 \rho_c \xi_1^2 \frac{dD_n}{d\xi}|_{\xi_1}$$

Numerically this is useful, but there are a few analytic solutions. Namely, $n = 0$, 1, and 5. For $n = 1$ the solution is:

$$D_1(\xi) = \frac{\sin \xi}{\xi}$$

where we only do it up to the first zero, $\xi_1 = \pi$. And for $n = 5$ there is no finite radius:

$$D_5(\xi) = \left(1 + \frac{\xi^2}{3}\right)^{-1/2}$$

with $\xi_1 = \infty$. However, the total mass is finite. For $n > 5$ the mass is infinite.

For adiabatic monatomic gas, $\gamma = 5/3$ and $n = 1.5$. This also works for white dwarfs in some cases.

$n = 3$ is useful, since this is what happens for a star in radiative equilibrium. Add radiative and gas pressure, $P_g = \rho k_B T/\bar{m} = \beta P$, $P_r = aT^4/3 = (1-\beta)P$. Eliminate T in favor of β:

$$\frac{a}{3}\left(\frac{\beta P \bar{m}}{\rho k_B}\right)^4 = (1-\beta)P$$

So from here you can see how $P = K\rho^{4/3}$ comes out.

IV.2.1 Minimum Mass

Need central conditions extreme enough to sustain *pp* burning. Consider a collapsing cloud of mass M. Kelvin-Helmholtz contraction, so all of energy is from contraction (gravity) not fusion. Looks like an ideal gas:

$$P_c = \frac{\rho_c}{\bar{m}} k_B T_c$$

The contraction will be slow and close to HSE if the pressure is almost enough to balance the star. Equating the two pressures:

$$k_B T_c \approx \left(\frac{\pi}{6}\right)^{1/3} G \bar{m} M^{2/3} \rho_c^{1/3}$$

So $T_c \propto \rho_c^{1/3}$, which goes up during contraction. Contraction will continue until T is enough for fusion or electrons become degenerate — either way the center will be supported against further contraction. So it will not be a star if center is degenerate before fusion.

Assume that electrons have become degenerate. Then:

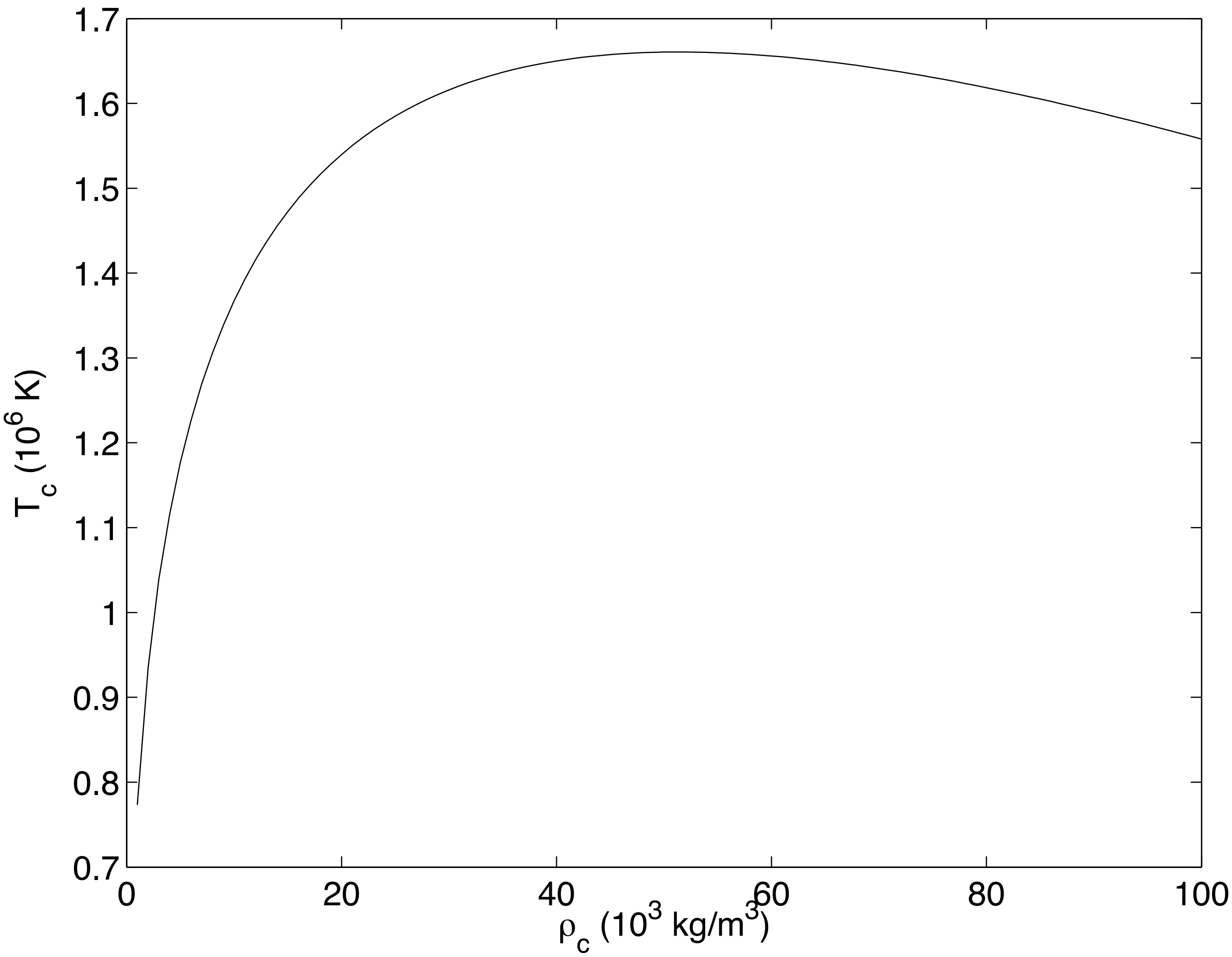
$$P_c = K_{\rm NR} n_e^{5/3} + n_i k_B T_c \approx K_{\rm NR} \left(\frac{\rho_c}{m_{\rm H}}\right)^{5/3} + \frac{\rho_c}{m_{\rm H}} k_B T_c$$

Set this equal to our P_c from before:

$$k_B T_c \approx \left(\frac{\pi}{6}\right)^{1/3} G m_{\rm H} M^{2/3} \rho_c^{1/3} - K_{\rm NR} \left(\frac{\rho_c}{m_{\rm H}}\right)^{2/3}$$

So this is the temperature when the electrons are degenerate but the ions are not. What is the maximum temperature that will be reached?

$$k_B T_c = A\rho_c^{1/3} - B\rho_c^{2/3}$$


1.7
1.6
1.5
1.4
1.3
1.2
1.1
1
0.9
0.8
0.7
T_c (10^6 K)
0
20
40
60
80
100
ρ_c (10^3 kg/m^3)

Can differentiate and find maximum. This is at $k_B T_c = A^2/4B$, and $\rho_c = (A/2B)^3$. Or:

$$k_B T_{c,\max} \approx \left(\frac{\pi}{6}\right)^{2/3} \frac{G^2 m_{\rm H}^{8/3}}{4K_{\rm NR}} M^{4/3}$$

Can then solve for $M_{\rm min}$ needed to have $T_c \geq T_{\rm ignition}$. For a rough estimate, use $T_{\rm ign} = T_{c,\odot}/10 = 1.5 \times 10^6$ K. This gives $M_{\rm min} = 0.05\, M_\odot$, which isn't bad. Real calculations say closer to 0.08 $M_\odot$.

IV.2.2 Maximum Mass

Things get tricky if pressure is from relativistic particles with $\gamma = 4/3$ (nearly unstable). Which will happen if radiation supplies most of the pressure.

$$P_g = \frac{\rho}{\bar{m}} k_B T_c = \beta P_c$$

and

$$P_r = \frac{a}{3} T_c^4 = (1-\beta) P_c$$

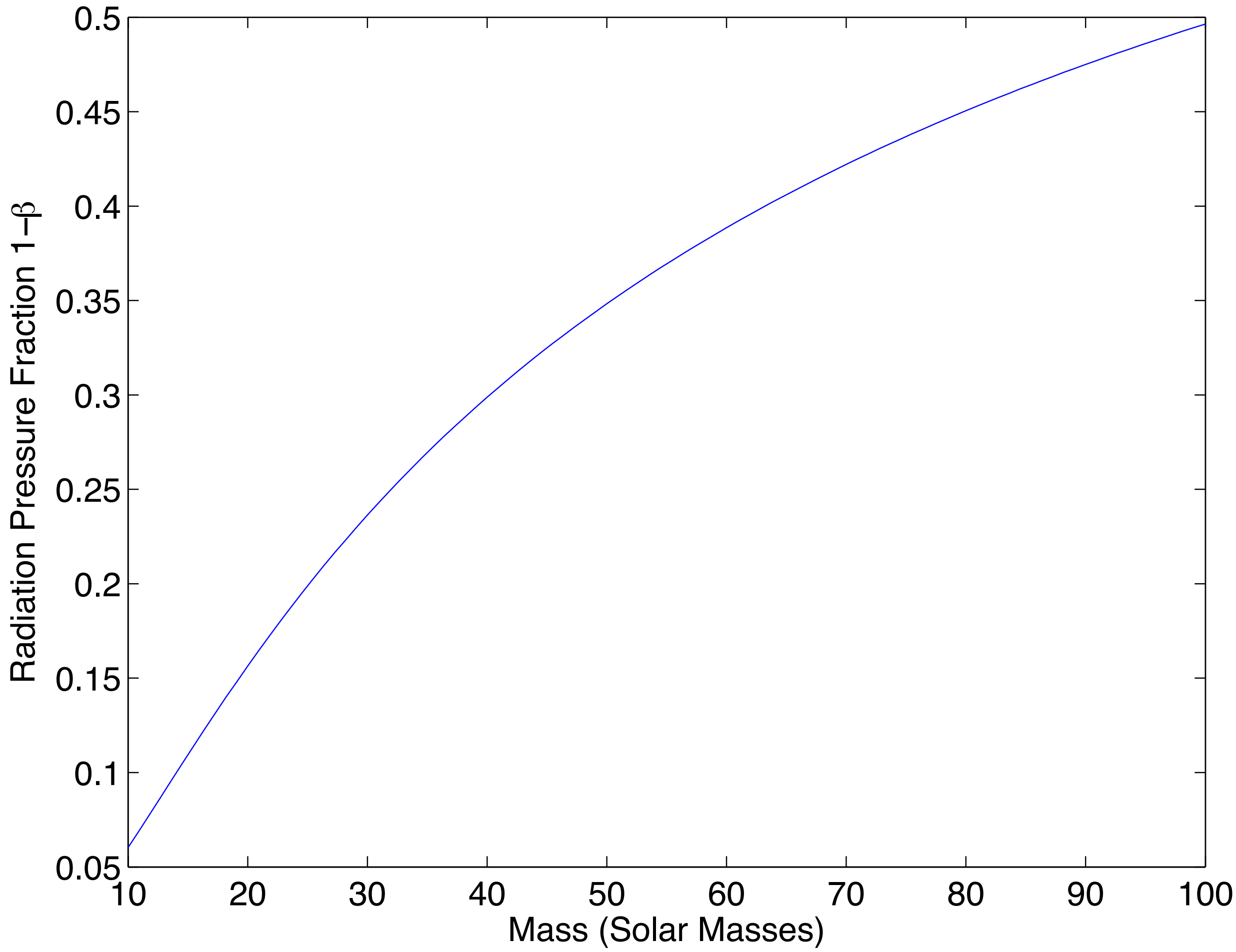
where β is the fraction of total pressure supplied by ions and electrons.

$$P_c = \left(\frac{3}{a}\frac{(1-\beta)}{\beta^4}\right)^{1/3} \left(\frac{k_B \rho_c}{\bar{m}}\right)^{4/3}$$

Equate this to pressure needed to support the star and get:

$$\left(\frac{\pi}{36}\right)^{1/3} G M^{2/3} = \left(\frac{3}{a}\frac{(1-\beta)}{\beta^4}\right)^{1/3} \left(\frac{k_B}{\bar{m}}\right)^{4/3}$$

Radiation pressure gets more important as the mass increases.



When $M > 100 M_\odot$, $1 - \beta > 0.5$ and the star is very unstable. Even $> 50 M_\odot$ is very rare, but then gain these stars do not live for a long time so they are hard to spot.

Lecture V Stellar Evolution

V.1.3 Low-Mass Stars

Main-sequence: core H fusion. If the star is $< 0.5\,M_\odot$ or so, will never fuse helium. May eventually become a red giant (degenerate He core, surrounded by H burning shell, surrounded by puffy envelope) and then a white dwarf, but this can take hundreds of billions of years.

V.1.4 Middle-Mass Stars

Main sequence lasts for Gyr. Eventually, only He in core. Contracts, increasing pressure and T but not enough for He to ignite. Becomes degenerate. Outside the core H burns in shell. Envelope puffs up, becomes red giant. Ascends the red giant branch (RGB).

H fusion continues to produce He. This "falls" into the core, making it contract further. Eventually, might get He fusion. Since this happens in a degenerate core (when $M < 1.5\,M_\odot$ or so), it will start as an unstable "flash" (raise T does not affect P), but the flash will take a long time to propagate through star so it will not really effect things too much. But extra energy will expand core, making things non-degenerate evetually. Moves to horizontal branch (hotter and smaller). This is basically a He-burning main-sequence, but it is much faster since the reaction is hotter and there is less fuel.

Eventually will exhaust He in core. Contracts again, looks a lot like RGB. Call this phase the asymptotic giant branch (AGB). Moves up again, things become somewhat unstable. Pulsations fling off outer layers, lead to planetary nebula (PN) and white dwarf (WD).

V.1.5 More Massive Stars

Core never becomes degenerate. So do not become much brighter on the RGB — mostly just become redder. Eventually ignite He, but it is a more gentle process. Move to horizontal branch (HB: helium burning main sequence). Exhaust He, then contract again up AGB. Get rid of outer layers, end up as PN and WD.

V.1.6 Massive Stars

Keep plowing through fusion, making more and more massive elements. Luminosity scales up steeply with mass: $L \propto M^3$ or M^4. Available fuel depends linearly on the mass. So timescale for evolution (nuclear timescale) is $\tau_{\rm nuc} \propto {\rm fuel}/{\rm consumption\,rate} \sim M/L \sim M^{-2.5}$. Which means that massive stars burn through their fuel very quickly.

For instance, for a $25\,M_\odot$ star:

Stage	Timescale	$T/10^9$ K	ρ
H burning	7×10^6 yr	0.06	5×10^1
He burning	5×10^5 yr	0.23	7×10^2
C burning	600 yr	0.93	2×10^5
Ne burning	1 yr	1.7	4×10^6
O burning	6 mo	2.3	1×10^7
Si burning	1 day	4.1	3×10^7

This will be followed by (in general) a core-collapse supernova because further fusion is unable to generate any energy: no exothermic reactions are possible.

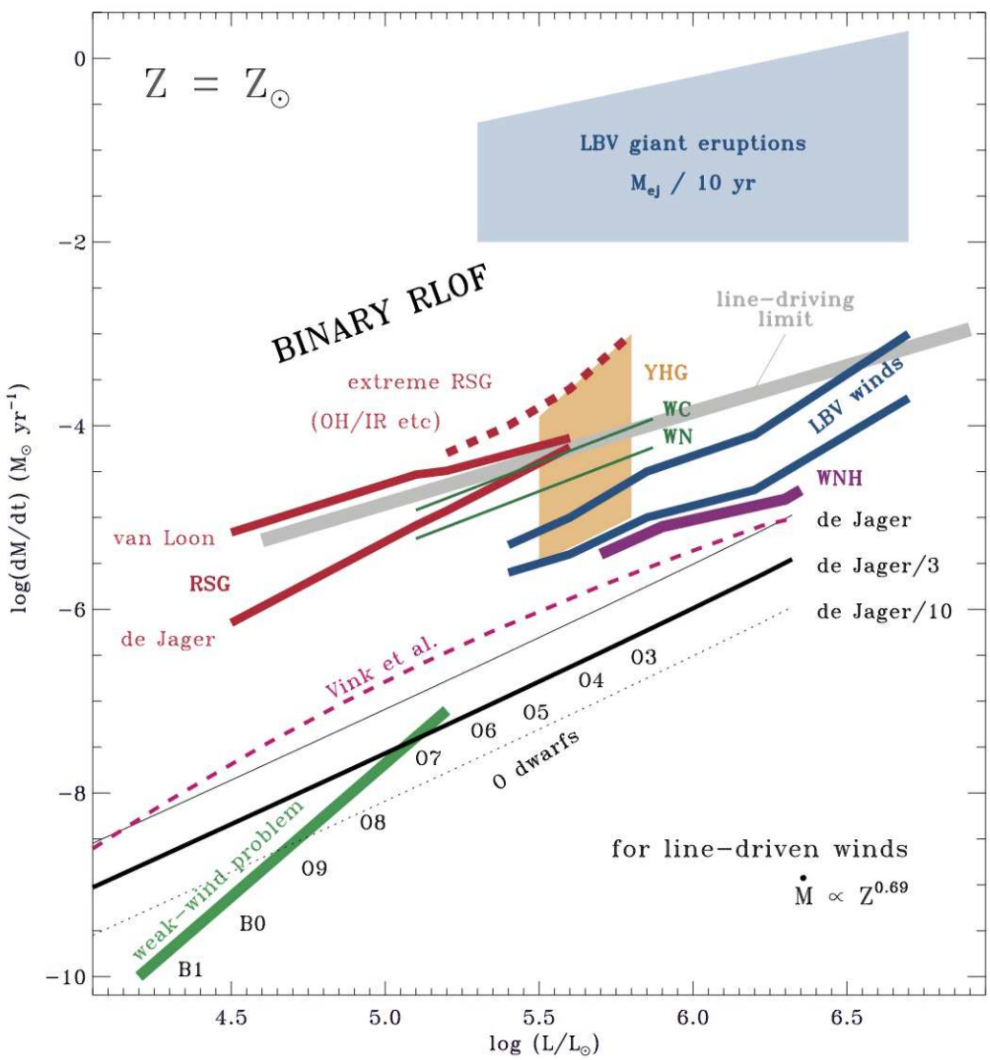
However, mass-loss also dominates the end products of these stars. The later stages of evolution are very large, so the outer regions are very loosely bound. Even in a $1\,M_\odot$ star this is significant. At $T = 5 \times 10^4$ K inside a star, the kinetic energy $(3/2)k_BT \sim 7$ eV. At a radius of 200 $R_\odot$ (AGB) the binding energy will be similar, and such particles can escape the star.

Overall mass-loss is complicated and we have to determine the results semi-empirically. For instance, there is an expression (Garmani & Conti 1984):

$$\dot{M} \approx -22 \times 10^{-8} \left(\frac{L}{10^3\,L_\odot}\right)^{3.7} \left(\frac{M}{M_\odot}\right)^{-3.1} \left(\frac{R}{10^2\,R_\odot}\right) M_\odot\,\mathrm{yr}^{-1} \tag{V.123}$$

This has a lot of free terms. But we can relate the mass to the radius and luminosity for different phases of evolution. In the end we find that mass-loss can be quite significant for high-mass stars.

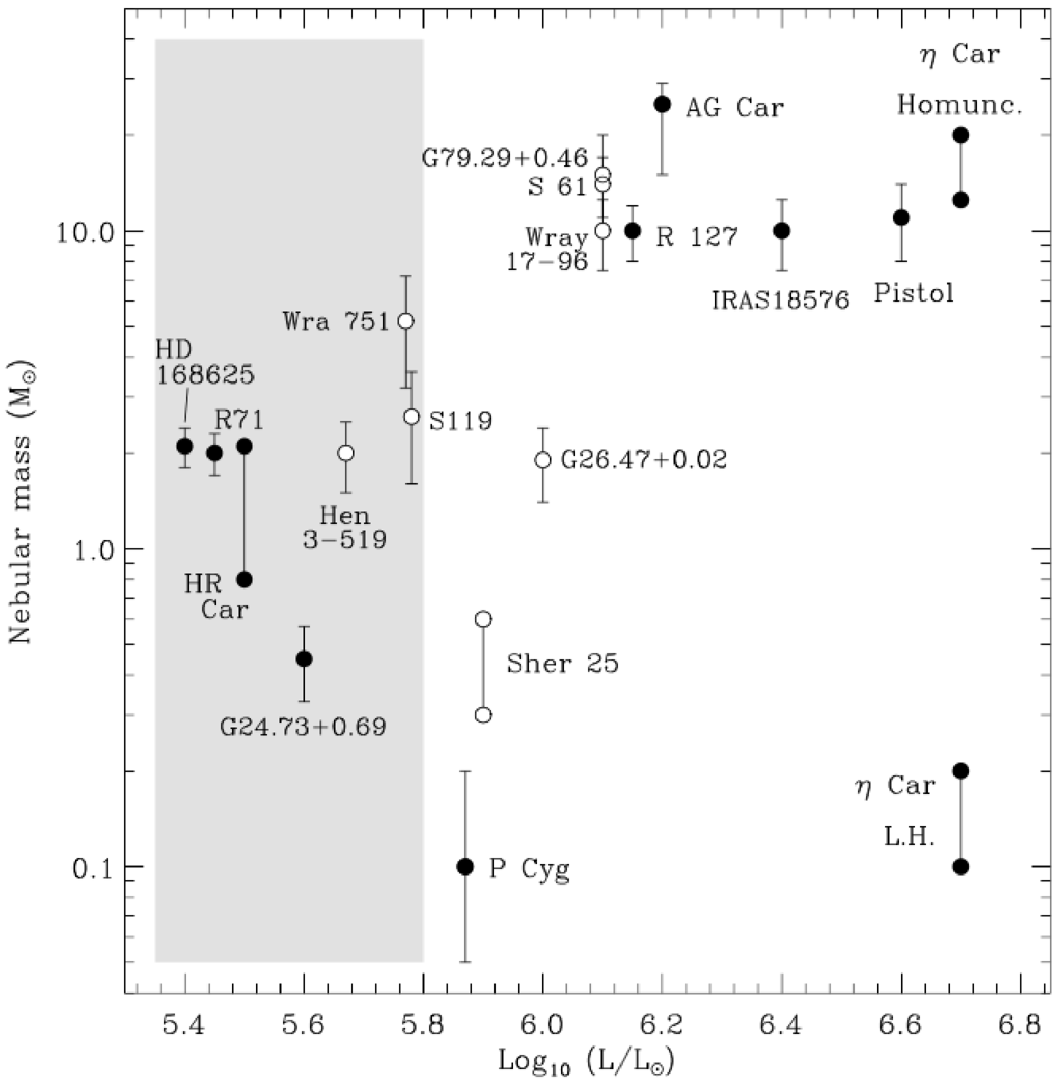
This plot, from Smith (2014, ARA&A), shows different prescriptions for mass-loss as well as typical observed values. Much mass-loss may not be steady, but may be through luminous blue variable (LBV) eruptions or clumpy winds, which make measuring and modeling much harder.



To complicate further, during the AGB phase we have large-amplitude radial pulsations driven by instabilities in the atmosphere, as well as a large increase in the luminosity.

The most massive stars that we know of are also very short-lived, rare, and far away. So hard to understand. But examples in the Milky Way are the Pistol Star and Eta Carina. Both of these are likely 10's of $M_\odot$ and have lost 10's of $M_\odot$ in significant outflows. So the mass of the star right before supernova is not necessarily very close to what it was when it was born. Binary interactions can further modify the result.

This plot from Smith (2014) shows the estimates of mass in shells around LBV stars. The stars are very luminous and the shells are very massive.



Very massive stars ($\gtrsim 100\,M_\odot$) will form. However, above $60\,M_\odot$ or so, pulsations occur driven by instabilities related to opacity changes and to nuclear power changes (κ and ϵ mechanisms) that drive a lot of mass-loss. Add to this the instability from a large amount of radiation pressure and they are not very stable.

Likely in the early universe at lower metallicity (fewer generations of stars) the mass-loss was less efficient. At very low metallicities the mass-loss prescriptions can change dramatically. That is because much of mass-loss is determined by very weakly bound sites in the outer atmosphere. Here parcels of gas are carried away by momentum exchange between the outgoing flux and gas due to opacity. If the gas is only H, then the opacity is low because there are very few atomic transitions. So we expect more massive stars early on. These most massive stars may not go through the normal channels to end up at core-collapse SN. Instead they may have a significant pair-instability mechanism, that can lead to violent pulsations and mass loss or even complete disruption through a supernova leading to no remnant (for a He core of $65\,M_{\odot}$, corresponding to an initial mass of $140\,M_{\odot}$).

The physics here is that radiation pressure is very significant. If the radiation field is sufficiently energetic such that $\gamma + \gamma \rightarrow e^{-} + e^{+}$ is favored, the pressure support will be reduced and the core will contract. Contraction will heat up the core, meaning more γ-rays and more pair production. This cascades until support is gone and the core collapses. The core has runaway thermonuclear reactions that release enough energy to disrupt the core, leaving nothing behind.

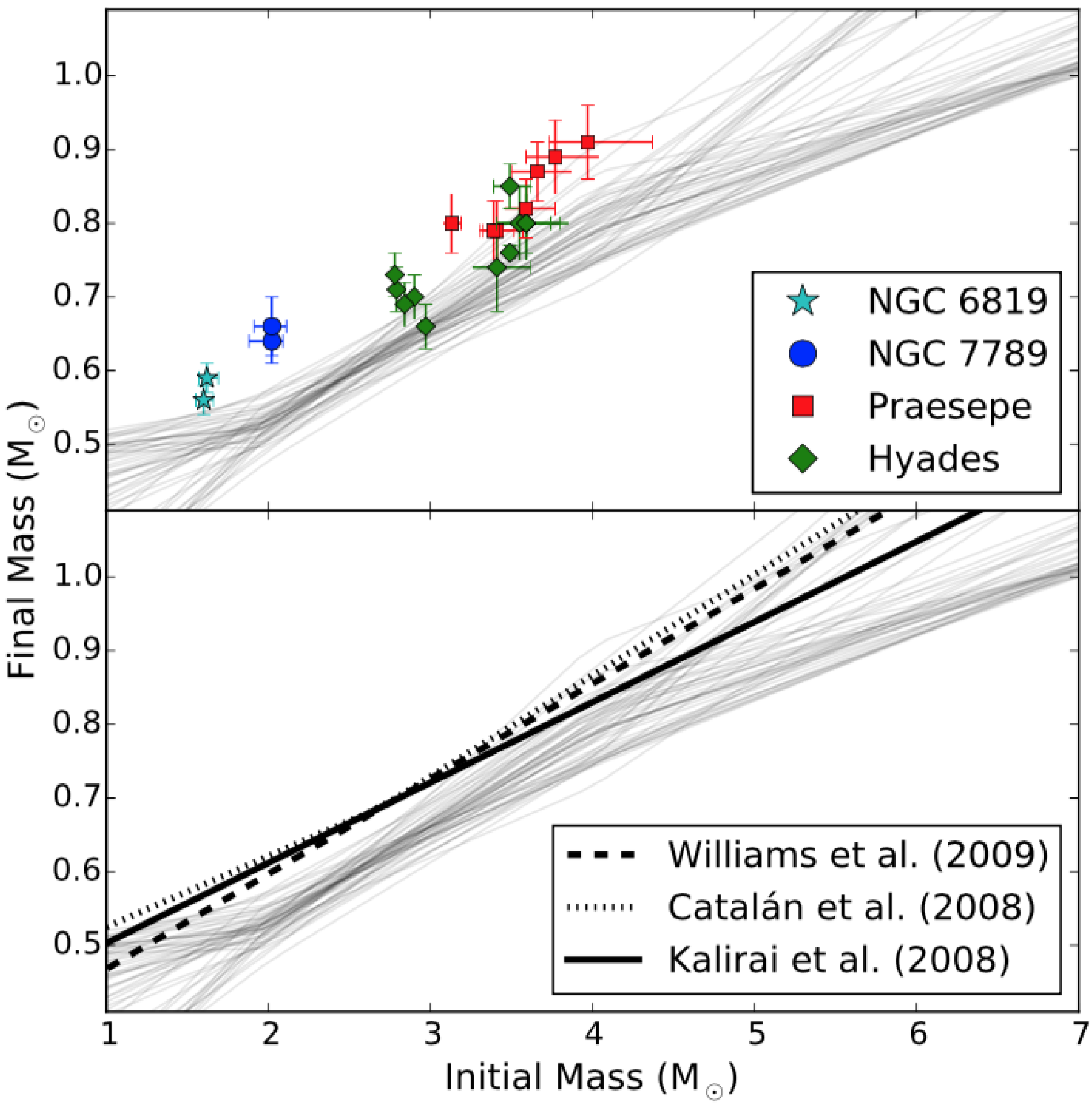
V.2 Diversion: How do we measure the mass of a star?

Methods:

1. Main-sequence fitting: requires knowledge that it is on the main sequence; requires estimate of foreground extinction (dust); requires assumption that it is a single star
2. Binaries: spectroscopic binaries have inclination degeneracy; eclipsing binaries (where we know inclination) are rare; can worry that binary evolution has altered single-star expectations
3. Asteroseismology: can be very precise, but only available for a small number of stars

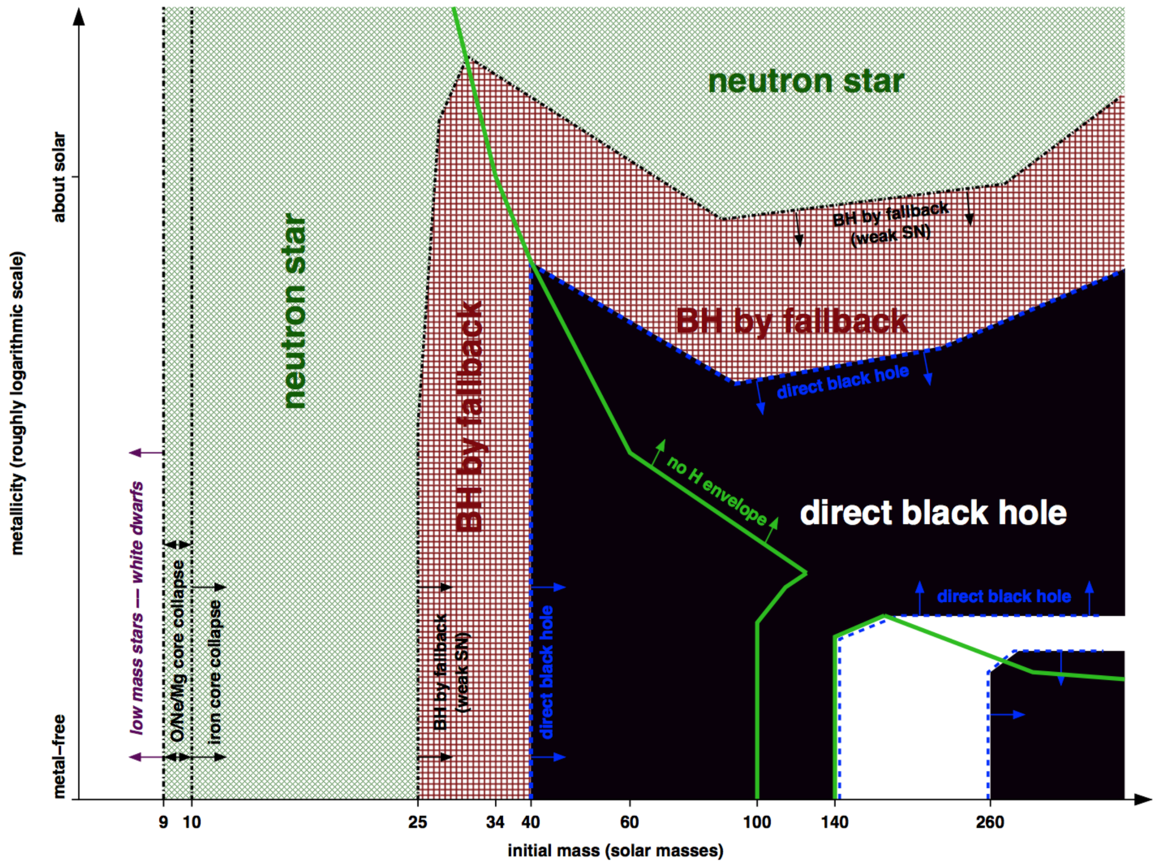
Lecture VI Stellar End Products

If the mass if $< 8\,M_\odot$ or so, degenerate core of the star ends up as a white dwarf.



Efforts to map the final WD product from the main-sequence mass are difficult. The main question is one of mass-loss: how much mass is lost in the main-sequence (small), RGB/AGB (considerably more) and PN formation (a lot) stages. In this plot from Andrews et al. (2015), the main differences between the different models (thin lines) are how they treat mass loss.

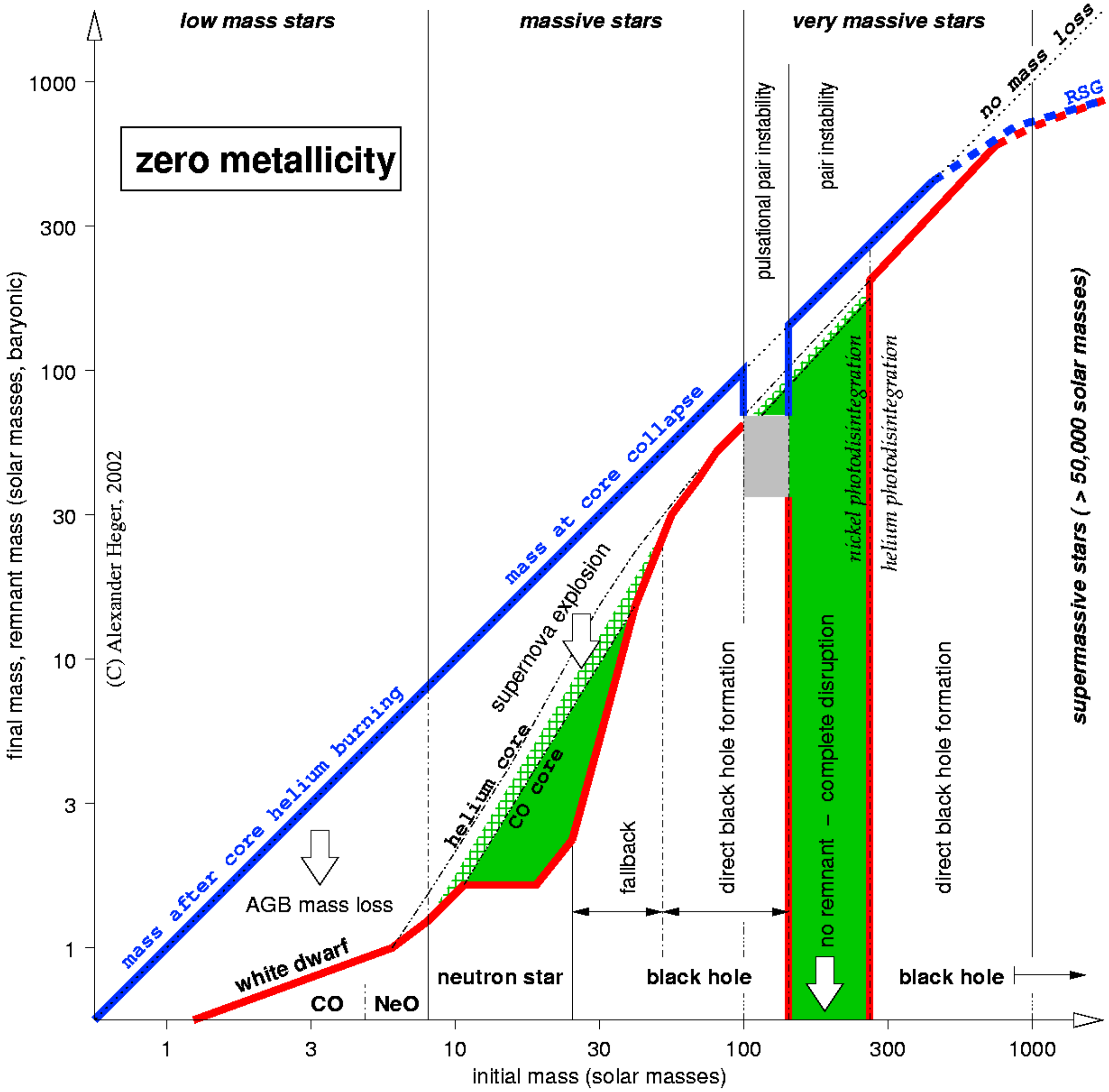
Extending to higher masses we can have counter-intuitive behavior from Heger et al. (2003):



At low masses the behavior is as expected: WDs transitioning to NSs. But eventually we need at least one more variable, which here is metallicity. That is because of the effects of mass-loss. There are possibly different regimes of black hole (BH) formation, both direct collapse and "fall-back", where there is an initial neutron star surrounded by a disk of material that drives it over its maximum mass limit. So there is the possibility of non-monatonic results, where more massive stars can have less massive products depending on metallicity. And there is also the regime of no product, in the pain-instabillity supernova gap.

Other uncertainties can come from the effects of metallicity, rotation, magnetic fields, etc. All of those can change the mass loss and are much harder to measure. So when inferring properties from small sub-samples can make systematic errors.

For example, see `http://2sn.org/firststars/`:



which may be relevant to the most massive stars produced in the early universe.

Lecture VII Core Collapse

VII.1.1 Advanced Burning

Carbon burning:

$$^4\text{He} +^{12}\text{C} \rightarrow^{16}\text{O} + \gamma \tag{VII.124}$$

(which is good for us). Straightforward reaction w/o resonance or unstable meta-state. Then

$$^4\text{He} +^{16}\text{O} \rightarrow^{20}\text{Ne} + \gamma \tag{VII.125}$$

Can have additional He captures to make other elements (Ne, Mg, Si), although this doesn't happen too much during He burning since the temperature isn't high enough.

These reactions bypass Li, Be, B. We do not see very much of these, and most of what we do see comes not from stars but from a cosmic ray hitting a heavy nucleus and splitting it (spallation).

Eventually hotter and hotter. Core of C and O builds up (if the star isn't massive enough, this ends up as WD). C burning starts at 5×10^8 K, with reactions like:

$$^{12}\text{C} +^{12}\text{C} \rightarrow^{20}\text{Ne} +^4\text{He} \tag{VII.126}$$

(or ^{23}Na or ^{23}Mg). If the star is a little more massive (8–10 $M_\odot$) things can ened here with O/Ne/Mg WD.

Then Ne burning if $> 10^9$ K, making ^{24}Mg. Important next step is

$$^{16}\text{O} +^{16}\text{O} \rightarrow^{28}\text{Si} +^4\text{He} \tag{VII.127}$$

($> 2 \times 10^9$ K) followed by Si burning (3×10^9 K). The reactions mostly involve heavy nuclei + light particles made from breaking up the heavier ones, gets rather complicated.

This break-up happens when photons have enough energy to split apart nuclei. E.g.,

$$\gamma +^{28}\text{Si} \rightarrow^{24}\text{Mg} +^4\text{He} \tag{VII.128}$$

Just like ionization, but at temperatures 10^6 times higher.

All this happens very quickly. And it needs a very massive star to keep going, where everything happens more quickly than in the Sun.

For a 15 $M_\odot$ star:

Stage	Timescale	Reaction	Product	$T/10^9$ K	ρ (cgs)	$L/L_\odot$	$L_\nu/L_{\nu,\odot}$
H burn	11 Myr	pp	He	0.035	5.8	28,000	1800
		CNO	He,N,Na				
He burn	2.0 Myr	3α	C	0.18	1390	44,000	1900
		$^{12}\text{C} + \alpha$	O				
C burn	2000 yr	$^{12}\text{C} +^{12}\text{C}$	Ne,Na,Mg,Al	0.81	2.8×10^5	72,000	3.7×10^5
Ne burn	0.7 yr	$^{20}\text{Ne} + \gamma$	O,Mg,Al	1.6	1.2×10^7	75,000	1.4×10^8
O burn	2.6 yr	$^{16}\text{O} +^{16}\text{O}$	Si,S,Ar,Ca	1.9	8.8×10^6	75,000	9.1×10^8
Si burn	18 d	$^{28}\text{Si} + \gamma$	Fe,Ni,Cr,Ti,...	3.3	4.8×10^7	75,000	1.3×10^{11}
Fe collapse	1 s	neutronization	neutron star	> 7.1	$> 7.3 \times 10^9$	75,000	$> 3.6 \times 10^{15}$

Remember that overall luminosity is limited by Eddington limit, $L_{\rm Edd} = 3.2 \times 10^4\,L_\odot(M/M_\odot)$, so we are at $\approx 15\%$ of Eddington. Neutrinos have no such limit, so can be much higher.

Star with $> 10\,M_\odot$ (or so). Will go through all stages of nuclear burning in $< 10\,$Myr. Eventually have Si burning making iron at $T = 3\times10^9$ K, surrounded by shells of lighter elements. Cannot get energy out of iron via fusion, so core contracts (just like RGB). Stabilized somewhat by degenerate electrons, but Si burning dumps increasing amounts of stuff on and electrons get increasingly relativistic. When the core is at $M_{\rm Ch} \approx 1.4\,M_\odot$, electrons have become ultra-relativistic and the core can no longer support itself.

VII.1.2 Onset of Collapse

During contraction T rises. If makes exothermic reactions possible, then T and pressure rise and collapse stops. But what if no exothermic reaction is possible? If only endothermic, reduces P, makes contraction into collapse. Once $kT > 1\,$MeV (10^{10} K) you can also have direct neutrino production via:

$$\gamma + \gamma \rightarrow e^+ + e^- \rightarrow \nu_e + \bar{\nu}_e \tag{VII.129}$$

which further increases the neutrino luminosity and destabilizes the star.

Possible reactions are photodisintigration of nuclei and electron capture (inverse β decay). Photodisintigration: KE is used to unbind nuclei. Electron capture: KE of electrons is converted into KE of neutrinos (and lost). These both suck up energy very effectively, turning contraction into free-fall. At this point $\rho \approx 10^9\,{\rm g\,cm^{-3}}$, and free-fall happens with:

$$\tau_{\rm ff} = \sqrt{\frac{3\pi}{32G\rho}} \approx 1\,ms \tag{VII.130}$$

VII.1.3 Photodisintigration

T rises enough such that photons have nuclear-scale energies. Takes a tightly-bound Fe nucleus and makes two or more loosely bound nuclei, absorbing binding energy. This can take many paths, but as an example:

$$\gamma +^{56}{\rm Fe} \leftrightarrow 13^4{\rm He} + 4n \tag{VII.131}$$

equilibrium between iron and helium + neutrons. This takes:

$$Q = (13m_4 + 4m_1 - m_{56})c^2 = 124.4\,{\rm MeV} \tag{VII.132}$$

So $1\,$kg of Fe can absorb 2×10^{21} erg (50 kton of TNT). Use Saha equation to determine relative fractions:

$$\mu_{56} = 13\mu_4 + 4\mu_1 \tag{VII.133}$$

with

$$\mu_A = m_Ac^2 - k_BT\ln\left(\frac{g_An_{Q,A}}{n_A}\right) \tag{VII.134}$$

and

$$n_{Q,A} = \left(\frac{2\pi m_A k_B T}{h^2}\right)^{3/2} \tag{VII.135}$$

which gives:

$$\frac{n_4^{13} n_1^4}{n_{56}} = \frac{g_4^{13} g_1^4}{g_{56}} \frac{n_{Q,4}^{13} n_{Q,1}^4}{n_{Q,56}} e^{-Q/k_B T} \tag{VII.136}$$

The g factors can be complicated, but we will assume $g_1 = 2$, $g_4 = g_{56} = 1$. From this we get that roughly 75% of the Fe is dissociated when $\rho = 10^9\,\mathrm{g\,cm^{-3}}$ and $T = 10^{10}$ K.

For higher temperatures still:

$$\gamma +^4 \mathrm{He} \leftrightarrow 2p + 2n \tag{VII.137}$$

Overall, in collapse of $1.4\,M_\odot$, absorb 4×10^{51} erg in breaking Fe and 1×10^{52} erg in breaking He, for a total of $E_{\text{photo}} \approx 1.4 \times 10^{52}$ erg. This is $L_\odot \times 10^{11}$ yr. Easy to see how this could lead to collapse.

VII.1.4 Electron Captures

Neutron can decay on its own (β decay):

$$n \to p + e^- + \bar{\nu}_e \tag{VII.138}$$

with half-life of 10.25 min. This produces electrons and neutrinos with total energy of $(m_p - m_n)c^2 = 1.3\,\mathrm{MeV}$, so the max electron energy is $1.3\,\mathrm{MeV}$. If electrons with that energy cannot be produces, neutrons cannot decay. For instance, if all of the low-energy spots are filled by other electrons (in a dense gas of degenerate electrons with $E_F > 1.3\,\mathrm{MeV}$) this cannot happen.

Moreover, if electrons with $E > 1.3\,\mathrm{MeV}$ are around, they can capture onto protons to form neutrons:

$$e^- + p \to n + \nu_e \tag{VII.139}$$

This can happen even if the protons are in nuclei. For instance, *neutronization* starts when:

$$e +^{56} \mathrm{Fe} \to^{56} \mathrm{Mn} + \nu_e \tag{VII.140}$$

is favorable, at $\rho > 1.2 \times 10^9\,\mathrm{g\,cm^{-3}}$. This happens when $E_F = m_e c^2 + 3.7\,\mathrm{MeV}$. The Mn would normally decay back in 2.6 hr, but here instead it will capture again to make ^{56}Cr. And so on as the density goes up past $10^{10}\,\mathrm{g\,cm^{-3}}$.

This speeds up further when $\rho > 10^{11}\,\mathrm{g\,cm^{-3}}$. Almost all of the energy in neutrinos is lost. So the pressure support goes away quickly. How much energy? Core has $\sim 10^{57}$ electrons, which could make 10^{57} neutrinos. Each capture will take an electron with $E \approx 10\,\mathrm{MeV}$, appropriate for $\rho > 2 \times 10^{10}\,\mathrm{g\,cm^{-3}}$. So total energy is:

$$E_{\text{cap}} \approx 10^{57} \times 10\,\mathrm{MeV} = 1.6 \times 10^{52}\,\mathrm{erg} \tag{VII.141}$$

which is similar to that from photodisintigration. But in this case it is carried from the star in a burst of neutrinos. If they could get out immediately, the burst would take $\sim$ ms. But in fact when the core density is $> 10^{11}\,\mathrm{g\,cm^{-3}}$ the mfp becomes comparable to the size of the core, a few km. They will get out, but it will take a few seconds.

VII.1.5 And Then...

The collapse will proceed on the free-fall timescale. What will stop it? It will stop when the bulk density is comparable to the nuclear density. For a nucleus with A nucleons, $R \approx r_0 A^{1/3}$ with $r_0 = 1.2$ fm. So $\rho_{\rm nuc} = 3m_n/4\pi r_0^3 = 2.3 \times 10^{14}\,\mathrm{g\,cm^{-3}}$. Once we are that this stage we need new physics (neutron degeneracy, nuclear forces). The collapse will stop when the density is a few times this as strong nuclear force comes in, and creates a "bounce". This propels a shock wave through the material, leading to a supernova.

Supernovae are observed to have 10^{51} erg of KE and 10^{49} erg of optical energy (over the first few years). Where does this come from? Gravitational binding energy:

$$E_G \sim \frac{GM^2}{R_{\rm core}} = 3 \times 10^{53}\,\mathrm{erg} \left(\frac{M}{M_\odot}\right)^2 \frac{10\,\mathrm{km}}{R} \tag{VII.142}$$

Which is orders of magnitude more than we see. We only see a small fraction of this, and we don't quite know exactly how the energy is partitioned. But this is plenty of energy compared to photodisintigration or electron capture. Most of the energy in fact comes via neutrinos, either right during collapse or later, as the neutron star cools. This happens over the diffusion timescale, $R^2/c\bar{l}$, and each flavor of neutrinos will carry $\sim E_G/6$.

On Feb 23, 1987, two neutrino detectors recorded excesses. They identify neutrinos via:

$$\bar{\nu}_e + p \to n + e^+ \tag{VII.143}$$

and if the positron has enough energy, it will be faster than the local speed of light in water, so it will emit Čerenkov radiation. That can be detected. Only ~ 1 in 10^{15} neutrinos is expected to be detected.

Saw about 20 neutrinos over $\sim 10\,\mathrm{s}$. Expect that this is the diffusion timescale, which makes sense if $R \approx 100$ km and $\bar{l} = 10^{-4}R$. This came from SN 1987A in the LMC, implying an energy of about $(0.3 - 0.5) \times 10^{53}$ erg for $\bar{\nu}_e$.

Energies of the neutrinos is consistent with $T_{\rm Eff} \approx 5 \times 10^{10}\,\mathrm{K}$. Compare to internal temperature, using the mfp:

$$T_{\rm Eff} \approx \left(\frac{\bar{l}}{R}\right)^{1/4} T_I \tag{VII.144}$$

implies an internal temperature of $10^{11-12}\,\mathrm{K}$.

Lecture VIII The Neutrino Mechanism

As we have discussed, when the core of a massive star reaches the Chandrasekhar mass, it can collapse. The collapse of the core to radii ~ 30 km releases a huge amount of energy – typically per nucleon

$$E \approx \frac{GMm_p}{R} \approx 62 \left(\frac{M}{1.4\,M_\odot}\right)\left(\frac{R}{30\,\mathrm{km}}\right)^{-1}\ \mathrm{MeV} \tag{VIII.145}$$

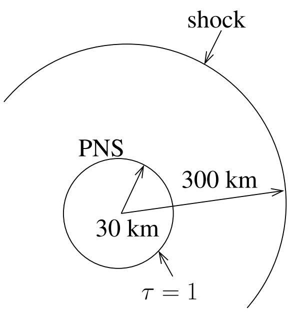
, which is much greater than the 7 MeV that typically binds the nuclei together. As a result, the iron is photo dissociated to alpha particles, i.e., helium nuclei.

The collapse core reaches nuclear densities and bounces to a radius of $r \sim 30$ km, which drive a shock wave into the in falling material and we have a hot protoneutron star. This shock wave moves out to about 300 km where it stalls – balanced by ram pressure and in falling material.

In the protoneutron star, it is so hot that everything dissociates to nucleons. The density is also so high that it is energetically favorable to combine protons and electrons to make neutrons in a process known a neutronization.

$$p + e \to n + \nu_e, \tag{VIII.146}$$

where the ν_e would normally escape to infinity. However, now the density of the protoneutron star is so high that the neutrinos don't immediately escape to infinity, but must diffuse out. So a picture of the system can be visualize as:



We can estimate the neutrino photosphere, i.e., the $\tau = 1$ surface by consider the cross section:

$$\sigma_\nu = 10^{-45}\left(\frac{E_\nu}{m_e c^2}\right)^2 A^2\,\mathrm{cm}^2, \tag{VIII.147}$$

where E_ν is the energy of the electron neutrino. Now the electrons are degenerate and have Fermi energy

$$E_F = 1\left(\frac{\rho}{10^6\,\mathrm{g\,cm^{-3}}}\right)^{1/3}\mu_e^{-4/3}\,\mathrm{MeV}, \tag{VIII.148}$$

where $\mu_e = A/Z$ is the nucleon to electron number densities. For a density of 10^{12} and $\mu_e = 2$, we find $E_F \approx 40$ MeV. The mean free path of these neutrinos is then

$$\lambda_{\mathrm{mfp}} = \frac{1}{n\sigma_\nu} = \frac{Am_p}{\rho\sigma_\nu} = 1\left(\frac{\rho}{10^{12}\,\mathrm{g\,cm^{-3}}}\right)^{-5/3}\left(\frac{A}{2}\right)^{-1}\ \mathrm{km} \tag{VIII.149}$$

The diffusion time is then

$$t_{\text{diff}} = \frac{R^2}{\lambda_{\text{mfp}} c}. \tag{VIII.150}$$

For nuclear density $\rho \sim 10^{14}\,\text{g}\,\text{cm}^{-3}$ and $R \sim 30$ km, gives $\lambda_{\text{mfp}} = 50$ cm and $t_{\text{diff}} = 6$ s, which is close to the right answer of 10 s. The photosphere is at $\tau = 1 = n\sigma_\nu R \to \lambda_{\text{mfp}} \sim R$. This gives $\rho_{\tau=1} = 1.3 \times 10^{11}\,\text{g}\,\text{cm}^{-3}$.

So we know that the diffusion time is ~ 10 s and the energy of the collapse is

$$E = \frac{GM^2}{R} \sim 10^{53}\,\text{ergs} \to L = \frac{E}{t_{\text{diff}}} \sim 10^{52}\,\text{ergs}\,\text{s}^{-1}. \tag{VIII.151}$$

So we can tap $\sim 1\%$ of the neutrino energy to get the energy of an observed SN, which is 10^{51} ergs.

To see if this is possible, let's look at how core collapse SN might proceed. The neutrinos from the photosphere illuminate the region behind the shock heating it up and adding to its energy. This region is known as the **gain region**.

The material that is falling into the gain region from the shock is order $0.1\,M_\odot\,\text{s}^{-1}$. Assuming that is comes in through 4π, we have

$$\dot{M} = 4\pi\rho r_s^2 v \tag{VIII.152}$$

For $v = \sqrt{GM_{\text{PNS}}} r_s$, we can show that $\rho \approx 10^{10}\,\text{g}\,\text{cm}^{-3}$ giving a mass of the gain region of $M_g = 4\pi r_s^3\rho/3 \approx 0.01\,M_\odot$. The amount of optical depth is then

$$\tau = n\sigma r_s = 6.3 \times 10^{-5} \left(\frac{r_s}{100\,\text{km}}\right) \left(\frac{\rho}{10^{10}\,\text{g}\,\text{cm}^{-3}}\right) \left(\frac{E_\nu}{m_e c^2}\right)^2 A \tag{VIII.153}$$

Now we can estimate average energy of the neutrino from the luminosity:

$$L_\nu = 10^{52}\,\text{ergs}\,\text{s}^{-1} = 4\pi R^2 \sigma T_e^4 \to T_e \approx 3\,\text{MeV} \tag{VIII.154}$$

This gives an average neutrino energy of $E_\nu \approx 4k_B T_e \approx 12$ MeV. The factor of 4 comes from Fermi-Dirac statistics. It would be 3 for Bose-Einstein statistics, i.e., photons. In any case, we find

$$\tau = 0.04 - 0.08 \left(\frac{r_s}{100\,\text{km}}\right) \left(\frac{\rho}{10^{10}\,\text{g}\,\text{cm}^{-3}}\right) \left(\frac{T_e}{3\,\text{MeV}}\right)^2 \tag{VIII.155}$$

This corresponds to a total heating rate of $\dot{E} = L_\nu \tau$ for $\tau \ll 1$ so in principle there is both plenty of heating and plenty of coupling to get SN energies of 10^{51} ergs. In fact, it should be possible to get energies of order 10^{52} ergs, but the best 3-d calculations get at most 10^{51} ergs. The reasons for this is that the stuff that is falling through that gain region is falling onto the star and does not stay there forever. If it would stick around for 10 seconds they this is a possibility, but it does not. Rather the time it stays in the gain region is called the residence time

$$t_{\text{res}} = \frac{M_g}{\dot{M}} = \frac{r_s}{v} = 0.1 s. \tag{VIII.156}$$

So the total energy absorbed is

$$E_{\rm SN} = L_\nu \tau t_{\rm res} \approx 4 - 10 \times 10^{-3} L_\nu, \tag{VIII.157}$$

If we say L_ν starts out a bit larger initially – say few $\times 10^{52}\,\mathrm{ergs\,s^{-1}}$, then we can almost get up to the 10^{51} ergs required for a SN explosion.

This neutrino mechanism is by far the most popular story by which SN explosion occur. This is because it is the greatest reservoir of energy and the fact that neutrinos were detected from SN 1987A. However, its coupling and hence efficiency is low. Other mechanisms include:

- Rapid spin and magnetic despinning – requires PNS to be born with large magnetic fields
- Jets from accreting material - Possible, but unclear how jets form in these environments – though this appear to be the case for long-GRBs
- Thermonuclear explosions - most recent idea, not yet well explored.

Lecture IX Supernova Explosions

Collapse of iron core of massive star.

10^{53} erg released, which is mostly the gravitational energy before collapse.

Of this:

- 1% goes into the kinetic energy of the explosion (10^{51} erg, so this is known as 1 foe).
- 0.01% goes into photons
- 99% goes into neutrinos

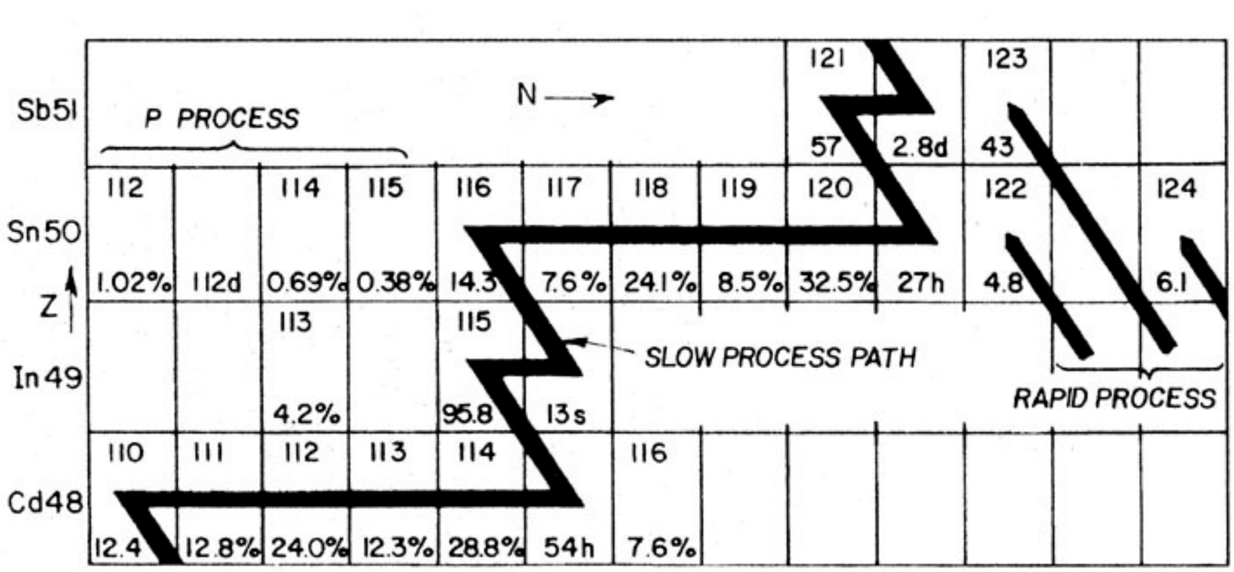
Late-time lightcurve of supernova depends mostly on what was surrounding the star (thick envelope or not) — the initial kinetic energy + radioactive decay does a good job of reproducing the energetics. But you have to be careful about observing at a specific wavelength because detailed processes (e.g., dust formation) can come into play and change the appearance.

IX.1.6 Chemistry

Sun produces mostly He from fusion. Massive stars can produce up to Fe, but still limits (i.e., little Li produced). How to make the rest? SNe.

Most of this happens through the *r-process* (rapid). Heavy elements + many neutrons → very heavy, unstable nuclei. Then decay to something stable (but still heavy).

The *s-process* (slow) can also occur, but that is mostly in post-main sequence evolution, where there is repeated n capture then α or β decay.



Core-collapse SNe produce a lot of α elements in addition, coming from α capture onto products of He burning (so mass is multiple of 4): Ne, Mg, Si, S, Ar, Ca, Ti. Type Ia supernovae produce a lot of Fe peak elements (V, Cr, Mn, Fe, Co, Ni) from explosive nucleosynthesis of C/O, but have much lower neutron flux around so don't get r-process.

Lecture X Long GRBs

Nice review paper: Piran (2004, Reviews of Modern Physics, 76, 1143).

1960s: Vela satellites are sent up to monitor nuclear test-ban treaty by looking for γ-ray flashes. Saw bursts of γ-rays from above rather than below.

Durations of bursts: 0.01 s–minutes, with structure down to <ms.

It was a major question as to where GRBs came from. Solar system? Galactic? Extragalactic? See fluence (integral of flux over time) of up to $10^{-4}\,\mathrm{erg\,cm^{-2}}$. Photon energies are typically MeV, stretching from keV to GeV. With unknown distance had unknown L (10^{32} erg up to 10^{52} erg), so didn't know mechanism.

Saw that bursts had very uniform distribution over the sky. Suggests that has to be isotropic. Could still be local (Oort cloud), but more likely distant (halo of MW or extragalactic).

Can try to figure out something about what the origin is even without knowing how far away sources are. Assume that they are homogeneous: same source density everywhere. If n bursts per volume per time and space is Euclidean, number with fluence $> S_0$:

$$N(S_0) = \frac{4}{3}\pi n r(S_0)^3 \tag{X.158}$$

with $r = \sqrt{E/4\pi S_0}$. So:

$$N(S_0) = \frac{4}{3}\pi n \left(\frac{E}{4\pi S_0}\right)^{3/2} \tag{X.159}$$

So $N(S_0) \propto S_0^{-3/2}$. We can plot the actual distribution of soures and we see it follows this at high S_0, but at low S_0 we don't see enough GRBs. Something is contributing to a paucity of sources. Maybe this is because of cosmology? (this can also be seen through a $V/V_{\max}$ test).

We see two classes of GRBs. Short/hard (< 1 s, higher energies) and long/soft (> 3 s, lower energies). Here we consider long/soft.

After the GRB, we see afterglow emission. It progresses from X-rays (keV) to optical, IR, and then radio. The afterglows enabled accurate positions to be found, which then led to actual redshifts and host galaxies. With redshifts, estimate energy $> 10^{52}$ erg. Optical declines $\propto t^{-\alpha}$ with $\alpha = 1.2$, but it often changes to $\alpha = 2$ across all wavelengths (achromatic) after a few days. Radio afterglow shows some very short timescale variability ($\sim$ days).

X.2 GRB Physics

General model is relativistic, jetted explosion coming from a massive star.

Can significantly lower E is we take the the emission to come from an anisotropic jet with bulk Lorentz factor Γ. We only see emission from $\theta < 1/\Gamma$. However, Γ decreases with time through the afterglow. Eventually it will reach the physical size of the jet θ_j. This is called a "jet break."

We correct the energy $E_\gamma = (\theta_j^2/2)E_{\gamma,\rm iso}$. Estimate the angle from the time until the achromatic break (also need to assume the local density):

$$\theta_j = 0.16(n/E_{k,\rm iso,52})^{1/8}t_{b,\rm days}^{3/8} \tag{X.160}$$

This uses the total kinetic energy rather than just the γ-ray energy. Find jet breaks of few to $10°$. Thus we end up with energies near 10^{51} erg, pretty uniformly. This also revises the rates to $1/3 \times 10^5$ yr per galaxy, roughly $1/3000$ the rate of SNe. Roughly tracks cosmic history of star formation.

We infer relativistic expansion from radio scintillation: need small source sizes to have refraction by ISM. Infer 10^{17} cm after 2 weeks, which would infer $v = 20c$. Instead $\Gamma > 100$ makes it work (actual size is $\sim \Gamma^2 ct$, but apparent size is $\sim \Gamma ct$).

Actual radiation (at all wavelengths) is synchrotron.

Association with star-forming galaxies and in some cases contemporaneous supernovae shows that these are coming from collapse of massive stars: "collapsar" model. This explains how to get so much energy out in so little time. Something like accretion onto BH in center of SN.

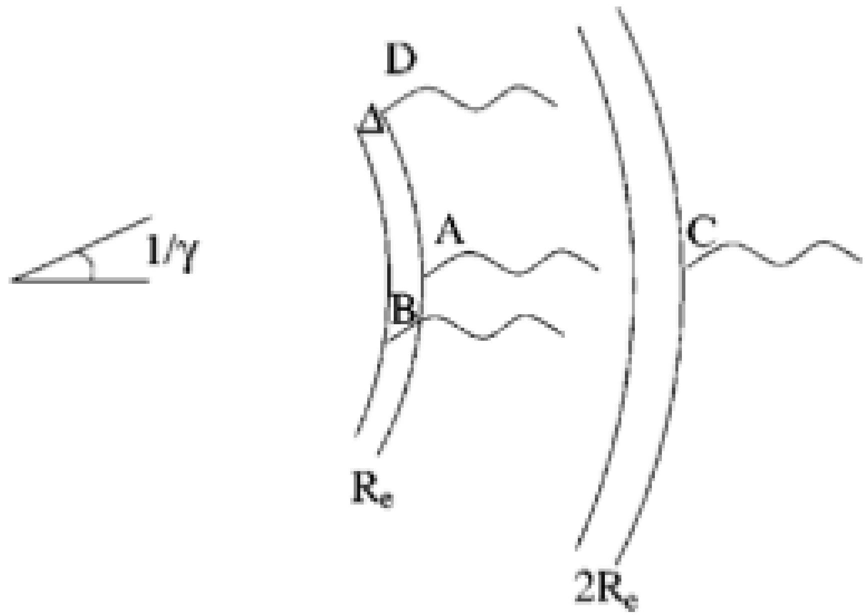
Relativistic motion also solves "compactness problem:" expect source to be optically thick to high-energy photons. Fluctuate with δt, so source is $< c\delta t$ in size. Would infer 10^{15} optical depth to $\gamma + \gamma \to e^+ + e^-$.

Instead, bulk motion with Γ. So the energy of the photons is reduced by Γ, so fewer can pair-produce. Size of inferred source is also smaller, $c\delta t\Gamma^2$ (observed time for emission from R is $R/2c\Gamma^2$). Makes it so that $\tau < 1$.

Emission from a shell is also influenced by Γ. For off-axis emission it comes at $R\theta^2/2c$, but since $\theta \sim 1/\Gamma$, comes at $R/2c\Gamma^2$. This is the same as the previous timescale.

X.2.1 Prompt Emission

Internal shocks in expanding shells. Duration of burst T from size of total jet, and substructure δt from the size of each shell. What is the cause of all this? People considered two models: external or internal shocks.



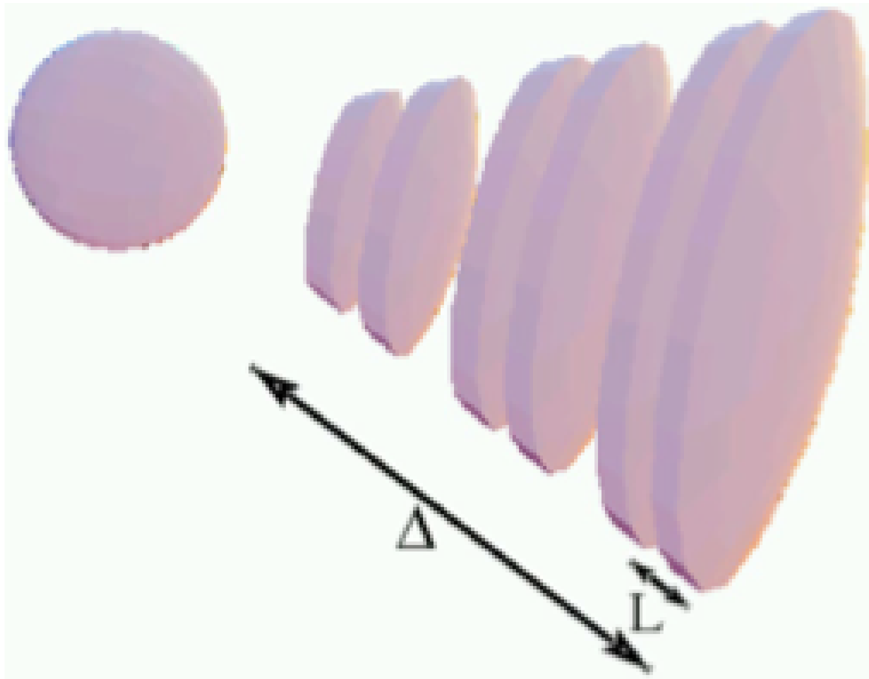
A model showing shell that emits from plowing into external medium and generates shock.

For the **external shock** model:
From Piran (2004): "Consider a quasispherical relativistic emitting shell with a radius R, a width Δ and a Lorentz factor Γ . . . Consider now photons emitted at different points along the shock. Photons emitted by matter moving directly towards the observer (point A) will arrive first. Photons emitted by matter moving at an angle $1/\Gamma$ (point D) would arrive after $t_{ang} = R/2c\Gamma^2$. This is also t_R, the time of arrival of photons emitted by matter moving directly towards the observer but emitted at $2R$ (point C). Thus $t_R \approx t_{ang}$."

From each shell will spread out emission over $t_\Delta = \Delta/c$.

Range of angles will smear out emission over t_{ang}. If $t_\Delta < t_{ang}$, then burst will be smooth in time with width t_{ang}. So the whole thing will not have substructure. Instead we need $t_\Delta = \Delta/c > t_{ang}$: angular smoothing should be much smaller than the thickness of the emitting region. Then $t_\Delta \sim T$, and $t_{ang} \sim \delta t$.

For the **internal shock** model:
Again, from Piran (2004): "Consider an 'inner engine' emitting a relativistic wind active over a time $t_\Delta = \Delta/c$ (where Δ is the overall width of the flow in the observer frame). The source is variable on a scale L/c. Internal shocks will take place at $R_s \sim L\Gamma^2$. At this place the angular time and the radial time satisfy $t_{ang} \sim t_R \sim L/c$. Internal shocks continue as long as the source is active, thus the overall observed duration $T = t_\Delta$ reflects the time that the 'inner engine' is active. Note that now $t_{ang} \sim L/c < t_\Delta$ is trivially satisfied. The observed variability time scale, δt, reflects the variability of the source L/c, while the overall duration of the burst reflects the overall duration of the activity of the 'inner engine.'"



Based on most observables the **prompt** emission is caused by internal shocks. However, the **after-glow** is caused by external shocks. So we have the internal/external model for long GRBs.

Internal shock has a faster shell overtake a slower one at:

$$R_{int} = c\delta t\Gamma^2 = 3 \times 10^{14}\,\text{cm}\Gamma_{100}^2\tilde{\delta t} \tag{X.161}$$

$\tilde{\delta t}$ is the time difference between when the shells were emitted. This timescale must be less than the timescale for external shocking. If Γ is too high this will happen too late.

Consider two shells with m_r and m_s with $\Gamma_r > \Gamma_s \gg 1$. In a collision the bulk Lorentz factor of the result is:

$$\Gamma_m \approx \sqrt{\frac{m_r\Gamma_r + m_s\Gamma_s}{m_r/\Gamma_r + m_s/\Gamma_s}} \tag{X.162}$$

We have $\mathcal{E}$ is the energy in the local frame and $E = \Gamma_m\mathcal{E}$ the energy in the observer frame. But E is the difference in total energies before and after collision:

$$E = m_r c^2(\Gamma_r - \Gamma_m) + m_s c^2(\Gamma_s - \Gamma_m) \tag{X.163}$$

So we can look at the conversion efficiency: how much of the original KE gets turned into shock energy:

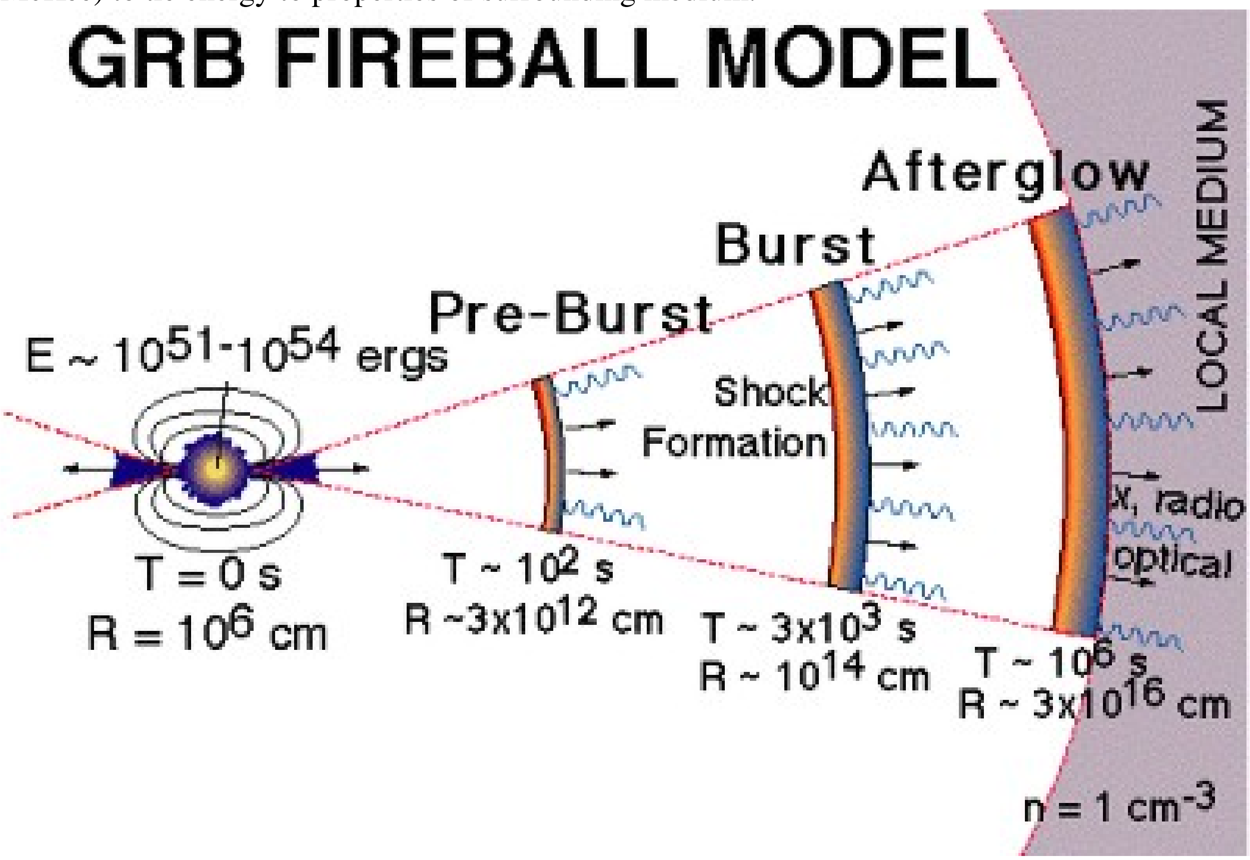
$$\epsilon = 1 - \frac{(m_r + m_s)\Gamma_m}{m_r\Gamma_r + m_s\Gamma_s} \tag{X.164}$$

To have high ϵ needs $\Gamma_r \gg \Gamma_s$ and comparable masses.

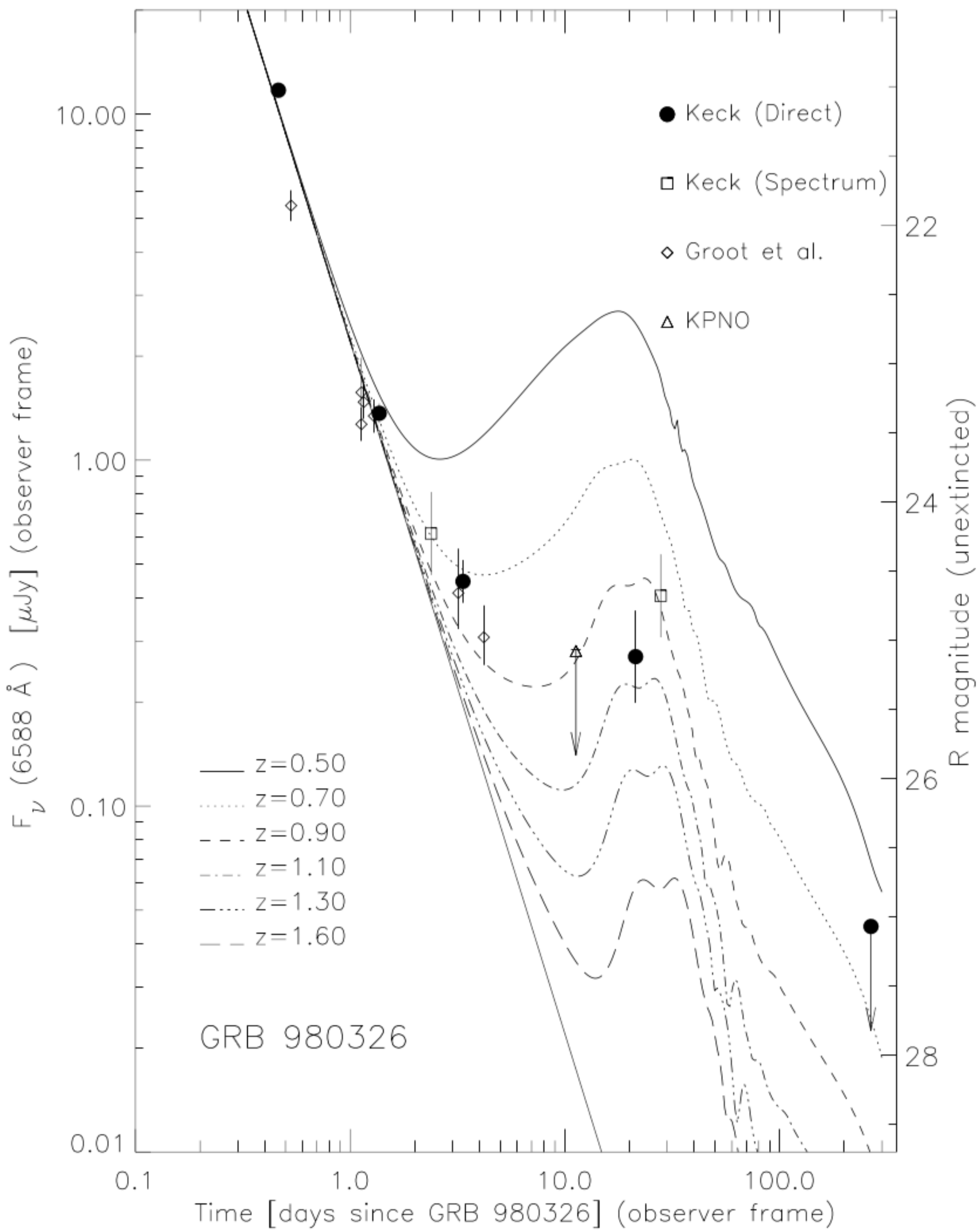
There are still questions about the actual acceleration mechanism, the nature of the central engine, the efficiency, and baryon loading. The last is that when the jet sweeps up $\Gamma m_j c^2$ it should slow down. But it doesn't. So the region above the jet should be pretty empty. We think that this is evacuated by the progenitor star.

X.2.2 Afterglow

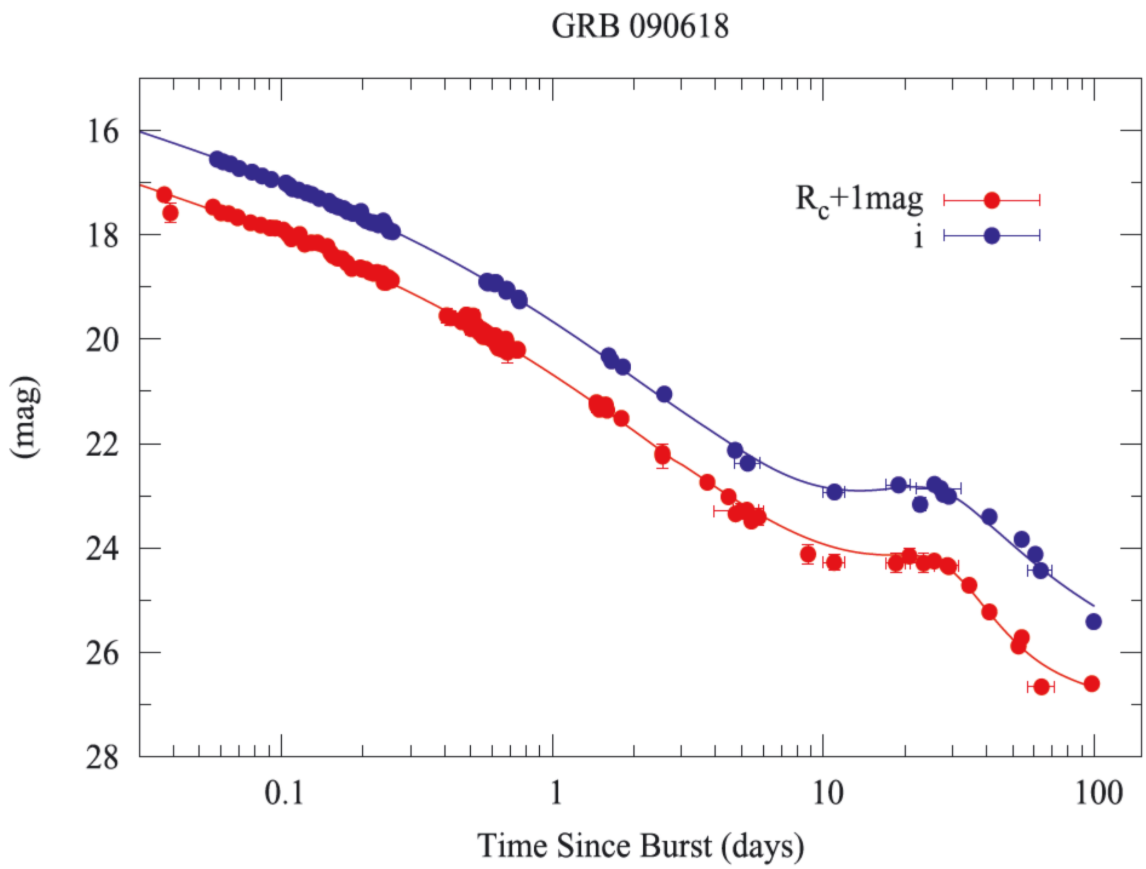
Relativistic jet collides with external medium. Progression in energy is from cooling and from optical depth considerations: at the beginning is optically thick at low energies. This $\tau = 1$ energy moves down from X-ray to optical to radio. Use relativistic variant of Sedov model (Blandford-McKee) to tie energy to properties of surrounding medium.



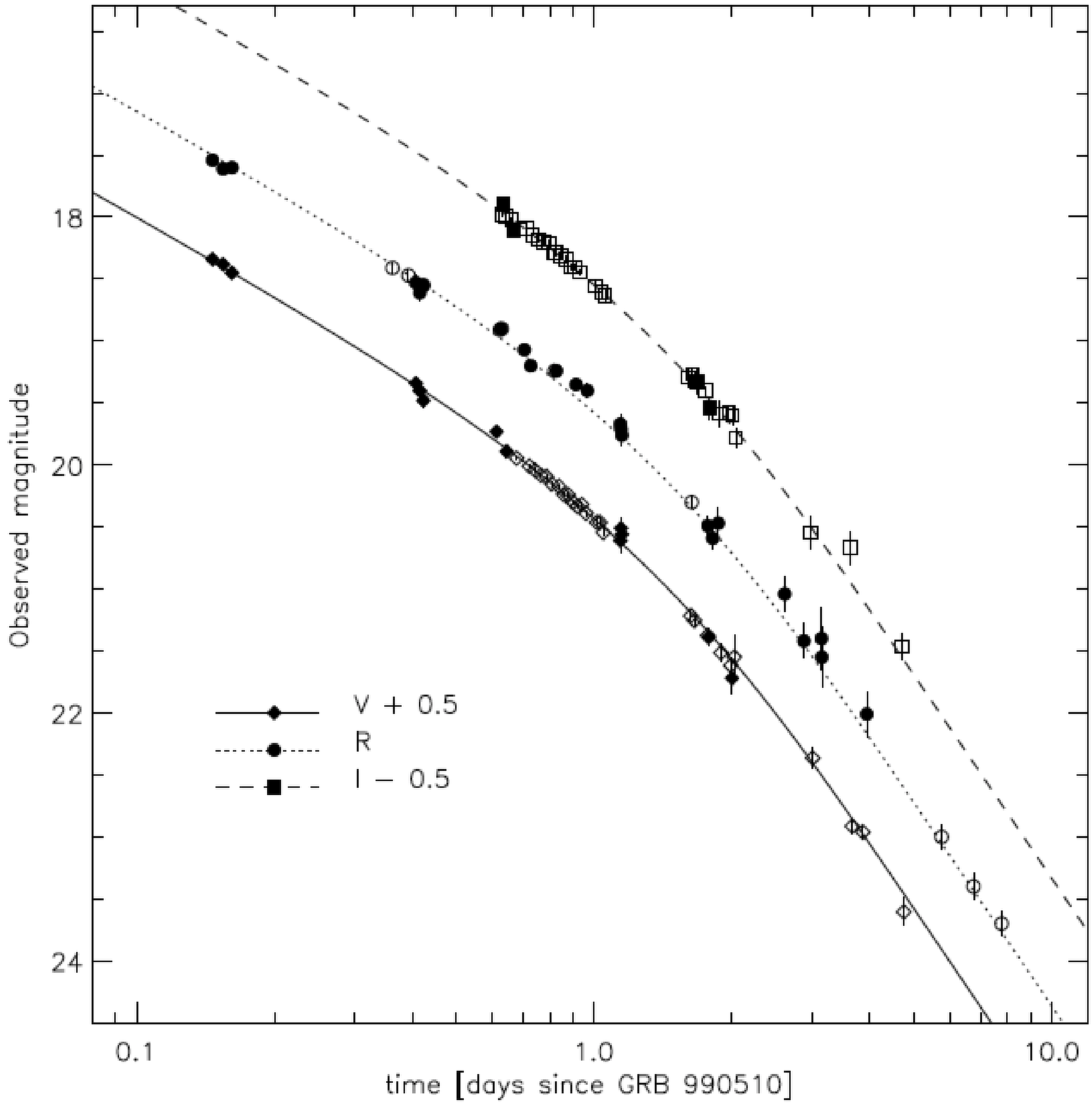
Blast wave evolution from
https://swift.gsfc.nasa.gov/about_swift/objectives/environment.html



An optical SN-like bump superimposed on the afterglow of GRB 980326. Models of SN 1998bw at different redshifts are shown. The color and light curve of the bump was found to be consistent with 1998bw at redshift of unity. From Bloom et al. (1999).



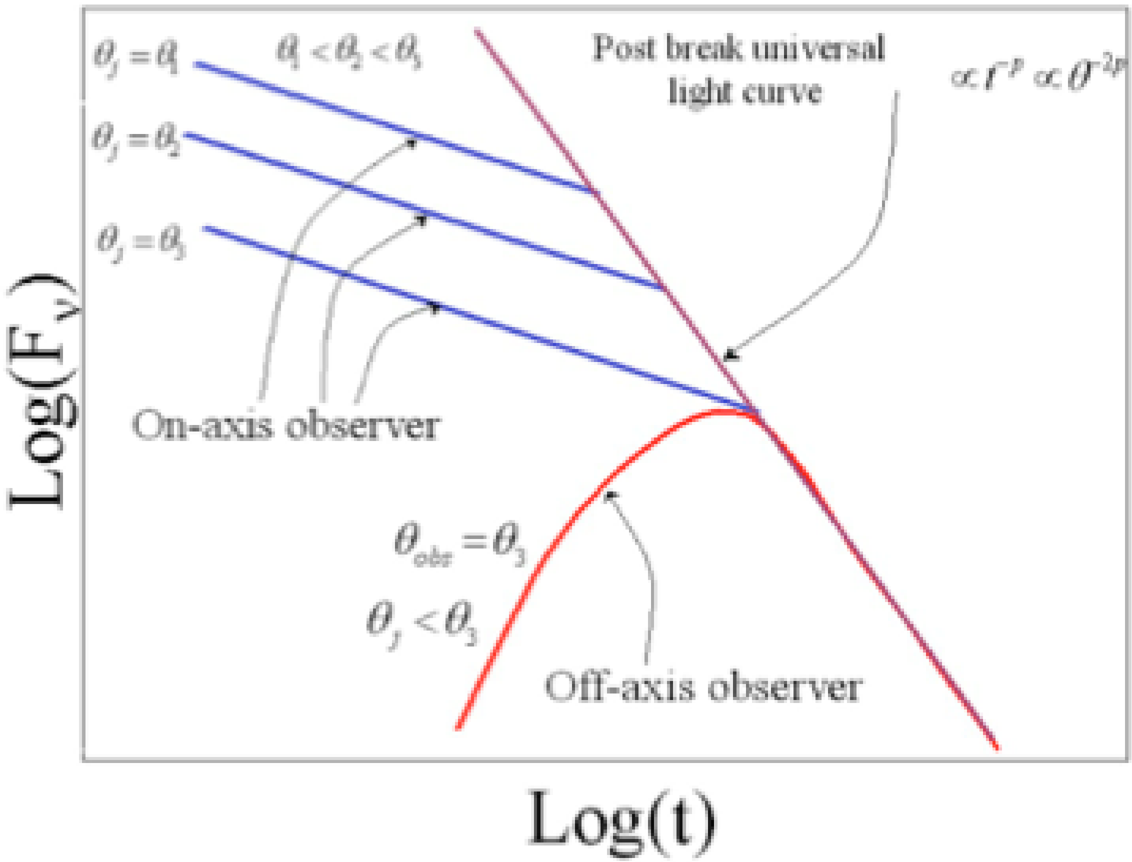
Late SN bump is visible at ~ 30 days after the GRB in the optical light curve of GRB 090618. From Cano et al. (2011).



Observed light curves of the optical afterglow of GRB 990510 in three filters (Harrison et al. 1999). Achromatic steepening of the light curve is seen.

X.2.3 Orphan Afterglows

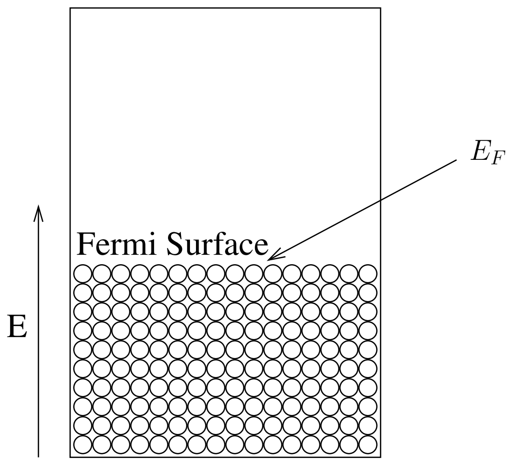
Can we see afterglow without GRB? Mostly no. This would be if we are at $\theta_{\rm obs} > \theta_j$: no prompt emission, but will see afterglow once $\Gamma < 1/\theta_{\rm obs}$. This is a major target of next-generation transient facilities. We *have* detected optical afterglow before we realized it was a GRB (work by Brad Cenko and Alex Urban), but eventually we got the GRB signal from satellites and realized it was pretty normal.



$\theta_j = \theta_1$
$\theta_1 < \theta_2 < \theta_3$
Post break universal light curve
$\propto t^{-p} \propto \theta^{-2p}$
$\theta_j = \theta_2$
$\theta_j = \theta_3$
Log(F$_\nu$)
On-axis observer
$\theta_{obs} = \theta_3$
$\theta_j < \theta_3$
Off-axis observer
Log(t)

Lecture XI Type Ia supernova

As you compress material, the electrons are forced closer together, i.e., they start to occupy similar states. But electrons are fermions and so Pauli's exclusion principle holds. As a result, electron fills up higher and higher energy states up to the Fermi energy E_F, which depends on the density:



This gives an energy density that produces a pressure, i.e., energy density is roughly $n_e E_F$. This is called degeneracy pressure or Fermi pressure. This can be fitted using

$$P = K\rho^\Gamma \quad \begin{cases} K = \frac{10^{13}}{\mu_e^{5/3}} & \Gamma = \frac{5}{3} \quad \text{for } \rho \ll 10^6\,\mathrm{g\,cm^{-3}} \\ K = \frac{1.24\times10^{15}}{\mu_e^{4/3}} & \Gamma = \frac{4}{3} \quad \text{for } \rho \gg 10^6\,\mathrm{g\,cm^{-3}} \end{cases} \tag{XI.165}$$

The difference in Γ is due to electrons being non-relativistic vs relativistic. The transition occurs at $\rho = 10^6\,\mathrm{g\,cm^{-3}}$. This in turn allows us to derive a mass-radius relation. To do this we use the equation for hydrostatic balance

$$\frac{dP}{dr} = -\frac{GM\rho}{r^2} \to -\frac{P_c}{r} = -\frac{GM^2}{r^5} \tag{XI.166}$$

where we have approximated $dP/dr = P_c/r$ and $\rho = M/r^3$. Now this gives us $P_c = GM^2/r^4$. Plugging in our fits for non-relativistic Fermi pressure, we find

$$K\left(\frac{M}{R^3}\right)^{5/3} = \frac{GM^2}{R^4} \tag{XI.167}$$

This gives a scaling $R \propto M^{-1/3}$, so as the mass increases, the star shrinks.

Now if we do this same exercise for relativistic electrons:

$$K\left(\frac{M}{R^3}\right)^{4/3} = \frac{GM^2}{R^4} \to M^{2/3} = \frac{K}{G} = 1.2\,M_\odot \tag{XI.168}$$

Note that we don't seem to find a relation between M and R. In fact, formally the radius approaches zero as M approaches a mass where the electron are relativistic. This mass is called the Chandrasekhar limit or Chandrasekhar mass and is estimated to be $1.4 M_\odot$.

This results from an equation of state like $P \propto \rho^{4/3}$, which is characteristic for pressure which are due to relativistic particles. In fact, these systems either collapse or explode. The other case there this pressure is important is radiation pressure which precludes stars with $M \gg 100\, M_\odot$.

Let use go ahead and estimate the binding energy of this star – non relativistic.

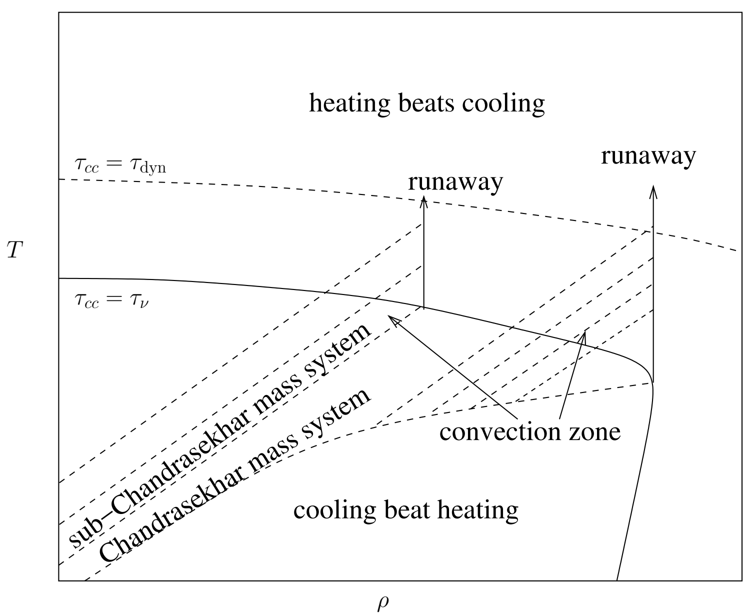
$$E = \int P dV = \int P \frac{dV}{dM} dM = \int c_s^2 dM = \int K \frac{M^{2/3}}{R^2} dM = 2.2 \times 10^{50} \left(\frac{M}{1M_\odot}\right)^{7/3} \quad \text{(XI.169)}$$

For $M = 1.4 M_\odot$, the binding energy is 0.5×10^{51} ergs.

Now if I burn $\sim 1M_\odot$ of C (1/2 to Ni and 1/2 to Si), we find that energy released is 2×10^{51} ergs greater than the binding energy. So burning all the material will unbind the star.

When this burning happens explosively, this results in what is called a type Ia supernova, which is the optically brightest supernova that we see. The "I" means no hydrogen lines, which suggests a star without hydrogen. The "a" means strong silicon lines which suggests lots burning of alpha elements.

The challenge is to get these stars to explode. To understand how this works consider a plot of a star in T-ρ space



To start these stars burning, we have to either compress things to very high densities or heat them up significantly. Once it cross the cc= ν cooling curve, then carbon burning will heat up the star, which causes more carbon burning which drives more heating. This runaway in the temperature will build up a convection zone above the burning region.

The formation of the convection zone occurs because there is no way for the heat to escape – the time it take the heat to escape if 1 Myr while the total burning time is at most 10,000 years. This convection zone is quite substantial. For a 1.4 solar mass star, you have have 1.1-1.2 solar masses in the convection zone.

Eventually burning region hits $\tau_{cc} = \tau_{dyn}$. At this stage, the carbon burning time hits the dynamical time and dynamical burning can occur. This dynamical burning will proceed by either

- deflagration
- detonation

Deflagation is like burning paper. A local region burns and raises it temperature significantly. Heat is conducted or radiate to its neighbors, which heat them up and cause them to burn. On the other hand, detonation occurs via a shock. Dynamite or TNT is consumed this way. A shock wave compresses, burns, and repowers itself with the burning material.

At this point, it is useful to discuss the various ways we can set up these stars to explode. There are two different models for SN Ia.

- Single degenerate model: A single white dwarf accretes material from its companion non-degenerate star.
- double degenerate model: The more massive white dwarf in a double white dwarf systems either accretes He or C/O from it less massive companion or tidally disrupts it companion.
- double detonation model: A helium layer on the surface triggers a detonation in the CO material by driving a shock into the material.

Up until 2010, the single degenerate model was the most popular model for SN Ia, but a few major problems remain with this model. these issues include

1. How do these WD gain mass – most of the material accreted is H or He. In order for the WD to have a change to increase mass, the accretion rate has to be larger than $10^{-7} M_\odot\,\mathrm{yr}^{-1}$. But in order for these stars to not prematurely explode, it has to be less than $10^{-6} M_\odot\,\mathrm{yr}^{-1}$. These systems should appear as supersoft x-ray sources, but the number observed in our galaxy and in other galaxies is much to low to account for the observed SN Ia rate.
2. How do these SD explode – to make the observed Si, you need a deflagration, but to make the observed Ni, you need a detonation. Hybrid models of explosion can get around this, but it is unclear what the physics behind it is.

Early double degenerate models suppose that 2 WDs merge to produce a super-Chandrasekhar mass WD. Recent numerical simulations of these systems find that these type of merger paradoxically gives SN that are less bright – too little Ni. Other models including by your truly, suppose that sub-Chandrasekhar mass systems can also explode, but it is not clear if they do and how they can get hot enough.

Lecture XII Light Curves of SNIa

Type Ia supernova is a perfect jumping off point to study light curves from transient events. The physics for radioactively power light curves are the same no matter what the situation. let's begin by stating some properties of SN Ia.

- Very bright optical explosion, $\sim 10^{43}$ ergs/s at maximum light. Galaxies are similar in brightness $\sim 10^{42} - 10^{43}$ ergs/s so hence then can outshing an entire galaxy.
- Reach maximum light at 10-15 days. Decays exponentially afterwards. The timescale is similar to the timescale of radioactive decay of ^{5}6Ni
- Powered by radioactive decay of ^{5}6Ni via the process: ^{5}6Ni $\rightarrow^{5}$ 6Co, with $\tau \approx 9$ d; ^{5}6Co $\rightarrow^{5}$ 6Fe, $\tau \approx 114$d
- They are what are called standardizable candles, not standard candles. Their width and peak correlate.
- Can be used the measure luminosity distance.
- dominant source of Fe in the universe.
- Occurs in both young and old populations. This is unlike core-collapse SN, which only occurs in young populations.

A few SN Ia light curves are shown in Figure 4. They all have the same general structure – rising in the beginning followed by falling at the end. They have all been zeroed at maximum light, so you might notice that the dimmer it is, the quicker it rises and falls. This correlation between the brightness and the overall length make them really standardizable candles and the fact that they are standardizable and bright make them crucially important in cosmology as they allow us to measure the luminosity distance. To see how this works consider the bottom plot in Figure 4. Here, by adjusting their overall width and normalization, we can see that they all belong to the same family.

The relation between the maximum brightness and the speed of the supernova is knowns as the "stretch-width" relation or the "Phillips" relation. Roughly this relation is empirical and is given by

$$M_{\rm B,max} = -21.726 + 2.698\Delta m_{\rm B,15}, \qquad \text{(XII.170)}$$

where $M_{\rm B,max}$ is the absolute B-band magnitude and $\Delta m_{\rm B,15}$ is the decline in the B band luminosity at 15 days after maximum light. Figure 5 shows graphically the definition of $\Delta m_{\rm B,15}$

The origin of this Phillips relation is though to be due to the physics of radiative transfers and energy deposition in SN Ia. Both of these processes crucially are due to the same thing, the Fe-group elements that supply the opacity and the radioactivity. In addition, this physics is generally applicable to almost any transient explosion.

The first crucial point is that the radiation is not a result of the original explosion, but from latent radioactive heat. To see why this is so, let us begin by considering if the gas is heated up during

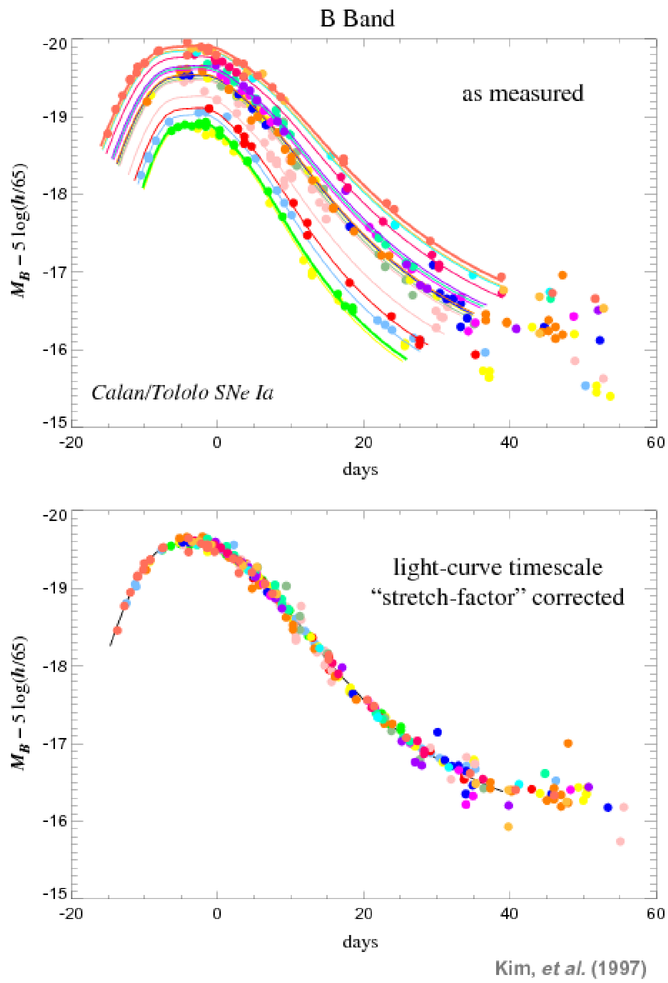


Figure 4: A bunch of SN Ia light curves uncorrected (top) and corrected (bottom)

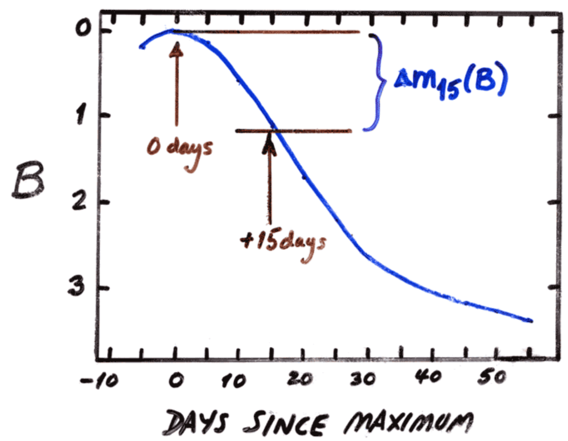


Figure 5: Definition of $\Delta m_{\mathrm{B},15}$

the SN explosion. For a 1000 km WD, the temperature of a gas heated to a thermal energy of 10^{51} ergs is

$$aT^4 = \frac{E}{V} = \frac{E}{4\pi R^3/3} \to T \approx 10^{10} K \left(\frac{E}{10^{51}\,\mathrm{ergs}}\right)^{1/4} \left(\frac{R}{1000\,\mathrm{km}}\right)^{-3/4}, \tag{XII.171}$$

where I assume (rightly) that radiation energy is the dominant energy. Now as this material expands, $P \sim aT^4 \sim \rho^{4/3} \to T \propto 1/r$. So in other words

$$T(t) = T_i \frac{R}{vt}, \tag{XII.172}$$

where $T_i \approx 10^{10}$ K is the initial temperature, R is the initial radius, and $v \approx 10^4$ km/s is the expansion velocity. At the maximum light is reached at $10 - 15$ days ($t \approx 10^6$ s), then the temperature of this material is roughly 1000 K, which is very cold and certainly would not emit in the optical.

So to understand the Phillips relation, let's calculate the time to maximum light. The time to maximum light is going to be related to the width, but I'll assume this relation is simple for now. To calculate the time to maximum light, let's consider the diffusion of photons in an expanding medium. In any material the rate $\mathcal{R}$ at which photons scatter is

$$\mathcal{R} = n\sigma c \tag{XII.173}$$

where n is the number density of scatterer (atoms) and σ is the cross section. The mean free path is then

$$\lambda = \frac{c}{\mathcal{R}} = \frac{1}{n\sigma}. \tag{XII.174}$$

Now consider a uniform sphere of radius R and mass M. In order for a random photon to escape, it must make N steps where

$$N = \frac{R^2}{\lambda^2}, \tag{XII.175}$$

where we assume a random walk. The total distance traveled and escape time are then

$$L = N\lambda = \frac{R^2}{\lambda^2}\lambda \quad \text{and} \quad t_{\text{esc}} = \frac{L}{c} = \frac{R^2}{\lambda c}, \tag{XII.176}$$

respectively. The resulting effective speed is then

$$v_{\text{esc}} = \frac{R}{t_{\text{esc}}} = c\frac{\lambda}{R} = \frac{c}{\tau}, \tag{XII.177}$$

where $\tau = n\sigma R$ is called the optical depth. Estimating $n = M/R^3 m_p$, and $R = v_{\text{exp}} t_{\text{exp}}$, where v_{exp} and t_{exp} are the expansion velocities and times, we have

$$t_{\text{esc}} = \frac{R^2}{R^3}\frac{M}{m_p}\frac{\sigma}{c} = \frac{M}{m_p}\frac{\sigma}{c v_{\text{exp}} t_{\text{exp}}}. \tag{XII.178}$$

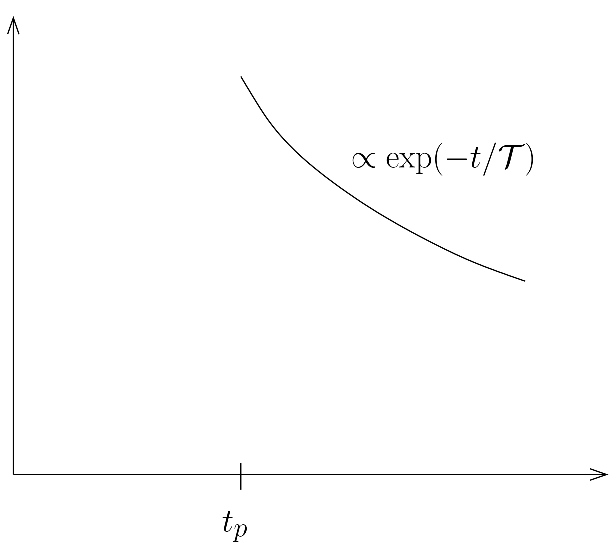
Now when $t_p = t_{\text{esc}} = t_{\text{exp}}$ then the photons (produced by radioactive decay) can escape before adiabatic expansion dilutes them. This gives

$$t_p = \sqrt{\frac{M}{m_p}\frac{\sigma}{c v_{\text{exp}}}} \approx 19 \left(\frac{M}{1\,M_\odot}\right)^{1/2} \left(\frac{\kappa}{0.04\,\text{cm}^2\,\text{g}^{-1}}\right)^{1/2} \left(\frac{v}{10^4\,\text{km}\,\text{s}^{-1}}\right)^{-1/2} \text{d}, \tag{XII.179}$$

where $\kappa = \sigma/m_p$ is the Rosseland mean opacity. The number is promising as it is similar to the timescale for SN Ia.

Interestingly, at this time $v_{\text{esc}} = v_{\text{exp}} = c/\tau \to \tau = c/v_{\text{exp}}$. As the expansion velocity is always less than the speed of light, this means that τ is generally between 10-30. I should warn you that alot of people will make statements like the peak time is determined by then the supernova is optically thin or calculate max light for when $\tau = 1$. Don't believe them. The real condition is when $v_{\text{exp}} = v_{\text{esc}}$.

Now that we have determined t_p, we need to figure out what happens next. Right now t_p is a meaningless timescale, but it does allow us to determine the next phase. In particular, light that is produced as a result of nuclear decay after this time will immediately escape. So roughly if $L \sim \dot{E} \sim \exp(-t/\mathcal{T})$, we know that for $t > t_p$, $L \propto \exp(-t/\mathcal{T})$, where $\mathcal{T}$ is some nuclear half-life.



Now we need to figure out what happens for $t < t_p$. Let's begin with a simple brain dead model. Suppose we have a fixed temperature photosphere, $R \propto v_{\rm exp} t_{\rm exp}$, then we have

$$L = 4\pi R^2 \sigma T_e^4 \to L \propto t^2, \tag{XII.180}$$

but this is quite unjustified. Why is it constant T_e in particular? However, because it suggest for $t < t_p$, L is rising, this would show that t_p is around where L peaks.

So let's do something more sophisticated. Suppose we look from infinity at an expanding supernova. You will only detect the photons from a depth of ΔR, where

$$v_{\rm esc} = \frac{c}{\tau(\Delta R)} = v_{\rm exp} = \frac{c}{n\sigma \Delta R} = \frac{c}{\frac{M\sigma}{R^3 m_p}\Delta R} \to \Delta R = \frac{cR^3}{\frac{M}{m_p}\sigma v_{\rm exp}} \tag{XII.181}$$

The associate amount of material that powers the supernova is then

$$\Delta M = M \frac{c}{v_{\rm exp}} \frac{m_p}{\sigma} \frac{R^2}{M} = M \frac{c v_{\rm exp}}{M\kappa} t^2 \approx 0.003 M_\odot \left(\frac{v}{10^4\,\mathrm{km\,s^{-1}}}\right) \left(\frac{\kappa}{0.04}\right)^{-1} \left(\frac{t}{1\,\mathrm{d}}\right)^2 \tag{XII.182}$$

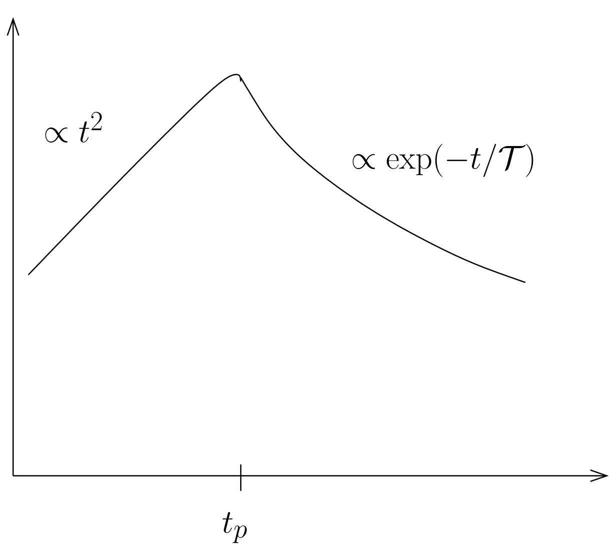
, which is the mass shell that powers what you see at early times. Now if we assume $t \ll \mathcal{T}$, i.e., early times, then the nuclear decay rate per gram is roughly constant, i.e., $\dot{q} = $ constant. So we have

$$L = \Delta M \dot{q} \propto t^2 \tag{XII.183}$$

as before.

Let's now go ahead and calculate the photosphere. Well if we are looking at early times then $\Delta R \ll R$, which implies that the $R_{\rm ph} \approx R = v_{\rm exp} t_{\rm exp}$. So we have if $L \propto t^2$ and $R \propto t$ then T_e is constant!!!

This is amazing and is the result of $\Delta R \propto R$, and $t \ll \mathcal{T}$. But in any case, we can go ahead and fill in the light curve, and gives t_p is the time of maximum light.



Now at maximum light, the luminosity is roughly

$$L_{\max} = M\dot{q}_0 \exp\left(-\frac{t_p}{\mathcal{T}}\right) \quad \text{(XII.184)}$$

which encodes our version of the Phillips relation. To see this, we need to emphasize an implicit assumption we have made up to this point, which is the M supplying the radioactive decay is the same M that is providing the opacity. It turns out that this is indeed the case as the M supplying the radioactive decay are iron group elements, i.e., ^{5}6Ni, and supplies most of the opacity, i.e., Fe group elements have alot of lines at these temperatures.

But this by itself is not enough. let's go and take the $\log$ of equation (XII.184) which gives

$$\log L_{\max} = \log M - \frac{t_p}{\mathcal{T}} \quad \text{(XII.185)}$$

Noting that $t_p \propto M^{1/2}$, we have

$$\log L_{\max} = 2\log t_p - \frac{t_p}{\mathcal{T}} \quad \text{(XII.186)}$$

This is a Phillips relation in disguise. It might seem that as I increase t_p that I have a positive contribution from the first term and a negative contribution from the second term. But in fact the first term usually dominates. This seems odd as it is a logarithmic contribution compared to the second contribution, but in fact this is the case.

In Figure 6, I show the above relation for different values of $\mathcal{T}$ (units are in days). The period of time we are interested in is for t_p between 10-15 days. Here, it is obvious that for $\mathcal{T} \gtrsim 10$ days, then there is a positive correlation between t_p and $\log L_{\max}$ which is a proxy for magnitude. This is due to the fact that while $\log t_p$ is normally a weak contribution, $t_p/\mathcal{T}$ is even smaller for $t_p \ll \mathcal{T}$.

XII.2 Early time light curves

Will maximum light is usually the easiest thing to observe, the advent of automated transient finders is allowing the capture of the early timescale observation of supernova. This is important as it is enormously powerful for constraining SNIa and other transients. For instance:

1. Repeated observations over an area where a SN occurred can constrain the type of projenitors – most useful for very nearly Ia's as progenitor are too dim for most Ia's.
2. The early time light curve can constrain the mechanism of explosion and the size of the exploding object.

In particular, if the explosion is powered by a shock then it is known that shocks accelerate as it moves up a density gradient. In particular for a adiabatic, radiation pressure ($\gamma = 4/3$) dominated shock, the velocity of the shock as it moves to lower density is (see Piro, Chang, & Weinberg 2010)

$$v_s = v_{s,0}\left(\frac{\rho}{\rho_0}\right)^{-0.1858}, \quad \text{(XII.187)}$$

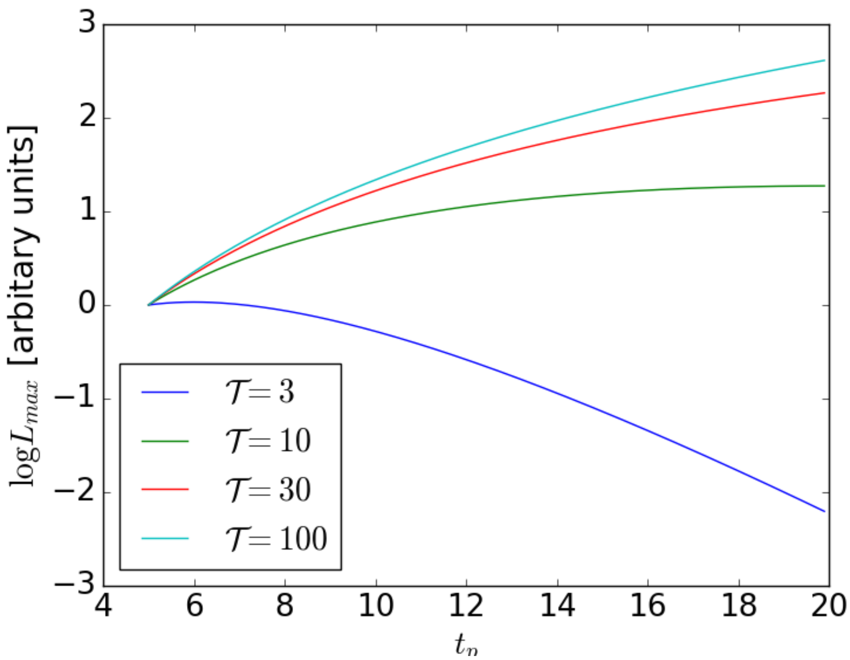


Figure 6: Plot of a simple phillips relation from theory

where $v_{s,0}$ and $\rho_{s,0}$ is the characteristic shock velocity and density. As this shock moves toward the surface of the WD, it produces a very bright x-ray flash with a characteristic time of ≈ 1 sec and a luminosity of $L \sim 10^{40}$ ergs/s, though at very short times it can be much higher than this. This is call a shock breakout.

This x-ray shock breakout is fairly short and low luminosity, but for larger objects, e.g., core-collapse supernova, it is far more promising. For instance SN2008D was discovered by the swift satellite while it was monitoring another object in the galaxy NGC2770. During the observation SN2008D went off and because the XRT was pointed at it, it was able to capture its rise and fall. The discovery image is shown in Figure 7.

You may ask for instance what why does this shock breakout result in x-rays. In fact this is a property of most shock breakout in that it produces flashes of high energy radiation and the timescale and energetics depends on the size of the object and the structure of the atmosphere in which the shocks are propagating. To get a flavor of how this works consider the breakout of a shock through a WD atmosphere, where I will need to tell you that $v_{s,0} = 10^9$ cm/s and $\rho_{s,0} = 10^6\,\mathrm{g\,cm^{-3}}$. Now as this propagates to the surface, $v_s = v_{\rm esc}$ occurs at a density of $\rho = 10^{-3}\,\mathrm{g\,cm^{-3}}$, where $v_{\rm esc} = c/\tau$ is the effective speed of escaping photons and τ is the optical depth measure from the surface. This gives a shock velocity of nearly $v_s \approx 3 \times 10^{10} = c$. You may be disturbed by this, but in fact $v_{\rm esc} \to \Gamma v_{\rm esc}$. I have dropped the Lorentz factor for computational ease.

In any case the energy density of the shock is

$$\epsilon = \rho v_s^2 = aT^4 \to T \approx 10^8\,\mathrm{K}, \tag{XII.188}$$

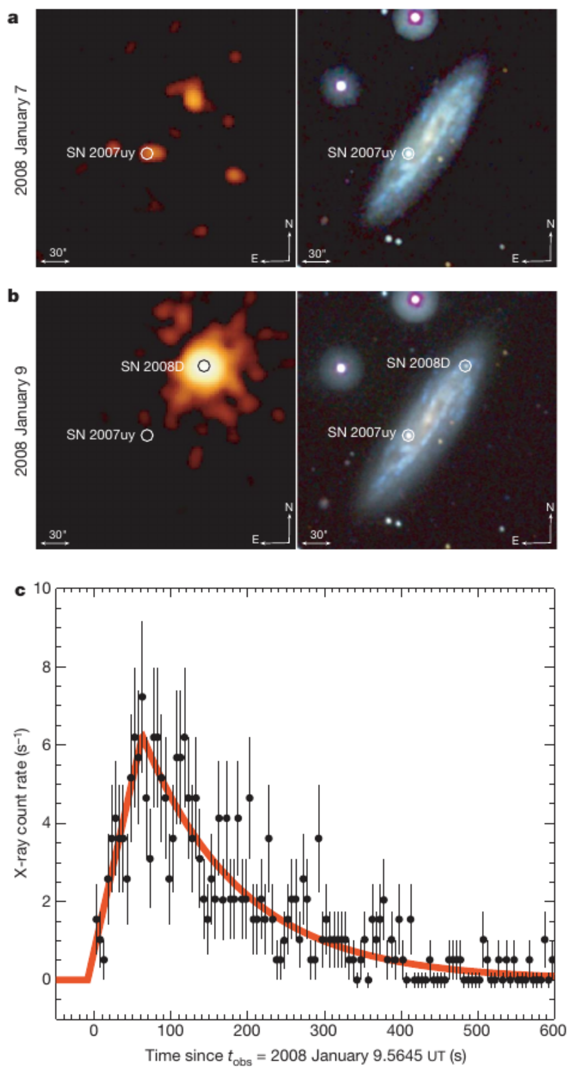


Figure 7: Plot of a simple phillips relation from theory

which is about 10 keV. The total energetics is also similarly argued. By noting

$$E = 4\pi R^2 \Delta R \epsilon = 4\pi R^2 H \epsilon, \tag{XII.189}$$

where we have used the height of the atmosphere as an estimate for the size of the shell. To get H, we recall the equation of hydrostatic balance and assume an initial isothermal atmosphere

$$\frac{dP}{dz} = -\rho g \rightarrow \frac{P}{H} = \frac{\rho k_B T}{m_p H} = \rho g \tag{XII.190}$$

for $g \approx 10^9$ appropriate for a WD, we find $H \approx 10^4$ cm for $T \approx 10^5$ K. This gives $E \approx 10^{41}$ ergs.

In the HW, you will do a simple analysis of the luminosity evolution. A sophisticated analysis of the luminosity evolution yields a very shallow power law shown in top panel of Figure 8 where the power law scales like $L \propto t^{-0.14}$. The effective temperature drops more steeply $T_e \propto t^{-0.44}$. In any case, one assume that the photosphere radiates like a blackbody and so

$$I_\lambda(T_e) = \frac{2hc^2}{\lambda^5} \frac{1}{\exp(hc/\lambda k_B T_e) - 1} \tag{XII.191}$$

where $I_\lambda(T_e)$ is the specific intensity and λ is the wavelength and integrate the flux in a band, $F(t) = \int_{\lambda_1}^{\lambda_2} I_\lambda(T_e(t)) d\lambda$, where λ_1 and λ_2 are the limits of the band to get the optical luminosity in some band, i.e.,

$$L_{\rm band} = 4\pi R(t)^2 F(t). \tag{XII.192}$$

This is also shown in the top plot of Figure 8 and in Figure 9 by the dotted lines.

Note that the optical luminosity has a light curve that rises and then falls even though the bolometric (totally EM flux) is a power law without any such features. This is a warning that it is not always the case the transitions and peaks in light curves have any meaning, i.e., radiation escapes or optical thinness, but is just the fact that we normally detect just a narrow sliver of EM radiation.

As mentioned in the previous class these models are really useful to measure the properties of the progenitor. In Figure 9, we show one such measurement that was made for the early time light curve of SN 2011 fe. This type Ia explosion occurred on M101, a nearby galaxy and was the nearest Type Ia since the time of Tycho Brahe. In any case, the light curve was captured by the Palomar Transient Factory at 12 hours after the explosion – the black dots. As you can see they rule out an initially stellar radii of the object to be $< 0.1\,R_\odot$ compared to the models of Piro et al. and Rabinak & Waxman (2011). Additionally, images taken 8 hours before, a mere 4 hours after explosion suggest that it was really dim, ruling out shock breakout of even a $0.02\,R_\odot$, which is about 2x the size of earth.

Interestingly, it also puts a constraint on possible companions stars as well. For instance, if the Ia was the result of Roche-lobe overflow accretion from a companion star, the shock wave from the exploding star will impact the companion and given a favorable viewing angle, would shock the material and produce radiation that is visible. This is shown by the solid line in Figure 9 and sets a constraint on size of the companion star. In any case, it also gives a constraint on the companion, which if were to be believe would rule out almost any Roche-filling system.

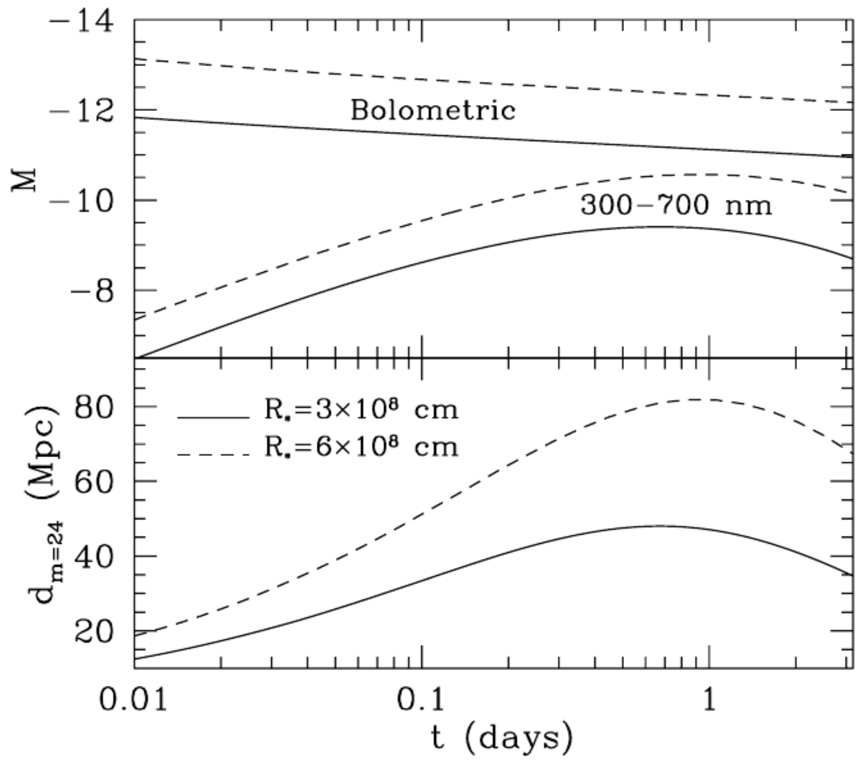


Figure 8: Plot of bolometric and band flux from shock heated expanding envelope

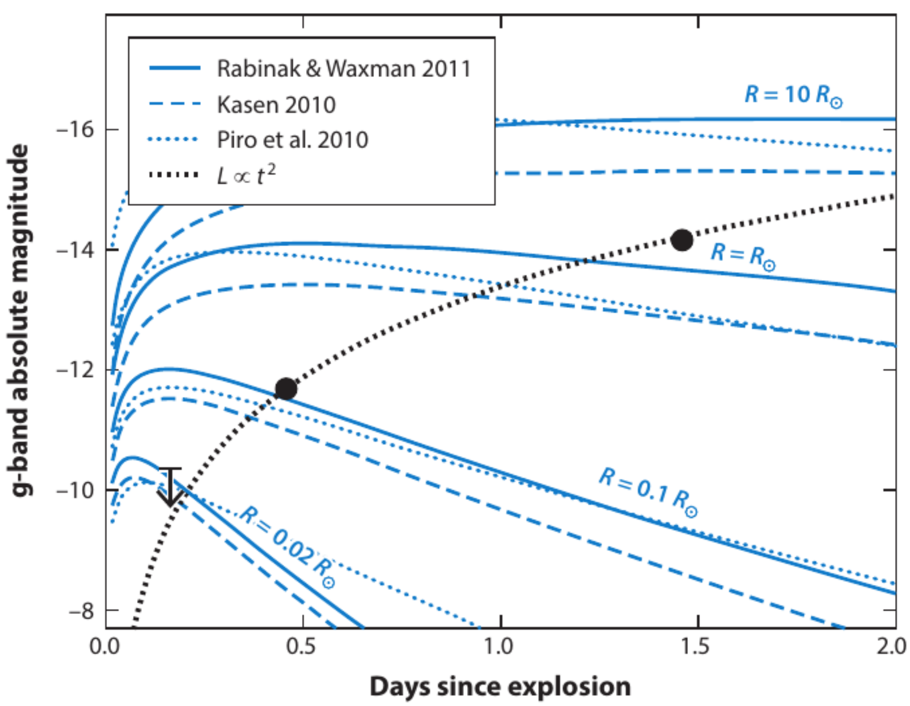


Figure 9: Constraints on early time emission compared to models.

XII.3 Late Time Radio Emission of SN Ia

Talking about radio emission from SN Ia's is a funny thing. This is in part because there is no radio emission from Ia's. We will talk about why this is significant in a bit. In any case, radio emission is increasingly an important part of constraining properties of the region around explosive events. It is also an important part of GRB research, but this is for another time.

In any case, radio emission from many astrophysical phenomenon is due to synchrotron radiation. Which is just relativistically boosted cyclotron radiation. To see how this works consider an electron moving in a magnetic field. From freshman physics we know

$$F = ma = m_e\frac{v^2}{r} = e\frac{v}{c} \times B \rightarrow \omega_L = \frac{v}{r} = \frac{eB}{m_e c}, \tag{XII.193}$$

where ω_L is the Larmor or cyclotron frequency. Now for a relativistic electron, just replace $\omega_L \rightarrow \omega_g = \omega_L/\gamma$, where γ is the Lorentz factor.

This accelerating electron will emit. Nonrelativistically it will emit at a frequency of $\nu = \omega_L/2\pi$, but relativistically this radiation will be beamed and be in a cone of angle $\sim 1/\gamma$. So as an electron moves in a circle, you will see a pulse when the electron is moving within $1/\gamma$ of you line of sight. Time dilation gives you another factor of γ. Length contraction gives yet another factor of γ, so all together, the frequency you observe is

$$\nu = \gamma^3\nu_g = \gamma^2\nu_L, \tag{XII.194}$$

where $\nu_g = \omega_g/2\pi$ and $\nu_L = \omega_L/2\pi$. This is great. Now the amount of power that an electron will emit is

$$\frac{dE}{dt} = \frac{4}{3}\sigma_T c\beta^2\gamma^2 U_B, \tag{XII.195}$$

where $U_B = B^2/8\pi$ is the energy density of the magnetic field. Look in Jackson to see how this is derived or take my word for it.

The expansion of a SN is a shock that plows into the circumstellar medium (CSM). As this shock moves through the CSM, the shock will accelerate electron to high energies. This distribution of particles is most definitely not thermal and gives a high energy tail, namely

$$\frac{dN}{dE} = N_0 E^{-p}, \tag{XII.196}$$

where E is the energy of the electrons. In this case, general arguments give you $p \approx 2 - 3$. These accelerated electrons will radiate via synchrotron radiation with $\nu = \gamma^2\nu_L \propto E^2$, i.e., their frequency depends on energy. Let see how this ensemble will radiate.

In particular, the intensity of radiation will be

$$j_{\nu(E)}d\nu = \frac{dE}{dt}\frac{dN}{dE}dE \tag{XII.197}$$

Noting that

$$\nu = \gamma^2\nu_L \rightarrow E = \sqrt{\frac{\nu}{\nu_L}}m_e c^2, \tag{XII.198}$$

we take the derivative $dE/d\nu$ to find

$$\frac{dE}{d\nu} = \frac{1}{2}\frac{m_e c^2}{\sqrt{\nu\nu_L}}, \tag{XII.199}$$

which gives

$$j_{\nu(E)} = \frac{dE}{dt}\frac{dN}{dE}\frac{dE}{d\nu} = \frac{4}{3}\sigma_T c\beta^2\gamma^2 U_B(-p)N_0 E^{p-1}\frac{1}{2}\frac{m_e c^2}{\sqrt{\nu\nu_L}} \propto \nu^{(1-p)/2} \tag{XII.200}$$

Note that for $p \approx 2-3$, this gives a falling spectrum.

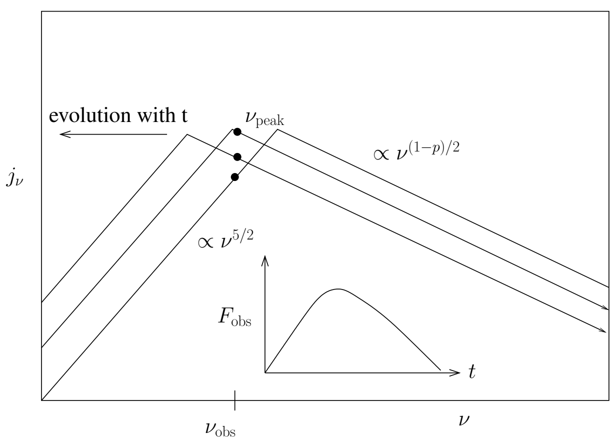
Now this emission is assumed to be optically thin, but it turns out that at low frequencies this radiation can be absorbed. Absorption and re-emission gives rise to a blackbody radiation. To see how this works consider the rayleigh-jeans tail of the Planck function

$$I_\nu = \frac{2k_B T}{c^2}\nu^2 \tag{XII.201}$$

for $h\nu \ll k_B T$. Now in this case the T is the temperature of the synchrotron electrons, i.e., $k_B T = E$. Using our calculation for E from above we see that

$$I_\nu \propto \nu^{1/2}\nu^2 = \nu^{5/2}, \tag{XII.202}$$

which is rising with time. Given these two limits one can then draw the spectra.



We are left with one last caveat, and that is the position of the peak. In this case, one can guess it is the boundary between optically thin and optically thick, i.e., $\tau = 1$. To estimate where this has to be, we need to know where

$$\tau = 1 = \kappa_\nu \rho r, \tag{XII.203}$$

where κ is the opacity. To determine κ, we use detailed balance, i.e., in equilibrium, emission of radiation is equal to absorption of the local radiation field. Namely

$$j_\nu = I_\nu\alpha_\nu \to \alpha_\nu = \frac{j_\nu}{I_\nu} \propto \frac{\nu^{(1-p)/2}}{\nu^{5/2}} = \nu^{-(p+4)/2} \tag{XII.204}$$

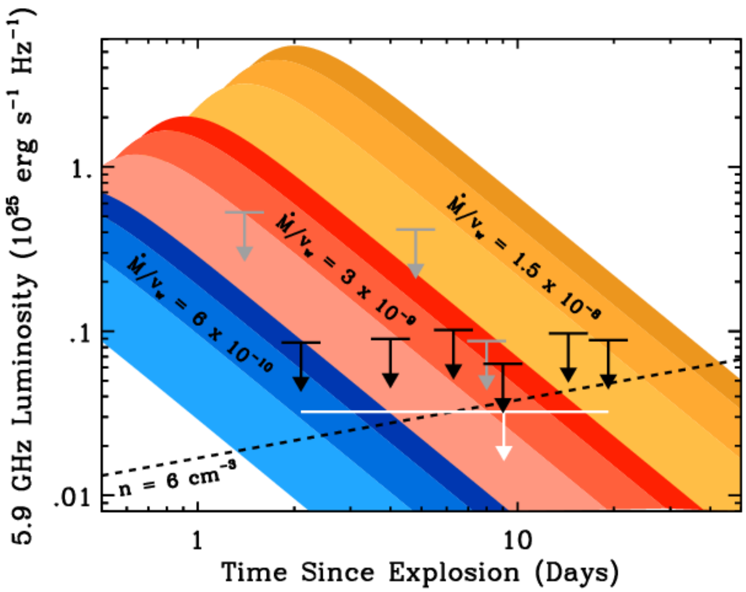


Figure 10: Light curves from shock interaction + radio limits

where α_ν is the absorption coefficient. The absorption coefficient is related to the opacity, so we can estimate

$$\tau = 1 \propto \alpha_{\nu_{\rm peak}} \rho r, \tag{XII.205}$$

so that we see that the peak frequency is related to the density and size of the local environment.

Now you may wonder what does any of this have to do with the CSM. Well in many scenarios for Ia, the companion was some kind of degenerate star that fills it Roche lobe. If it were an evolved star, then these stars would have given off powerful winds that pollute the CSM. For a constant velocity wind at a constant mass loss rate, we can find

$$4\pi r^2 \rho v_w = \dot{M} \to \rho \propto \frac{\dot{M}}{v_w} r^{-2} \tag{XII.206}$$

Using this result, we find

$$\nu_{\rm peak}^{-(p+4)/2} \rho r \propto 1 \to \nu_{\rm peak} \propto r^{-2/(p+4)} \propto t^{-2/(p+4)} \tag{XII.207}$$

For $p \approx 2-3$, we find $\nu_{\rm peak} \sim t^{-0.3}$, which means that the peak moves to lower and lower frequencies as a function of time. If we look at a particular frequency starting in the optically thick regime, we will see an initial rise follow by a fall. And the amplitude constrains $\dot{M}/v_w$. Alas, no such radio detection has been observed in an Ia. In Figure 10, we show the strong constraints from SN 2011fe. As you can see the observations constrains $\dot{M}/v_w$ as low as 6×10^{-10} in units of $M_\odot\,{\rm yr}^{-1}/100\,{\rm km\,s}^{-1}$, which appear to exclude any sort of non-degenerate Roche filling companion, though caveats remain.

Lecture XIII Equation of state of degenerate matter

XIII.0 White Dwarfs: Elementary Estimates

XIII.0.1 Degeneracy pressure

Cold matter is stable against collapse, because two fermions cannot be in the same quantum state. If N electrons (or N identical fermions of any kind) are in a volume V, their average momentum is as large as if each occupied a volume V/N. (In a collection of bosons in its ground state, each boson has the ground state energy for a box of volume V) The spacing between electrons is $l = (V/N)^{1/3} = n^{-1/3}$, where n is the particle density, $n = N/V$. An electron confined to a box of side l has a minimum momentum given by the uncertainty relation,

$$p \sim \frac{\hbar}{l} = \hbar n^{1/3}.$$

An ordinary gas is too hot for the minimum momentum allowed by the uncertainty relation to be important; the particles are widely separated, and their average thermal momentum, of order $\sqrt{mkT}$, is much larger than this minimum momentum. As a result, the translational kinetic energy per particle is the thermal energy $E = \frac{3}{2}kT$, and their pressure is very close to thermal pressure of an ideal gas, $P = nkT$. In a liquid or solid, however, the pressure is provided by the resistance of electrons against being crushed to a length smaller than their ground state spacing. In the ground state, their kinetic energy is not zero; the minimum momentum given above implies a minimum kinetic energy,

$$E = \frac{p^2}{2m} = \frac{\hbar^2}{m} n^{2/3},$$

and the corresponding minimum pressure is

$$P = n\langle p_x v_x \rangle = \frac{1}{3} n\langle pv \rangle = \frac{p^2}{3m} n \sim \frac{\hbar^2}{m} n^{5/3}. \tag{XIII.1}$$

In a metal, the outer electrons are free to move, but they are still in or near a ground state, with the spacing between electrons equal to atomic spacing. For what density n is this minimum pressure comparable to the kinetic pressure $P = nkT$? The degeneracy energy must be comparable to kT:

$$E = \frac{\hbar^2}{m} n^{2/3} = kT. \tag{XIII.2}$$

At room temperature, with $m = m_e$, we have,

$$n^{2/3} = \frac{kTm_e}{\hbar^2} = \frac{\left(\frac{1}{40}\text{eV}\right)[.511\ \text{MeV}/(3\times 10^{10}\text{cm/s})^2]}{(6.48\times 10^{-22}\text{MeV-s})^2} \tag{XIII.3}$$

$$= 3\times 10^{13}\text{cm}^{-2} \Rightarrow \tag{XIII.4}$$

$$\ell = n^{-1/3} = 2\times 10^{-7}\text{cm}. \tag{XIII.5}$$

This is well above atomic spacing, so metals are degenerate. At atomic spacing, $\ell = 5 \times 10^{-9}$ cm, matter is degenerate when

$$
\begin{aligned}
kT &< \frac{\hbar^2}{m_e} n^{2/3} = \frac{\hbar^2}{m_e \ell^2} \sim 10\text{eV}, && \text{(XIII.6)}\\
\Rightarrow T &< 10^5 K. && \text{(XIII.7)}
\end{aligned}
$$

10 eV is the kinetic energy of electrons confined to an atomic-size volume. (It is on the order of the binding energy – e.g., 13.6 eV for hydrogen). In particular, the Sun and Jupiter have electrons at atomic spacing (each has density of about 1 g cm^{-3}) so the Sun, with average temperature far above 10^5 K is not degenerate, while Jupiter, with temperature well below 10^5 K, is.

XIII.0.2 White Dwarfs

The following estimates supplement David's summary in XI.1 .

XIII.0.2.1 Structure: Mass-Radius Relation

A white dwarf is the final state of a star whose *initial* mass is less than about 4 $M_\odot$. At the end of its evolution, the star blows off its outer envelope of hydrogen, and the core that remains contracts and eventually cools to a dead ball of He, or of He and C, depending on the star's initial mass. (The most massive stars that end as white dwarfs leave dwarfs with the heavier elements O, Ne and Mg.) Because the nuclear reactions have turned off, the dead star is held apart by its degeneracy pressure. The size of such a star turns out to *decrease* as its mass increases: adding baryons increases the gravitational attraction enough that more baryons are packed in a smaller total volume. This relation between mass and radius can be found from our equations of hydrostatic equilibrium and the equation of state of a degenerate gas. Here we'll again obtain an estimate based on the averaged form of the equation of hydrostatic equilibrium with dP/dr approximated by $-P/R$, $m(r)$ by $M \sim \rho R^3$, with M the total mass.

The rest you've already seen in Eq. (XI.167)

$$
\begin{aligned}
\frac{P}{R} &= \frac{GM\rho}{R^2}, \qquad P = \frac{\hbar^2}{m_e} n^{5/3} && \text{(XIII.8)}\\
\rho &= m_p n = \frac{M}{R^3} \Rightarrow n = \frac{M}{m_p R^3} \\
(XIII.8) \Rightarrow P &= \frac{GM^2}{R^4} = \frac{\hbar^2}{m_e}\left(\frac{M}{m_p R^3}\right)^{5/3} && \text{(XIII.9)}\\
\frac{GM^2}{R^4} &= \frac{\hbar^2}{m_e} \frac{M^{5/3}}{m_p^{5/3} R^5}
\end{aligned}
$$

$$
R = \frac{\hbar^2}{G m_e m_p^{5/3}} \cdot \frac{1}{M^{1/3}}. \qquad \text{(XIII.10)}
$$

With the right numerical factors for a Helium dwarf,

$$R = 1.4 \frac{\hbar^2}{G m_e m_p^{5/3}} M^{-1/3}$$

or

$$\frac{R}{R_\odot} = 0.014 \left(\frac{M_\odot}{M}\right)^{1/3}. \tag{XIII.11}$$

(When $M = M_\odot$, have $R = .014 R_\odot$). So we obtain the relation $R \propto M^{-1/3}$, valid when the star is

- Dense and cold enough for degeneracy: $M > M_{\text{Jupiter}} = \frac{1}{1000} M_\odot$.
- Not so massive that it collapses: $M < 1.5 M_\odot$.

XIII.0.2.2 Chandrasekhar limit

> *The star has to go on radiating and radiating and contracting and contracting until, I suppose, it gets down to a few km. radius, when gravity becomes strong enough to hold in the radiation, and the star can at last find peace. Dr. Chandrasekhar had got this result before, but he has rubbed it in in his latest paper; and, when discussing it with him, I felt driven to the conclusion that this was almost a* reductio ad absurdum *of the relativistic degeneracy formula.* A. S. Eddington (1935), published version of comments that followed a talk by Chandra on the upper mass limit.

Notice that Chandrasekhar and Eddington recognized the possibility of collapse to a black hole resulting from the upper limit on a mass supported by degeneracy pressure, but they failed to make the connection between collapse and supernovae. Baade and Zwicky had proposed in the previous year that supernovae were the result of collapse to a neutron star, but didn't relate collapse to the upper mass limit. The connection between the limiting mass of a degenerate core (or a white dwarf) and the collapse to a neutron star did not appear in print until 1939 articles by Gamow and by Chandrasekhar.

For a non-relativistic gas, $v = p/m$. As the gas becomes more relativistic, the energy per particle, $m\gamma v$, rises, but the velocity is limited by the speed of light: $v < c$. This limit on velocity implies a limit on the pressure per unit density: $p = m\gamma v$ implies $P/\rho = \frac{1}{3} pvn/(m\gamma n) < \frac{1}{3} c^2$. But the gravitational attraction has no bound. Therefore, as one increases the mass, the gravitational attraction inevitably overcomes the pressure, and this sets an *upper limit on the mass of white dwarfs*:

$$P = \frac{1}{3} pvn = \frac{1}{3} pcn < \frac{1}{3} \hbar n^{1/3} cn = \frac{1}{3} \hbar c n^{4/3}. \tag{XIII.12}$$

Repeating the argument from Eq. (XI.168), we again use

$$n = \frac{M}{m_p R^3}, \text{ and}$$

equation (XIII.12) to write

$$\begin{aligned} P &= \frac{GM^2}{R^4} < \hbar c \frac{M^{4/3}}{m_p^{4/3} R^4} \\ &\quad R^4\text{'s cancel} \Longrightarrow \\ M &< \left(\frac{\hbar c}{G}\right)^{3/2} \frac{1}{m_p^2} = 1.9 M_\odot \end{aligned} \tag{XIII.13}$$

Again we have neglected numerical constants, and a more precise upper limit is

$$M < 0.78 \left(\frac{\hbar c}{G}\right)^{3/2} \frac{1}{m_p^2} = 1.4 M_\odot. \tag{XIII.14}$$

This mass is the Chandrasekhar limit. As the mass of a cold star increases, its radius decreases according to the mass-radius relation (XIII.10), until its electrons become relativistic. When the electrons become relativistic, the pressure they contribute rises more slowly than the gravitational attraction, until, for $M > 1.4M_\odot$, the electrons would have to travel faster than light to support the star. If accreting matter drives a white dwarf, or the dead core of a star close to this upper mass limit, it collapses.

XIII.1 Some Thermodynamics: Newtonian and Relativistic

We will follow Chapter 2 of S&T, with emphasis on Sections 2.1-2.3.

XIII.1.1 Gibbs Condition for Phase Equilibrium

The Gibbs condition relates numbers N_i of particles of different species when physical, chemical or nuclear reactions relate the species. Thermodynamic variables are defined for equilibrium states, but we will need to assume that one can define the thermodynamic variables P, T, S and V for a system with any values of the N_i, even though chemical equilibrium has not been established. That is, one must be able to define equilibrium configurations in which the reactions are frozen out. This is often the case. For example, in a mixture of reacting gases, the number of molecules that, at any instant, are reacting is so small, even in a fast reaction, that they do not appreciably affect the thermodynamic parameters of the mixture (see, e.g., Pippard, *Classical Thermodynamics*, pp. 106-7).

More precisely, one assumes the existence of a function

$$E(S, V, N_i), \tag{XIII.15}$$

and one defines temperature T, pressure P and chemical potentials μ_i as the partial derivatives

$$T = \frac{\partial E}{\partial S}, \quad P = -\frac{\partial E}{\partial V}, \quad \mu_i = \frac{\partial E}{\partial N_i}, \tag{XIII.16}$$

so that

$$dE = TdS - PdV + \sum \mu_i dN_i. \tag{XIII.17}$$

When the reactions reach equilibrium, the number N_i of particles of each species are related, and the system lies on a submanifold of the larger space of configurations $\{(S, V, N_i)\}$.

A fluid element in a star is defined to move with the fluid, so that no fluid flows across its boundary. Then energy is transferred to its surroundings only by heat flow and work, not by the outflow or inflow of the particles N_i. The surrounding star imposes a fixed temperature and pressure on the fluid element. We thus model the local thermodynamic equilibrium of a fluid element as a system

(i) at fixed T and P, for which

(ii) the only energy exchange with its surroundings is in the form of heat ΔQ and work $P\Delta V$.

Condition (ii) implies that if the system absorbs heat δQ from its surroundings, and does work PdV on its surroundings, then

$$dE = \delta Q - PdV. \tag{XIII.18}$$

The change in any system's entropy satisfies the inequality $dS \geq \frac{\delta Q}{T}$ (in which both δQ and dS may be negative if heat is lost to the surroundings), with equality holding for an equilibrium process. Thus

$$dE \leq TdS - PdV. \tag{XIII.19}$$

In particular, under our assumption that reactions proceed slowly enough to let us define the function $E = E(S, V, N_i)$, then from Eq. (XIII.17), we conclude

$$\sum \mu_i dN_i \leq 0, \tag{XIII.20}$$

with

$$\sum \mu_i dN_i = 0, \tag{XIII.21}$$

when chemical equilibrium is established.

The Gibbs potential,

$$G = E - TS + PV,$$

satisfies, in general,

$$dG = -SdT + VdP + \sum \mu_i dN_i. \tag{XIII.22}$$

For our system, assumption (i) implies $dG = \sum \mu_i dN_i$, and (XIII.20) becomes

$$dG \leq 0. \tag{XIII.23}$$

Thus, on the way to chemical equilibrium, G decreases, and chemical equilibrium is a state of minimum G, with

$$dG = \sum \mu_i dN_i = 0, \tag{XIII.24}$$

for any set of changes dN_i away from chemical equilibrium that are allowed by the chemical reactions. In other words, for fixed T and P, G is an extremum at chemical equilibrium.

The restrictions imposed on the particle numbers N_i depend on the reactions that are in equilibrium. We begin with an example and then consider the general case. The reaction

$$H + e^- \leftrightarrow H^- \tag{XIII.25}$$

is important near the surface of the sun: Light emitted by this reaction is a key part of the Sun's continuous spectrum. Changes in N_H, N_e and N_{H^-} produced by the reaction satisfy

$$dN_H = dN_e = -dN_{H^-}.$$

At equilibrium, Eq. (XIII.21) implies

$$\begin{aligned} 0 &= \mu_H dN_H + \mu_e dN_e + \mu_{H^-} dN_{H^-} = (\mu_H + \mu_e - \mu_{H^-}) dN_{H^-} \\ &\Rightarrow \mu_H + \mu_e - \mu_{H^-} = 0 \end{aligned} \tag{XIII.26}$$

To describe a general reaction, let c_i be the coefficient of any reactant ($c_H = c_e = 1$ in (XIII.22)), $-c_i$ the coefficient of any product ($c_{H^-} = -1$). Then the chemical potentials at equilbrium satisfy

$$c_i \mu_i = 0. \tag{XIII.27}$$

If there are k species, each reaction defines a vector $\mathbf{c}$ in a k-dimensional space, $\mathbb{R}^k$. Independent reactions correspond to linearly independent vectors. If there are l independent reactions, then l

linearly independent relations among the μ_i leave $k-l$ independent chemical potentials — and $k-l$ independent conserved functions of particle numbers.

For example, at a time in the early universe when there is equilibrium among $e^+, e^-, \nu_e, \bar{\nu}_e, \nu_\mu, \bar{\nu}_\mu, \mu^+, \mu^-$, and a set of baryons and anti-baryons, reactions conserve only baryon number, charge, and lepton number. Then all chemical potentials will be linear combinations of, say, μ_n, μ_p and μ_e. If the expansion time is short compared to the weak interaction timescale, equilibria will not be established between muonic leptons $(\mu^\pm, \mu_\mu, \bar{\nu}_\mu)$ and electron leptons $(e^\pm, \nu_e, \bar{\nu}_e)$, and there will be four independent chemical potentials μ_n, μ_p, μ_e and μ_μ.

XIII.1.2 Extensive Quantities and Euler's Theorem

The quantities G, E, S, V and N_i are all extensive, all proportional to the size of the system. That is, when one considers two systems that differ only by overall size, all extensive quantities scale in the same way. Consequently,

$$E(\lambda S, \lambda V, \lambda N_i) = \lambda E(S, V, N_i). \tag{XIII.28}$$

Theorem (Euler). Let $f(x^1, \ldots, x^k)$ satisfy

$$f(\lambda \mathbf{x}) = \lambda f(\mathbf{x}). \tag{XIII.29}$$

Then

$$f = x^i \frac{\partial f}{\partial x^i}. \tag{XIII.30}$$

Proof. Left as an exercise.
From Euler's theorem we have

$$E = TS - PV + \mu_i N_i, \tag{XIII.31}$$

implying

$$G = \mu_i N_i. \tag{XIII.32}$$

XIII.1.3 Relativistic Thermodynamics

In the formalism so far presented, only one real change arises in going from the Newtonian limit to relativistic thermodynamics: The relativistic energy density includes the mass

$$E = M_0 + E_{\text{Newtonian}} \quad \text{(taking } c = 1\text{)}.$$

Energy, entropy, volume, and particle numbers remain well defined, if one specifies that they are to be measured by a comoving observer, an observer whose 4-velocity agrees with the 4-velocity u^α of the fluid. One can imagine measurements made by an inertial observer instantaneously at rest relative to a fluid element.

A set of intensive, local thermodynamic quantities that characterize a fluid is listed below (all measured by a comoving observer):

$$\begin{aligned} T &= \text{temperature} \\ P &= \text{pressure} \\ \epsilon &= \text{energy density} \\ n &= \text{baryon density} \\ \rho_0 &= nm_B \simeq \text{rest mass density (when antibaryons are not present)} \\ n_i &= \text{density of } i^{th} \text{ species of particle} \\ Y_i &= \frac{n_i}{n} \end{aligned}$$

The energy density ϵ is denoted by ε in Shapiro-Teukolsky, and S-T write $\varepsilon = \rho c^2$.

For a fluid element of volume V, conserved baryon number N, energy E, entropy S, we have

$$\epsilon = \frac{E}{V}, \quad s = \frac{S}{N}, \quad n = \frac{N}{V}, \quad Y_i = \frac{N_i}{N}, \tag{XIII.33}$$

N conserved $\Rightarrow dN = 0$. Then, from

$$dE = TdS - PdV + \sum \mu_i dN_i, \tag{XIII.34}$$

we have

$$d\left(\frac{E}{N}\right) = Td\left(\frac{S}{N}\right) - Pd\left(\frac{V}{N}\right) + \mu_i d\left(\frac{N_i}{N}\right) \tag{XIII.35}$$

$$d\left(\frac{\epsilon}{n}\right) = Tds - Pd\left(\frac{1}{n}\right) + \mu_i dY_i. \tag{XIII.36}$$

Equivalently,

$$d\epsilon = nTds + (\epsilon + P)\frac{dn}{n} + n\mu_i dY_i. \tag{XIII.37}$$

With a neutral system and enough reactions in equilibrium that only electron lepton number,muon lepton number, and baryon number are conserved, one can take as independent variables s, n, and two Y_i's:

$$\epsilon = \epsilon(s, n, Y_1, Y_2). \tag{XIII.38}$$

More generally

$$\epsilon = \epsilon(s, n, Y_1, \ldots, Y_m), \tag{XIII.39}$$

and

$$\begin{aligned} P &= -\frac{\partial(\epsilon/n)}{\partial(1/n)} = n\frac{\partial\epsilon}{\partial n} - \epsilon, & \text{(XIII.40)} \\ T &= \frac{\partial(\epsilon/n)}{\partial s} = \frac{1}{n}\frac{\partial\epsilon}{\partial s} & \text{(XIII.41)} \\ \mu_i &= \frac{1}{n}\frac{\partial\epsilon}{\partial Y_i} & \text{(XIII.42)} \end{aligned}$$

Eqs. (XIII.31) and (XIII.32) imply

$$\epsilon = nTs - P + \mu_i n_i, \qquad g := \frac{G}{N} = \mu_i Y_i \tag{XIII.43}$$

XIII.2 From Kinetic Theory and Stat Mech

In statistical mechanics, an ideal gas is described by a distribution function $\mathfrak{f}(x,p)$ that gives the number of particles per unit volume of phase space, so that

$$n = \int \mathfrak{f}(x,p) d^3p \tag{XIII.44}$$

is the number of particles per unit volume of physical space. The expectation value of a quantity $Q(x,p)$ is then

$$\langle Q(x) \rangle = \frac{\int Q\, \mathfrak{f}\, d^3p}{\int \mathfrak{f}\, d^3p}$$

or

$$n \langle Q \rangle = \int Q\, \mathfrak{f}\, d^3p. \tag{XIII.45}$$

The distribution function has dimension $1/$(volume in phase space), and S&T write it as

$$\mathfrak{f} = \frac{g}{h^3} f, \tag{XIII.46}$$

with h Planck's constant and f dimensionless. Here g counts the spin degrees of freedom: $g = 2S+1$ for massive particles and $g = 2$ for photons (spin is either along or opposite to photon momentum). For a collection of free particles at a fixed temperature T, f has the familiar form

$$f = \frac{1}{e^{(E-\mu)/kT} \pm 1}, \text{ with the sign} \begin{cases} +, & \text{fermions} \\ -, & \text{bosons} \end{cases} \tag{XIII.47}$$

When the temperature is high and the particle density low, this becomes the Maxwell-Boltzmann distribution, $f = e^{(\mu - E)/kT}$.

Given a distribution function, we can compute the energy density ϵ by writing $\epsilon = n\langle E \rangle$, with $E = p^2c^2 + m^2c^4$ the energy per particle,

$$\epsilon = n \langle E \rangle = \int \sqrt{p^2c^2 + m^2c^4}\, \mathfrak{f}\, d^3p. \tag{XIII.48}$$

To compute the pressure, insert a wall and note that the pressure on the wall is the change in momentum per unit time from particles colliding with the wall: For a wall perpendicular to the x-axis, the change in momentum of a particle is $2p_x$, when $v_x > 0$, and half the particles have $v_x > 0$. (Particles with $v_x < 0$ move away from the wall.) For $v_x > 0$, nv_x particles collide per unit area per unit time. The pressure is then

$$P = n\langle v_x 2p_x \rangle \frac{1}{2} = n\langle v_x p_x \rangle = \frac{1}{3} n \langle v_x p_x + v_y p_y + v_z p_z \rangle = \frac{1}{3}\langle vp \rangle.$$

Using $p = m\gamma v$, $E = m\gamma c^2$ to write $v = pc^2/E = pc^2/\sqrt{p^2c^2 + m^2c^4}$, we have

$$P = \frac{1}{3} \int \frac{p^2 c}{\sqrt{p^2 + m^2c^2}}\, \mathfrak{f}\, d^3p. \tag{XIII.49}$$

XIII.3 EOS of a Completely Degenerate Fermi Gas

Because the temperatures of white dwarfs and neutrons stars are far below the Fermi temperature of their electrons and nucleons, respectively, they can be accurately approximated as supported by the pressure of degenerate fermions. In neutron stars, however, the interactions between nucleons are important; the ideal gas distribution function (XIII.47) ignores interactions and gives a poor approximation to the neutron degeneracy pressure. In white dwarfs, on the other hand, interactions between elections are a small correction, and we can use the ideal gas distribution function at zero temperature to obtain the electron degeneracy pressure and infer the equation of state.

In the limit $T \to 0$, the distribution function becomes a step function, with all states below $\mu(T = 0)$ occupied and no particles with higher energy. That means $\mu(T = 0)$ is the Fermi energy, $\mu(T = 0) = E_F$.

$$f = \begin{cases} 1, & E < E_F \\ 0, & E > E_F \end{cases} \tag{XIII.50}$$

We begin with the number density of electrons, n_e. Defining the Fermi momentum p_F by $E_F^2 = p_F^2 c^2 + m_e^2 c^4$, we have

$$\begin{aligned} n_e &= \int \mathfrak{f}(x,p) d^3p = \frac{2}{h^3}\int_0^{p_F} 4\pi p^2 dp = \frac{8}{3h^3}\pi p_F^3 \\ &= \frac{1}{3\pi^2}\left(\frac{p_F}{\hbar}\right)^3. \qquad (h \to \hbar \text{ here}) \end{aligned} \tag{XIII.51}$$

We next find the pressure and energy density. Note first that the only dimensionful constants here are $\hbar, c$ and the electron mass m_e. Because these have three independent dimensions, we can construct exactly one quantity with a given dimension from these constants. In particular, S&T use the unique combination $m_e c$ with dimension of momentum to replace p_F by the dimensionless version

$$x := \frac{p_F}{m_e c}. \tag{XIII.52}$$

The combinations with dimensions of length and energy are

$$\lambda_e := \frac{\hbar}{m_e c}, \quad \text{and } m_e c^2. \tag{XIII.53}$$

Because n_e has dimension L^{-3} and depends only on $\hbar, m_e, c$ and p_F, it must have the form $\lambda_e^{-3} F(x)$, and it does: Eq. (XIII.51) is

$$n_e := \frac{1}{3\pi^2 \lambda_e^3} x^3. \tag{XIII.54}$$

Equivalently, we can adopt units with $\hbar = m_e = c = 1$. In these units, $p_F = x$ and $n_e = F(x)$; we return to conventional units as usual by multiplying n_e by the combination of $\hbar, m_e, c$ with dimension L^{-3}.

Because P and ϵ each have dimension of energy density, they can be written in the forms

$$P = \frac{m_e c^2}{\lambda_e^3}\phi(x), \qquad \epsilon = \frac{m_e c^2}{\lambda_e^3}\chi(x). \tag{XIII.55}$$

Our job is to find $\phi(x)$ and $\chi(x)$.

From Eq. (XIII.49), we have

$$P = \frac{1}{3}n\langle pv\rangle = \frac{1}{3}\frac{2}{(2\pi\hbar)^3}\int_0^{p_F}\frac{p^2c}{\sqrt{p^2+m^2c^2}}4\pi p^2 dp = \frac{m_e c^2}{\lambda_e^3}\frac{1}{3\pi^2}\int_0^x \frac{x^4}{\sqrt{1+x^2}}dx, \tag{XIII.56}$$

so

$$\phi(x) = \frac{1}{3\pi^2}\int_0^x \frac{x^4}{\sqrt{1+x^2}}dx. \tag{XIII.57}$$

Similarly,

$$\epsilon = n\,\langle E\rangle = \frac{2}{(2\pi\hbar)^3}\int_0^{p_F}\sqrt{p^2c^2+m^2c^4}\,4\pi p^2 dp = \frac{m_e c^2}{\lambda_e^3}\frac{1}{\pi^2}\int_0^x x^2\sqrt{1+x^2}dx \tag{XIII.58}$$

$$\chi(x) = \frac{1}{\pi^2}\int_0^x x^2\sqrt{1+x^2}dx. \tag{XIII.59}$$

S&T give ϕ and χ in closed form, but the integrals themselves more clearly show the behavior of the functions. The Newtonian regime has $x \ll 1$, and the highly relativistic regime has $x \gg 1$. In the Newtonian case, $\sqrt{1+x^2}$ becomes 1 and we have

$$\phi(x) = \frac{1}{15\pi^2}x^5 \qquad \text{(Newtonian)} \tag{XIII.60}$$

implying $P \propto n^{5/3}$ as anticipated. In the ultrarelativistic case, the integrals are dominated by the contribution with $x' \gg 1$ (writing the integrand in terms of x' instead of x to be clear), and we can replace $\sqrt{1+x'^2}$ by x'. Then

$$\phi(x) = \frac{1}{12\pi^2}x^4 \qquad \text{(ultrarelativistic)} \tag{XIII.61}$$

again agreeing with $P \propto n^{4/3}$ from our previous uncertainty-principle argument.

The EOS $P = P(n)$ at all densities follows from the expression for $P(x)$ by using Eq. (XIII.54) to replace x by $\lambda_e(3\pi^2 n_e)^{1/3}$. Explicitly,

$$P = \frac{m_e c^2}{\lambda_e^3}\frac{1}{24\pi^2}\left[3\sinh^{-1}x - x(3-2x^2)\sqrt{1+x^2}\right]\Big|_{x=\lambda_e(3\pi^2 n_e)^{1/3}}. \tag{XIII.62}$$

At zero temperature, the first law of thermodynamics implies

$$\frac{d\epsilon}{\epsilon+P} = \frac{dn}{n}. \tag{XIII.63}$$

That means one can compute $\epsilon(n)$ from $P(n)$ without explicitly computing ϵ as the expectation value (XIII.58). Checking that Eqs. (XIII.63) and (XIII.58) are consistent will be an exercise in the next problem set. Note that the thermodynamic relation (XIII.63) relies *only* on a temperature much smaller than the Fermi energy ($kT \ll E_F$), an excellent approximation for both white dwarfs and neutron stars.

In contrast, as noted earlier, the ideal degenerate Fermi gas is a poor approximation for neutron stars. We now look at its main errors for white dwarfs.

XIII.4 Electrostatic Correction: Electron Binding Energy

S&T now discuss the two primary corrections to the ideal Fermi gas EOS for white dwarfs. The first is the Coulomb binding energy of the electrons. The second (next two sections) arises from the changing fraction of neutrons to protons in nuclei as the density of the dwarf increases. The Coulomb binding energy changes the total energy E per electron and so changes the functions $\epsilon(n)$ and $P(n)$ that constitute the equation of state.

We can estimate the size of the correction by comparing the Coulomb energy $Ze^2/r \sim e^2 n^{1/3}$ to the Fermi energy. In the Newtonian regime, $E_F = \dfrac{p_F^2}{2m_e} = \dfrac{(3\pi^2)^{2/3}\hbar^2 n_e^{2/3}}{2m_e}$ giving

$$\begin{aligned}\frac{E_c}{E_F} &\sim \frac{2Ze^2 m_e n_e^{1/3}}{(3\pi^2)^{2/3}\hbar^2 n_e^{2/3}} = \frac{2}{(3\pi^2)^{2/3}}\frac{Z}{a_0 n_e^{1/3}},\\ &= Z\left(\frac{6\times 10^{22}\mathrm{cm}^{-3}}{n_e}\right)^{1/3} = Z\frac{n_e^{-1/3}}{2.6\times 10^{-8}\mathrm{cm}}\end{aligned} \tag{XIII.64}$$

Here $a_0 = \hbar^2/(m_e e^2)$ is the Bohr radius.

To estimate a typical white dwarf number density, use the mass-radius relation, Eq. XIII.11, in the form $(R_\odot/R)^3 = (0.014)^{-3} M/M_\odot$.

$$\begin{aligned}\frac{\rho}{\rho_\odot} &= \frac{M}{M_\odot}\left(\frac{R_\odot}{R}\right)^3 = (0.014)^{-3}\left(\frac{M}{M_\odot}\right)^2\\ \rho &\approx 5\times 10^5\left(\frac{\mathrm{M}}{\mathrm{M}_\odot}\right)^2 \mathrm{g/cm}^3\end{aligned} \tag{XIII.65}$$

For a dwarf with 2 nucleons per electron (e.g., a He or He-C dwarf),

$$n_e = \frac{\rho}{2m_p} = 1.5\times 10^{29}\left(\frac{M}{M_\odot}\right)^2 \mathrm{cm}^{-3}, \tag{XIII.66}$$

giving $n_e \approx 2\times 10^{28}\mathrm{cm}^{-3}$ and a spacing between nuclei $n_e^{-1/3} \approx 4\times 10^{-10}\mathrm{cm} = 0.07a_0$ for a typical dwarf with $M \sim M_\odot/2$: Our estimate (XIII.64) of the fractional Coulomb correction to the energy is then of order 0.03.

Now for the actual computation of the Coulomb binding energy of the electrons: Assign Z electrons to each nucleus, uniformly distributed in a sphere of volume $1/n_N$, n_N the number density of nuclei. The radius of r_0 the sphere is given by $\frac{4}{3}\pi r_0^3 = \frac{1}{n_N} = \frac{Z}{n_e}$, implying

$$r_0 = \left(\frac{3Z}{4\pi n_e}\right)^{1/3}. \tag{XIII.67}$$

The uniform charge distribution gives within a radius r the charge $q = -Ze\frac{r^3}{r_0^3}$. The work to bring a shell of charge $dq = 3\frac{r^2}{r_0^3}dr$ in from infinity is $\frac{q\,dq}{r}$, and we have

$$E_{e-e} = \int_0^{Ze} \frac{q}{r} dq = \frac{3}{5}\frac{Z^2e^2}{r_0}. \tag{XIII.68}$$

We subtract from this the work done by the ion (the point nucleus),

$$E_{e-i} = -\int_0^{r} \frac{Ze}{r} dq = -\frac{3}{2}\frac{Z^2e^2}{r_0}, \tag{XIII.69}$$

to obtain a binding energy per electron

$$\frac{E_c}{Z} = \frac{E_{e-e} + E_{e-i}}{Z} = -\frac{9}{10}\left(\frac{4}{3}\pi\right)^{1/3} Z^{2/3}e^2 n_e^{1/3}. \tag{XIII.70}$$

The contribution to the pressure $P(n_e)$ is given by Eq. (XIII.40)

$$P_c = n_e^2 \frac{d}{dn_e}(E_c/Z) = -\frac{3}{10}\left(\frac{4}{3}\pi\right)^{1/3} Z^{2/3}e^2 n_e^{4/3}; \tag{XIII.71}$$

For relativistic electrons, this just lowers the coefficient of $n_e^{4/3}$ by the fractional amount

$$-\frac{2^{5/3}}{5}\left(\frac{3}{\pi}\right)^{1/3} \alpha Z^{2/3} \approx -0.01 \tag{XIII.72}$$

for $Z = 2$, where $\alpha = e^2/\hbar c$ is the fine structure constant. Adding the details has given us a result consistent with but somewhat smaller than our estimated 3% correction to energy per electron for nonrelativistic electrons.

We have assumed uniform density, and the error in that assumption is again of order α - the equilibrium electron distribution n_e minimizes an energy that has an order α error. Now changing $n_e(x)$ by order α changes the degeneracy energy by order α^2 for the following reason (which you've encountered in first-order perturbation theory in QM): An eigenstate (here the ground state) is an extremum (here an absolute minimum) of $\langle\psi|H|\psi\rangle$, and changing the wave function by an amount ϵ therefore changes the energy only at order ϵ^2.

There is also a correction to the electrostatic correction itself: The correction is of order alpha, and the correction to that correction is then of order α^2. S&T compute the correction to n_e in the last part of Sect. 2.4.

XIII.5 Inverse β-decay: The Ideal, Cold, n-p-e Gas.

The second correction to the ideal Fermi gas for the white dwarf EOS comes from change in composition as a function of density, changing Y_e the number of electrons per baryon. To understand how this works, we start first with the simplest system of protons, electrons and neutrons and then, in the next section, look at the real case, where the nuclei are not primarily protons. The n-p-e system is not a bad approximation for the interior of neutron stars, so we'll get a decent estimate of the proton/neutron ratio as a function of density in neutron stars and an intuitive understanding of why that ratio increases as the density increases in both white dwarfs and neutron stars.

We want the equilibrium concentrations of n, p and e for the reactions

$$n \to p + e + \bar{\nu}_e, \quad p + e \to n + \nu_e. \tag{XIII.73}$$

The neutrinos leave the star without reacting: Only at the end of collapse, when the neutron star is forming, or in the merger of two neutron stars (or a neutron star and a black hole) is the cross section for neutrino capture high enough to give a mean free path less than a km. Then the number density of neutrinos remains unchanged at 0 and the equilibrium is determined by
(1) the Gibbs condition (XIII.24), $\sum \mu_i dN_i = 0$, with $dN_\nu = 0$, namely

$$\mu_p + \mu_e - \mu_n = 0; \tag{XIII.74}$$

(2) charge neutrality

$$n_e = n_p. \tag{XIII.75}$$

Now μ_i is the energy needed to add a particle of the ith species. Here, at zero temperature, all energy levels are filled up to the Fermi level, none above, and an additional particle is therefore added at its Fermi energy:

$$\mu_i = E_F^i = \sqrt{p_F^2 + m_i^2} \qquad (c = 1). \tag{XIII.76}$$

Neutrons can be created only if the sum of the Fermi energies of an electron and a proton is greater than the rest mass of a neutron. That is, at low density, $\mu_e = m_e,\ \mu_p = m_p,\ \mu_n = m_n$, and the relation (XIII.74) can't be satisfied – The Gibbs condition

$$0 = \mu_e dN_e + \mu_p dN_p + \mu_n dN_n = (\mu_e + \mu_p - \mu_n) dN_e$$

holds only if there is no reaction, $dN_e = 0$.

As one increases the density of an p-e gas, neutrons first form when $\mu_e + \mu_p = m_n$. Here $\mu_n = m_n$ because the neutron density is zero. Now charge neutrality together with the relation (XIII.51) $n \propto p_F^3$ implies $p_F^e = p_F^n$. The equilibrium condition becomes

$$\sqrt{p_F^{e\,2} + m_e^2} + \sqrt{p_F^{e\,2} + m_p^2} = \sqrt{p_F^{n\,2} + m_n^2}. \tag{XIII.77}$$

(This is S&T (2.5.5), written with p_F instead of x to make the physics clearer.) When neutrons first form, the electrons must have p_F^e of order m_e, because the mass difference between neutron and (proton + electron) is of order m_e. The protons are still nonrelativistic $p_F^p \ll m_p$, since $m_e \ll m_p$, so $E_F^p = m_p$ and we can write the condition for neutrons to first form as

$$E_F^e \equiv \sqrt{{p_F^e}^2 + m_e^2} = m_n - m_p : \tag{XIII.78}$$

The electron Fermi energy must be equal to the mass difference between neutron and proton. The critical density is then

$$n_e = \frac{1}{3\pi^2}\left(\frac{p_F^e}{\hbar}\right)^3 = \frac{1}{3\pi^2\hbar^3}[(m_n - m_p)^2 - m_e^2]^{3/2} = \frac{1}{3\pi^2\lambda_e^3}\left[\left(\frac{m_n - m_p}{m_e}\right)^2 - 1\right]^{3/2} \tag{XIII.79}$$

$$= 7\times 10^{30}\mathrm{cm}^{-3},$$

$$\rho = m_p n_p = m_p n_e = 1.3\times 10^7\mathrm{g/cm}^3, \tag{XIII.80}$$

This number density is an order of magnitude above that (XIII.66) of a typical dwarf but seven orders of magnitude below nuclear density.
Exercise. Check the consistency of our assumption $E_F^p = m_p$: Taylor expand $E = \sqrt{p^2 + m^2}$ to obtain the Newtonian energy correction to the rest mass, and find the fractional error $(E_F^p - m_p)/m_p$ in using m_p. Use the value of $p_F^p = p_F^e$ from (XIII.78) or (XIII.79). (Don't forget to either restore c or use energy units for m.)

For densities intermediate between the critical density for neutron formation and densities high enough that protons and neutrons are relativistic the Gibbs condition gives the ratio S&T (2.5.17). With $Q := m_n - m_p$, the expression is

$$\frac{n_p}{n_n} = \frac{1}{8}\left\{\frac{1 + \frac{4Q}{m_n x_n^2} + 4\frac{Q^2 - m_e^2}{m_n^4 x_n^4}}{1 + 1/x_n^2}\right\}^{3/2} \tag{XIII.81}$$

and we'll go over the somewhat lengthy derivation. The high density limit of this equation is the large x limit: $n_p/n_n = 1/8$. It turns out to be easy to find this high-density ratio directly from charge neutrality and the form of $\mu = E_F$ for ultrarelativistic particles, $p_F \gg m$.
Exercise. Do that: Use charge neutrality and the Gibbs condition for n-p-e equilibrium to show that the high density limit of n_p/n_n is

$$\frac{n_p}{n_n} = \frac{1}{8}. \tag{XIII.82}$$

(This is a quick calculation. Do *not* use Eq.(XIII.81)$\equiv$ (2.5.17) of S&T.)

Doing the easy exercise first is useful to see what's happening; now we embark on the more detailed, less transparent calculation leading to (XIII.81). The Gibbs condition (with the charge-neutrality condition $p_n = p_e$) is

$$\sqrt{p_p^2 + m_e^2} + \sqrt{p_p^2 + m_p^2} = \sqrt{p_n^2 + m_n^2} \quad \text{or} \quad \sqrt{A} + \sqrt{B} = \sqrt{C}. \tag{XIII.83}$$

Our goal is to write this as an equation for n_p/n_n as a function of p_n. Because $n_p/n_n = p_p^3/n_p^3$ and the Gibbs condition involves p_p^2 and p_n^2, we find p_p^2 in terms of p_n^2 and then divide by p_n^2. Now

$$\sqrt{A} + \sqrt{B} = \sqrt{C} \Rightarrow A + B + 2\sqrt{AB} = C \Rightarrow A^2 + B^2 + C^2 - 2(AB + AC + BC) = 0,$$

$$\text{or} \qquad 2(A+B)C = (A-B)^2 + C^2. \tag{XIII.84}$$

This last form is helpful because only the left side involves p_p. We have

$$\begin{aligned}2(2p_p^2+m_e^2+m_p^2)(p_n^2+m_n^2) &= (m_p^2-m_e^2)^2+(p_n^2+m_n^2)^2\\ 4p_p^2(p_n^2+m_n^2) &= (m_p^2-m_e^2)^2-2(m_e^2+m_p^2)(p_n^2+m_n^2)+(p_n^2+m_n^2)^2\\ &= p_n^4+2p_n^2(m_n^2-m_p^2-m_e^2)+m_e^4+m_p^4+m_n^4-2(m_e^2m_p^2+m_e^2m_n^2+m_p^2m_n^2)\\ &= p_n^4+2p_n^2(m_n^2-m_p^2-m_e^2)+(Q^2-m_e^2)[(m_n+m_p)^2-m_e^2]. \qquad \text{(XIII.85)}\end{aligned}$$

Now $Q \sim m_e^2 \ll m_p^2 \sim m_n^2$. To an accuracy of one part in 2000, we can replace $m_n^2 - m_p^2 - m_e^2 = Q(m_n+m_p) - m_e^2$ by $2Qm_n$ and we can replace $(m_n+m_p)^2 - m_e^2$ by $4m_n^2$:

$$4p_p^2(p_n^2+m_n^2) = p_n^4 + 4p_n^2 Q m_n + 4(Q^2-m_e^2)m_n^2.$$

Dividing by $4p_n^2(p_n^2+m_n^2)$ gives p_p^2/p_n^2, and we have

$$\frac{n_p}{n_n} = p_p^3/p_n^3 = \frac{1}{8}\left\{\frac{p_n^4+4p_n^2Qm_n+4(Q^2-m_e^2)m_n^2}{p_n^2(p_n^2+m_n^2)}\right\}^{3/2}, \qquad \text{(XIII.86)}$$

equivalent to Eq. (XIII.81) after the replacement $p_n \equiv m_n x_n$. Note that we are not entitled to discard the terms involving Q and m_e in the numerator here, because p_n can have any value: Qm_n will be larger or of order p_n^2 when $p_n \ll m_n$ - when the density is near the critical density for neutron production.

XIII.6 Beta-Equilibrium Between Relativistic Electrons and Nuclei: The Harrison-Wheeler EOS

In white dwarfs, as in the progenitor stars, Coulomb repulsion between nuclei is high enough to prevent the star from turning into iron ($^{56}_{26}$Fe), the lowest energy state at zero pressure. As a result, the composition of a dwarf depends on the composition of its progenitor star. When neutron stars form, photons are energetic enough to destroy the iron core of the star (photodisintegration of iron nuclei) and the energy and density is high enough that the matter is baked to an equilibrium state. As one proceeds inwards from the outer crust of the star, the density rises from the density of iron to 10^{15} g/cm^3, a few to several times nuclear density of $2.7 \times 1014\mathrm{g/cm}^3$. For ordinary nuclei, increasing the size increases the average nuclear binding energy, because a larger fraction of nucleons are in the interior, bonded to all nearest neighbors. For larger nuclei, however, the Coulomb repulsive energy grows as Z^2, overcoming the total nuclear binding energy, which is proportional to the number of nucleons (because only adjacent nucleons bond). This is what makes the binding energy per nucleon decrease after iron.

As the density increases, however, the electron Fermi energy rises, and the stablest nucleus shifts to one with more neutrons. Finally, the Fermi energy of neutrons in neutron-rich nuclei becomes high enough that it is energetically favorable to just add free neutrons: This is neutron drip, the density above which electrons combine with protons in nuclei to form free neutrons, leaving a smaller nucleus. At this density, the free neutrons do not decay because of the high Fermi energy of free electrons.

The equilibrium configuration of protons, neutrons and electrons is specified by finding the number of neutrons and protons in the stablest nucleus together with the number density of free electrons and, once the density is high enough, the number density of free neutrons.

Terminology:

$$\begin{aligned} A &= \text{number of baryons in a nucleus} \\ Z &= \text{number of protons in a nucleus} \\ n &= \text{number density of baryons} \\ n_N &= \text{number density of nuclei} \qquad & Y_N = n_N/n \\ n_e &= \text{number density of electrons} \qquad & Y_e = n_e/n \qquad \text{(XIII.87)} \\ n_n &= \text{number density of neutrons} \qquad & Y_n = n_n/n \qquad \text{(XIII.88)} \\ & & \text{(XIII.89)} \end{aligned}$$

The configuration is determined by A, Z, Y_n and n: That is, the fraction of baryons in nuclei is $1 - Y_n$, so the number densities of nuclei and electrons are

$$n_N = (1 - Y_n)/A \tag{XIII.90}$$

$$n_e = (1 - Y_n)\frac{Z}{A}. \tag{XIII.91}$$

Because $T = 0$, for a fixed number density n, minimizing the Gibbs free energy is equivalent to minimizing ϵ:

$$d\epsilon = Tds + (\epsilon + P)\frac{dn}{n} + n\sum \mu_i Y_i = n\sum \mu_i Y_i.$$

Our goal then is to write ϵ as a function of A, Z, Y_n and to write

$$\nabla\epsilon = 0, \quad \text{i.e.,} \ \frac{\partial\epsilon}{\partial Z} = 0, \ \frac{\partial\epsilon}{\partial A} = 0, \ \frac{\partial\epsilon}{\partial Y_n} = 0, \tag{XIII.92}$$

as three equations that determine the three quantities A, Z, Y_n as functions of n: We want the point in configuration space at which ϵ is a minimum.

Let $M(A, Z)$ be the mass of nucleus of A baryons and Z neutrons plus the rest mass of Z electrons (including the electrons follows historical and S&T convention). Then

$$\begin{aligned} \epsilon &= n_N M(A, Z) + (\epsilon_e - m_e) + \epsilon_n \\ &= n(1 - Y_n)\frac{M}{A} + \varepsilon_e(n_e)\big|_{n_e=(1-Y_n)Z/A} + \epsilon_n(n_n)\big|_{n_n=nY_n}. \end{aligned} \tag{XIII.93}$$

The heart of the problem is to find $M(A, Z)$. The approximation we use is essentially the *liquid drop model* in which A and Z are assumed large enough that the nucleus can be regarded as a sphere whose radius is proportional to $A^{1/3}$ and with the number of surface nucleons proportional to the surface area $\propto A^{2/3}$. Then, with E_b the average nuclear binding energy of a bulk nucleon (a nucleon bonded to all its nearest neighbors), the binding energy is smaller than AE_b by an amount proportional to the surface area,

$$-\text{binding energy} = -AE_b + \beta_2 A^{2/3},$$

with β_2 soon to be related to S&T b_2. For a given number of baryons, the sum of the proton and neutron Fermi energies is minimized by an equal division of protons and neutrons; this preference for an equal division is enhanced by the fact that the $n-p$ bond is stronger than the $p-p$ or $n-p$ nuclear bond. Because the nuclear-interaction binding energy is a minimum for $Z/A = 12$, departures from the minimum increase that binding energy by an amount quadratic in the departure, by a term $\beta_4(1/2 - Z/A)^2$. Finally, The Coulomb repulsion of protons adds an energy $\beta_5 \dfrac{Z^2}{A^{1/3}}$

$$M(A,Z) = (A-Z)m_n + Z(m_e + m_p) - E_b + \beta_2 A^{2/3} + \beta_4 A\left(\frac{1}{2} - \frac{Z}{A}\right) + \beta_5 \frac{Z^2}{A^{1/3}}.$$

S&T follow convention in factoring out the mass per baryon m_u in ^{12}C and almost grouping terms by powers of A

$$M(A,Z) = m_u\left[b_1 A + b_2 A^{2/3} - b_3 Z + b_4 A\left(\frac{1}{2} - \frac{Z}{A}\right)^2 + b_5 \frac{Z^2}{A^{1/3}}\right] \tag{XIII.94}$$

$$b_1 = \frac{m_n - E_b}{m_u}, \qquad b_3 = \frac{m_n - m_p - m_e}{m_u}.$$

The contribution of the binding energy to b_1 then measures a difference between bulk binding energy and nuclear binding energy per nucleon in ^{12}C; similarly b_2 and b_5 measure the difference of surface energy per baryon and Coulomb energy from their ^{12}C values.

The energy density of a neutral collections of (A, Z) nuclei, is

$$\epsilon(Z, A, Y_n) = n_N M(A,Z) + \varepsilon_e + \epsilon_n. \tag{XIII.95}$$

Here $\epsilon_n = \epsilon_n(n_n)$ is the energy density of free neutrons, $\varepsilon_e = \varepsilon_e(n_e) = \epsilon_e - n_e m_e$ is the internal energy density of the electrons (the electron rest mass density was already counted in M). As usual, adding an extra neutron or electron adds it at the Fermi energy, because all lower levels are filled:

$$\frac{d\epsilon_n}{dn_n} = E_F^n, \qquad \frac{d\varepsilon_e}{dn_e} = E_F^n - m_e.$$

The equilibrium condition that ϵ is a minimum means that nearby configurations with the same number of baryons distributed differently (and with the electron number determined by charge neutrality) have the same ϵ to linear order in ΔA, ΔZ and ΔY_n:

$$\frac{\partial \epsilon}{\partial Z} = \frac{\partial \epsilon}{\partial A} = \frac{\partial \epsilon}{\partial Y_n} = 0, \tag{XIII.96}$$

or, as a function on the three-dimensional configuration space with coordinates Z, A, Y_n, $\nabla\epsilon = 0$. These three conditions determine the values of the three variables Z, A, Y_n – the compostion of the nuclei (the values of A and Z and the relative numbers of nuclei and free neutrons. The energy of the equilibrium configuration as a function of n then determines the EOS.

The two most interesting conditions are (1) the relation determining the ratio Z/A (equivalently, the proton/neutron ratio in nuclei) and (2) the relation giving the critical density at which free neutrons appear – the onset of neutron drip. In S&T, the first relation is obtained from several lines of calculatieon that obscure what turns out to be a simple meaning. Here is a quicker and more physically transparent derivation: ϵ is a minimum among configurations with the same number n of baryons in a fixed volume. In particular ϵ is a minimum for rearrangements of the protons and neutrons from a set of nuclei with A baryons to a set with $A+1$ baryons per nucleus (and fewer total nuclei): In each nucleus, we are fixing Z/A while increasing A. Now a line of constant Z/A in the $Z-A$ plane is a radial line: In the $x-y$ planea line of constant x/y is a line of constant ϕ, changing only r. Demanding that a function $f(x,y)$ be a minimum at a point along this line means $\vec{r}\cdot\nabla f = 0$, or $(x\partial_x + y\partial_y)f = 0$. So we have, at equilibrium,

$$(A\partial_A + Z\partial_Z)\frac{M}{A} = 0. \tag{XIII.97}$$

Now

$$\frac{1}{m_u}\frac{M}{A} = b_1 + b_4\left(\frac{1}{2} - \frac{Z}{A}\right)^2 - b_3\frac{Z}{A} + b_2 A^{-1/3} + b_5\frac{Z^2}{A^{4/3}}$$

The first three terms are functions only of Z/A, so their radial derivative is zero. We are left with

$$\begin{aligned} -\frac{1}{3}b_2 A^{-1/3} + \frac{2}{3}b_5\frac{Z^2}{A^{4/3}} &= 0, \\ Z &= \sqrt{\frac{b_2}{2b_5}}A^{1/2}. \end{aligned} \tag{XIII.98}$$

The condition for neutron drip is easy to see and derive, and here S&T give the obvious interpretation. At equilibrium, taking one neutron from each nucleus and setting it free keeps the energy of the system unchanged. The number of nuclei and electrons are unchanged: The neutron is added to the existing sea of neutrons at the Fermi energy, so the change in mass of the nucleus from losing one neutron is then is then equal to the Fermi energy of the neutron. Formally, $\Delta A = -1$ for each of the n_N nuclei; and the number of free neutrons in a unit volume is increased by $\Delta n_n = n_N$. We then have

$$\begin{aligned} 0 = n_N\frac{\partial M}{\partial A}\Delta A + \frac{d\epsilon_n}{dn_n}\Delta n_n &= n_N\left(-\frac{\partial M}{\Delta}A + E_F^n\right) \Longrightarrow \\ \frac{\partial M}{\partial A} &= E_F^n. \end{aligned} \tag{XIII.99}$$

Lecture XIV Neutron Stars

XIV.1 Neutron star estimates

We have already seen that the spacing between nucleons in a nucleus is on the order

$$\ell = \frac{\hbar}{m_\pi c} = 1.3 \times 10^{-13}\mathrm{cm}.$$

Actual nuclear spacing is very close to this:

$$\ell = 1.8 \times 10^{-13}\mathrm{cm}.$$

The corresponding number density is

$$n = \frac{1}{\ell^3} = 1.7 \times 10^{38}\mathrm{cm}^{-3},$$

and the mass density is

$$\rho = m_p n = 2.8 \times 10^{14}\mathrm{g\ cm}^{-3}.$$

A few months before Chadwick discovered the neutron in 1932 Landau was visiting Bohr in Coperhagen. Landau thinking that nuclei must involve some neutral combination of protons and electrons suggested the possibility of stars with cores of nuclear density. Using the value of nuclear density that we just found, he quickly computed the size of a 1 $M_\odot$ star of nuclear density:

$$\frac{4}{3}\pi R^3 \rho = M_\odot \Rightarrow R = \left(\frac{M_\odot}{\frac{4}{3}\pi\rho}\right)^{1/3} = 12\mathrm{km}.$$

After Chadwick's discovery, Bohr, Landau and others, were thinking about the possibility of stars with neutron cores.

Energy of a supernova

Armed with Landau's suggestion and with the observations of the previous 10 years showing that observed supernova occurred in other galaxies, Baade and Zwicky wrote a paper in 1934 in which they wrote, "With all reserve we advance the view that a super-nova represents the transition of an ordinary star into a neutron star, consisting mainly of neutrons." Their suggestion was due in part to the following computation of the energy that would be produced in such a collapse, although they were unaware that Chandrasekhar's upper limit on white dwarfs would imply that a collapse of this kind was in fact possible.

Supernovae are explosions that can outshine the entire galaxy in which they occur, with brightness as great as 10^{43} erg/s ($> 10^9 L_\odot$) and total energy emitted as large as 10^{53} erg. (Energy in light is $10^{51} - 10^{52}$ erg, in neutrinos about 10^{53} erg). If one could convert the entire mass of the Sun to light, the energy emitted would be $M_\odot c^2 = (2 \times 10^{33}\mathrm{g})(3 \times 10^{10}\mathrm{cm\ s}^{-1})^2 = 2 \times 10^{54}$ erg, so the total energy emitted by a supernova is nearly $\frac{1}{10} M_\odot c^2$.

The Baade-Zwicky calculation is simple: What is the gravitational energy released in the collapse of a star of mass $M_\odot$ to the size of a neutron star, 10 km?

$$\Delta E = E_{\rm initial} - E_{\rm final} = -\frac{GM_\odot^2}{R_{\rm initial}} - \left(-\frac{GM_\odot^2}{10\ \rm km}\right) \tag{XIV.100}$$

$$= \frac{GM_\odot^2}{10\ \rm km}. \tag{XIV.101}$$

The first term is negligible compared to the second here, because the initial radius $R_{\rm initial}$, is 1000 times the final 10 km radius for stellar core (at white dwarf size) or a white dwarf collapsing to a neutron star. Let's write ΔE as a fraction of $M_\odot c^2$:

$$\Delta E = \frac{GM_\odot}{10\ {\rm km}\ c^2} M_\odot c^2 \tag{XIV.102}$$

$$\frac{GM_\odot}{10\ {\rm km}\ c^2} = \frac{(6.7\times10^{-8}{\rm g}^{-1}{\rm cm}^3{\rm s}^{-2})(2\times10^{33}{\rm g})}{(10^6{\rm cm})(3\times10^{10}{\rm cm\ s}^{-1})^2} = 0.1 \Longrightarrow \tag{XIV.103}$$

$$\Delta E = \frac{1}{10} M_\odot c^2 \tag{XIV.104}$$

Thus the collapse of an ordinary star (really a white-dwarf-like stellar core of iron) to a neutron star produces total energy equal to that observed in supernovae.

Iron cores collapse to neutron stars instead of simply fusing to make heavier elements because iron is the stablest element - the element with the greatest binding energy per nucleon. White dwarfs, on the other hand, are the endpoints of evolution of stars too light to have formed iron. If they accrete enough matter from a companion to reach their upper mass limit (or if two degenerate stellar cores of elements lighter than iron merge) their initial collapse leads to explosive nucleosynthesis well before neutron-star density is reached. As a result, much less energy is emitted: The total energy is about 10^{51} erg. Instead of collapsing to a neutron star, the explosion blows the star apart.

Spin-down of a neutron star

The hypothesis that pulsars were neutron stars was strikingly confirmed when it was found that the Crab pulsar was spinning down and that the spin down rate agreed with what one would predict from the hypothesis that it was a rotating neutron star. The Crab nebula is a supernova remnant from a supernova that occurred in 1054 and was seen and recorded by Asian astronomers (only one known mention of it in Europe). After Jocelyn Bell and Tony Hewish's discovery of a set of pulsars and the suggestion that they might be neutron stars, two sets of astronomers started to take data from the Crab nebula in the hope of seeing a pulsar inside it. Once they looked, they quickly saw a pulsar blinking 30 times a second. This is much too fast to be a rotating or pulsating white dwarf. As we have seen, to be able to rotate 30 times a second, an object has to be much denser than a dwarf. The speed of the blinking and the fact that the pulsar was found in the center of a supernova remnant, as would be expected for a neutron star, strongly suggested that pulsars were neutron stars.

The clinching evidence was obtained when the pulsar's period was seen to change: It was spinning down at a rate $\dot P/P = \frac{1}{2500}$ yr. If this energy were going into heating up the Crab nebula, could it account for the observed luminosity of 3×10^{38}erg/s? Tommy Gold did the quick calculation and

found that the predicted and observed luminosities agreed.
Exercise. Do that calculation, estimating the NS moment of inertia and checking that the observed luminosity agrees with the rate at which the pulsar loses rotational energy.

Magnetic field

The average magnetic field at surface of the Sun is ~ 1 G. If the Sun were to collapse to 10 km (this will never happen) while conserving flux, its magnetic field would be

$$B(10 \text{ km })^2 = 1\text{G} \cdot (7 \times 10^5 \text{ km})^2 \Rightarrow B = 5 \times 10^{10} \text{ G}$$

This is an underestimate of what one would expect for neutron star magnetic fields from the collapse of stellar cores. Magnetic fields in the cores of sun and of other stars are expected to be perhaps 100 times larger than this, reflected in the 100–1000 G magnetic fields of sunspots. In addition, fluid motion during collapse (e.g., differential rotation, or a dynamo, or a magneto-rotational instability) magnifies seed magnetic fields. Neutron star magnetic fields are typically observed in the range 10^9–10^{12} G, with the smaller value corresponding to old neutron stars, whose magnetic fields have gradually faded (but are still enormous by earthly standards, where 10^6 G can only be sustained for an instant). The high-magnetic field tail of young neutron stars extends to about 10^{15}G, and the class of neutron stars with surface fields above 10^{14} G are called magnetars. Magnetar interior fields may exceed 10^{15} G.

Black holes and the upper limit on the mass of a neutron star (or of any matter above nuclear density)

Like white dwarfs, neutron stars have an upper limit on their mass. In fact there is a rigorous upper limit on the mass of any object at nuclear density or higher.

Upper limit on mass of a star at or above nuclear density

Density greater than nuclear:

$$\rho > \rho_{\text{nuclear}} = 2.7 \times 10^{14} \text{g cm}^{-3}. \tag{XIV.105}$$

This fiducial density, $2.7 \times 10^{14} \text{g/cm}^3$, is nuclear saturation density, the density inside large nuclei, a density for which all nucleons are surrounded by other nucleons.

No black hole:

$$R > \frac{2\,GM}{c^2} = 3 \text{ km}\,\frac{M}{M_\odot}. \tag{XIV.106}$$

Then

$$\begin{aligned} M = \int \rho 4\pi r^2 dr \quad &> \quad \frac{4}{3}\pi R^3 \rho_{\text{nuclear}} \\ &> \quad \left[\frac{4}{3}\pi (3 \text{ km})^3 \rho_{\text{nuclear}}\right] = \frac{1}{16} M_\odot \frac{M^3}{M_\odot^3}, \end{aligned} \tag{XIV.107}$$

using Eq. (XIV.106) above in the last line. We then have

$$\begin{aligned} M^2 \quad &< \quad 16\, M_\odot^2 \\ &\text{or} \\ M \quad &< \quad 4\, M_\odot. \end{aligned} \tag{XIV.108}$$

Including the surrounding lower-density envelope gives

$$M < 5\ M_\odot.$$

Note that, by keeping the minimum density in the form ρ in Eq. (XIV.108), instead of using its numerical value, we get an upper limit on a spherical self-gravitating mass whose density is everywhere greater than ρ

$$M < 4\ M_\odot \sqrt{\frac{2.7 \times 10^{14}}{\rho}} \tag{XIV.109}$$

The calculation assumed almost nothing about the matter—only that its density was greater than nuclear. If, in addition, one assumes that the speed of sound is less than the speed of light, one can improve the upper mass limit on dense stars: Taking $\frac{dP}{d\rho} \le c^2$ for ρ above some matching density ρ_m and using the (almost) known equation of state below $\rho_{\rm match}$ gives

$$M < 4.1\sqrt{\frac{2.7 \times 10^{14} \mathrm{g/cm^3}}{\rho_{\rm match}}} M_\odot, \tag{XIV.110}$$

with equality for $\frac{dP}{d\rho} = c^2$, the stiffest equation of state consistent with causality. This was first found by Rhoades and Ruffini and was written in a form that shows the dependence on matching density by Hartle and Sabbadini.

XIV.2 Equation of State Above Nuclear Density

S&T Chapter 8 is still a good reference. The notes comment on more recent work and on observational constraints.

Description from Chamel, Fantina, Zdunik, and Haensel, Phys Rev C 91, 055803:

"According to our current understanding, a neutron star contains qualitatively distinct regions. A thin atmospheric plasma layer of light elements (mainly hydrogen and helium) possibly surrounds a Coulomb liquid of electrons and ions. Below these liquid surface layers, the matter consists of a solid crust made of a crystal lattice of fully ionized atoms. With increasing density, nuclei become progressively more neutron rich due to electron captures until neutrons start to drip out of nuclei at some threshold density $\rho_{\rm drip}$. This neutron-drip transition marks the boundary between the outer and inner crusts. The crust dissolves into an homogeneous liquid mixture at about half the density prevailing in heavy atomic nuclei."

As we have seen in S&T Chap. 2, the condition for the onset of neutron drip is that the equilibrium nucleus with A nucleons and Z protons be in equilibrium with a nucleus with one fewer neutron together with a free neutron:

$$M(A, Z) = M(A - 1, Z) + m_n. \tag{XIV.111}$$

Finding the density requires the detailed form of $M(A,Z)$, but can find a lower bound as follows: At any density, extremizing $M(A,Z)$ means that the mass is unchanged by electron capture: The electron Fermi energy is equal to the difference $M(A,Z) - M(A,Z-1)$. A necessary condition for neutron drip to occur is then that the electron Fermi energy be higher than the loss of binding energy from capturing an electon and then removing a neutron: At nuclear density, the binding energy per nucleon is about 10 MeV. For the Fermi energy of an electron to be this high, the electron must be highly relativistic (electron rest mass = 0.511 MeV), so its Fermi energy is given by $E = p_e c$. The electron number density must then be at least

$$n_e = \frac{p_F^3}{3\pi^2\hbar^3} = \frac{E^3}{3\pi^2(\hbar c)^3} \gtrsim \frac{(10\text{MeV})^3}{3\pi^2(\hbar c)^3} \sim 4\times 10^{33} cm^{-3}. \tag{XIV.112}$$

using $\hbar c = 2.0\times 10^{-11}$ MeV-cm. The nuclei are neutron-rich. Assuming about 3 nucleons per electron gives

$$\rho \gtrsim 3 m_p n_e \sim 2\times 10^{10}, \tag{XIV.113}$$

about 20 times smaller than the actual neutron drip density.

XIV.2.1 Allowing nuclei to compress

Near and above neutron drip, the volume of each nucleus is smaller as a result of the increased pressure. To correct the EOS presented in S&T Chapter 2, Baym and Pethick introduce the volume V_N of each nucleus as a variable. The baryon density is then

$$n = A n_N + (1 - V_N n_N) n_n, \tag{XIV.114}$$

where n_n now means the number density of free neutrons in the volume outside the nuclei. The free-neutron contribution to the total energy density is similarly

$$\epsilon_n (1 - V_N n_N), \tag{XIV.115}$$

and the mass of each nucleus is a function of three variables $M(A,Z,V_N)$. Extremizing the energy density with respect to the additional variable V_N gives a fourth condition on the four variables A, Z, Y_n, V_N,

$$\frac{\partial\epsilon}{\partial V_N} = 0. \tag{XIV.116}$$

This is the requirement that the pressure of the neutron gas balances the pressure of the nuclei – essentially $dE = -PdV$ for each contribution to the pressure.

Adding an additional degree of freedom means the energy is lower than it would be with V_N frozen, and that softens the equation of state. The pressure is also smaller: As the number density n increases, the energy per baryon ε/n increases more slowly than it would without the additional freedom, so $d(\frac{\epsilon}{n}) = Pd\frac{1}{n}$ implies P increases more slowly.

n, p, e, μ matter
As the density rises above nuclear density, the electron Fermi level rises above the mass of a muon:

$$E_F = m_\mu = 105.7\mathrm{MeV} \gg \mathrm{m_e} \Longrightarrow \text{highly relativistic electrons}$$
$$n_e = \frac{p_F^3}{3\pi^2\hbar^3} = \frac{(m_\mu c^2)^3}{3\pi^2(\hbar c)^3} = \frac{1}{3\pi^2}\left(\frac{105.7\mathrm{MeV}}{2.0\times 10^{-11}\mathrm{MeV-cm}}\right)^3 = 5\times 10^{36}\mathrm{cm}^{-3} \qquad \text{(XIV.117)}$$

(This is equivalent to the value $x_e = 0.206$ on p. 223 of S&T.) Neutrons now outnumber electrons (and protons) by a factor of 20-40 (Yakovlev, Kaminker, Gnedin, Haensel, Phys. Reports **354**, 1, '01), so the density at which muons first appear is about

$$\rho = m_n n_n \approx 1.5\times 10^{38}\mathrm{cm}^{-3}(1.7\times 10^{-24}\mathrm{g/cm^3}) \approx 3\times 10^{14}\mathrm{g/cm^3} \qquad \text{(XIV.118)}$$

S&T quote a higher value, coming from a smaller proton/neutron ratio for an ideal degenerate Fermi gas.

XIV.2.2 Nuclear Pasta

As one goes inward and the crust approaches nuclear density, there is a transition from small spherical nuclei in a sparse gas, to a dense gas with giant embedded spherical nuclei (gnocchi), to a final state where the gas has reached nuclear density and there is no distinction between nuclei and gas. When the nuclei occupy greater than about 1/2 the total volume, the matter turns inside out, with bubbles of less dense matter immersed in the dense nuclear matter (swiss cheese). Along the way, as the pressure and density increase toward the swiss-cheese state, the shape of the embedded nuclei progress through to rods (spaghetti), and sheets (lasagna).

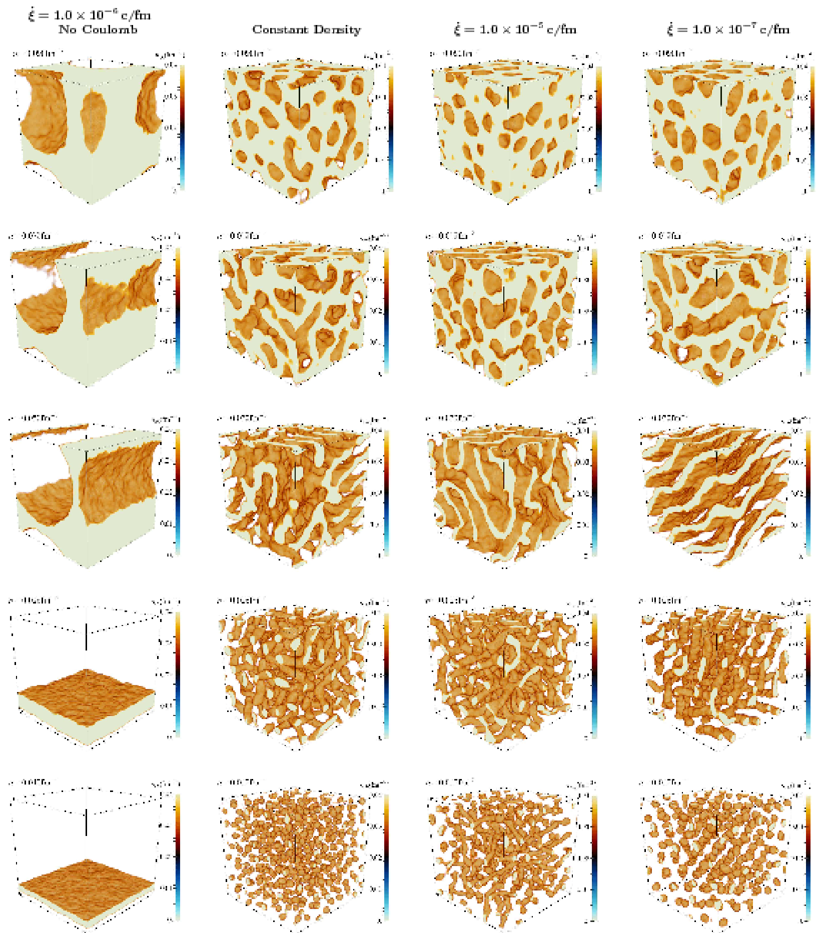


Figure 11: Nuclear pasta, from Chamel et al.

XIV.2.3 Core EOS

In the inner crust and outer part of the core, the neutron-star is an n, p, e, μ collection. The equation of state is found by many-body theory in a quantum-mechanical or quantum-field-theory context. Parts 8.3-8.11 of S&T present the Bethe-Johnson EOS, which uses a Hartree-Fock nonrelativistic quantum-mechanical treatment. Glendenning's book, *Compact Stars* gives a clear treatment of the quantum field theory (relativistic mean field) approach. We will not cover this.

In the last parts of Chapter 8, S&T summarize key unresolved issues associated with the equation of state of the neutron star core. They all amount to the way quarks are grouped and whether strange quarks (heavier than the up and down quarks of protons and neutrons) appear. Here's a summary of the particles. The listed quark masses are almost irrelevant to the masses of the particles they comprise. Here's a table of particles and their quark composition.

name	quarks	charge	mass
up quark	u	+2/3	2.3 MeV
down quark	d	-1/3	4.8 MeV
strange quark	s	-1/3	95 MeV
proton	uud	1	938.3 MeV
neutron	udd	0	939.6 MeV
Lambda Λ^0	uds	0	1115.7 MeV
pion π^-	$\bar{u}$ d	-1	139.6 MeV
kaon K^-	$\bar{u}$ s	-1	493.7 MeV

As a mnemonic, you can think of a strange quark as a heavy version of a down quark. A kaon is then a heavy version of a pion, and a Λ^0 is a heavy version of a neutron. (And for leptons, a muon is a heavy version of an electron.)

The alternative quark groupings in the core are summarized in the following diagram by Fridolin Weber:

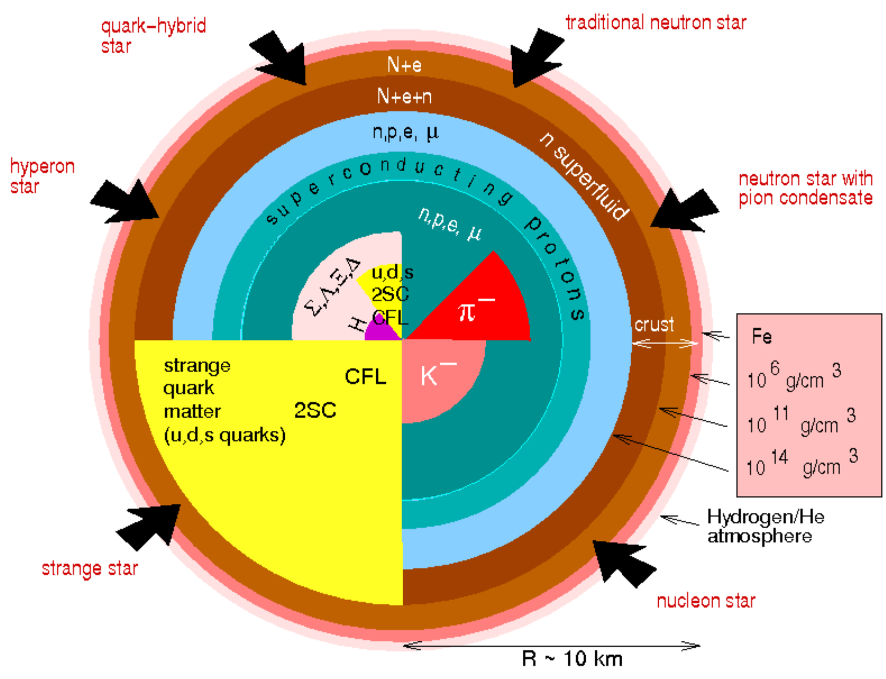


Figure 12: Alternate core compositions

The figure has all of the uncertainties listed in S&T, together with a possibility not known to S&T, that strange quark matter, a combination of unconfined up down and strange quarks in a single bag (a single giant nucleon) is the true ground state of matter at zero pressure. S&T discuss the more likely alternative, that strange quark matter is the composition at the highest densities occurring in neutron stars, when nucleons overlap. We briefly discuss the alternatives, following the order in S&T, beginning with pion condensation.

Pion and Kaon Condensation

Pions are quark-anti-quark pairs with mass 140 MeV, not too much larger than a muon. Weak interactions $e^- \to \pi^- + \nu_e$, $\mu^- \leftrightarrow \pi^- + \nu_\mu$ can replace electrons and muons by pions when their Fermi energies exceed the pion energy. Because pions are bosons, they can, in principle, all be in the same ground state, forming a pion condensate. [7] Because muons and electrons interact only via the weak and electromagnetic forces, the Fermi gas approximation is very good. Pions, however, are strongly interacting, and that means the density at which they appear is not just given by (electron Fermi energy) = (pion rest mass). Calculations extrapolating from low-density, with nucleon interactions and from high density with quark and gluon interactions still leave unresolved the energy at which they appear and the question of whether they condense.

Kaons are versions of pions with strange quarks replacing down quarks K^-=s$\bar{\mathrm{u}}$. Because strange quarks have mass of about 100 MeV, when particles have Fermi energies more than 100 MeV above their rest mass, it is sensible to ask whether strange particles are present. The possibility of kaon condensates was predicted in early calculations by Kaplan and Nelson and using features of the standard model. The basic idea is that the energy of K^- is lowered by interaction with nucleons, lowering the minimum electron Fermi energy needed to produce a kaon. More recent calculations by Pandharipande et al. (Phys.Rev.Lett. **75**,4567, '95) find that the attractive interaction at neutron-star densities is lower by about a factor of two. This raises the energy needed to create kaons and reduces the likelihood of a condensate.

A note that may make the terminology you will in the literature less opaque: Both of these calculations involved features of the standard model – of quantum chromodynamics. Calculations using using quarks and gluons and the fundamental Lagrangian of the standard model are impossible for neutron stars, but that Lagrangian restricts the possible terms that can arise in the approximate Lagrangians used to compute the EOS and composition of neutron stars. This restricted set of terms is due to what is called chiral symmetry: Without a Higgs particle, all particles in the standard model are massless and have spins either aligned or anti-aligned with their momentum - they have either left- or right-handed chirality (helicity). Chiral symmetry of the Lagrangian itself then restricts the allowed terms.[8]

Hyperons

Hyperons are baryons with at least one quark that is not an up or down quark. The lowest-mass hyperon is Λ^0, the neutral Lambda baryon (uds), differing from a neutron (udd) by down quark $\to$ strange quark. Typical reactions associated with Λ^0 equilibrium in neutron stars are

$$n \to \Lambda^0 + \pi^0$$
$$\Lambda^0 \to n + \pi^0 \qquad \Lambda^0 \to p + \pi^-$$

(followed by $\pi^- \to \mu^- + \bar{\nu}_\mu$, $\pi^0 \to \gamma\gamma$). We can estimate the density needed for Λ^0 to appear by

[7] Proposed in different forms by Bahcall and Wolf, Phys. Rev. **140**, B1445-1451, '65; Migdal; Sawyer; and Scalapino. A recent review by Pethick is at https://arxiv.org/pdf/1507.05839.pdf.

[8] Intrinsically massive particles have no definite helicity, but there are still two different representations of the Poincare group for massive particles of spin 1/2, and chirality is the term used to distinguish them. In the massless limit it becomes helicity.

demanding that the Fermi energy E_F of a neutron be at least as large as the sum of the rest masses of a Λ^0 and a neutral pion, $m_\pi + m_\Lambda =$ 140 MeV+ 1115 MeV = 1255 MeV. Use

$$\hbar c = 2.0 \times 10^{-11}\text{Mev-cm}, \quad n_n \sim \frac{1}{3\pi^2}\left(\frac{p_F c}{\hbar c}\right)^3, \quad E_F^2 = (p_F c)^2 + m_n^2 c^4.$$

Then $E_F = 1255$ MeV gives $p_F c = 834$ MeV,

$$n_n \sim \frac{1}{3\pi^2}\left(\frac{834\text{ MeV}}{2.0\times 10^{-11}\text{Mev-cm}}\right)^3 = 2.4\times 10^{39}\text{cm}^{-3}, \quad \rho = m_n n_n \sim 4\times 10^{15}\text{g/cm}^3, \tag{XIV.119}$$

about 15 times nuclear density. This is probably higher than the central density of the most massive possible neutron star, but we have ignored the binding energy of the Λ and the nuclear repulsion between neutrons at high density, and each of these reduces the density at which the hyperons appear. So the high-density nuclear interaction uncertainty leaves the question of hyperons open.

Strange quark matter

In 1968, scattering experiments at SLAC found the proton had pointlike constituents. These "partons" could be interpreted as the quarks predicted by Gell-Mann and Zweig in 1964. Where Landau had predicted stars at nuclear density a few months before Chadwick's discovery of the neutron, it was two years after the SLAC discovery that Naoki Itoh first suggested the existence of quark stars a super-nuclear densities (*Hydrostatic Equilibrium of Hypothetical Quark Stars*, Prog. Theor. Phys. **44**, 291,'70).

When the spacing between nucleons is smaller than the size of a nucleon, the quarks that are confined to nucleons are free to move. The size of a proton is about 10^{-13}cm, and if we suppose that they lose their identity when the spacing is about half that, then nuclei dissolve into a collection of quarks at a number density $n \sim 10^{40}$ cm^{40} or $\rho \sim 1.7\times 10^{16}$ g/cm^3. The corresponding Fermi energy per quark is then of order $(3\pi^2 n)^{1/3}\hbar c \sim 1300$ MeV or 1.3 GeV, high enough for a down quark $\to$ strange quark transition. At the Relativistic Heavy Ion Collider at Brookhaven, gold nuclei collide with energies of 100 GeV/nucleon, momentarily giving densities well above neutron-star density and creating a quark-gluon plasma. (If only the matter had time to cool at that density, we could measure the neutron star EOS. As it is, whether the highest density parts of neutron star cores are are a collection of free up, down and strange quarks is still unresolved.

A year after Itoh's paper, Bodmer ('71) and later but independently Witten ('84) pointed out that experimental data does not rule out the possibility that the ground state of matter at zero pressure and large baryon number is not iron but strange quark matter. If this is the case, all "neutron stars" may be strange quark stars, with roughly equal numbers of up, down and strange quarks, together with electrons to give overall charge neutrality.

XIV.3 Theoretical and observational constraints on the EOS

See e.g. Lattimer AIP Conference Proceedings 1645, 61 (2015); https://doi.org/10.1063/1.4909560; Lattimer, *The Nuclear Equation of State and Neutron Star Masses*, arXiv:1305.3510; Read *et al.*, Phys. Rev. D**79**, 124032,'09.

Causality gives a maximally stiff and maximally soft EOS consistent with causality. Both have $v_{\rm sound} = c$ in a central core. Maximally stiff:

$$P = \begin{cases} P_{\rm known}(\rho), & \rho \le \rho_{\rm match} \\ P_{\rm known}(\rho_{\rm match}) + (\rho - \rho_{\rm match})c^2, & \rho \ge \rho_{\rm match}. \end{cases} \quad \text{(XIV.120)}$$

This gives

$$M_{\rm max} = 4.1\sqrt{\frac{2.7\times 10^{14}{\rm g/cm^3}}{\rho_{\rm match}}} \quad \text{(XIV.121)}$$

The maximally compact EOS (Koranda, Stergioulas, JF) consistent with a maximum mass $M_{\rm max}$ paradoxically also has a maximally stiff core. The reason is that one needs a stiff core to support a massive star, while the most compact star has the softest possible EOS. By having the stiffest possible central core, one can have the largest possible part of the star with a soft EOS. The softest EOS is the EOS with the smallest pressure, so the maximally compact EOS is given by

$$P = \begin{cases} 0, & \rho \le \rho_0 \\ (\rho - \rho_0)c^2, & \rho \ge \rho_0, \end{cases} \quad \text{(XIV.122)}$$

where the ρ_0 is the largest value of rho that still allows a star with mass $M_{\rm max}$. This EOS gives a lower limit on the radius of any neutron star

$$R > 8.5\ {\rm km}\frac{\rm M_{max}}{\rm 2M_\odot} \quad \text{(XIV.123)}$$

and a lower limit $R \ge 2.82GM/c^2$ for a star of mass M. Matching to a known low-density EOS instead of to zero pressure raises this radius by only a few percent.

This EOS gives a minimum period of rotation

$$P_{\rm min} = 0.4\,\frac{M_{\rm max}}{2M_\odot}\ {\rm ms}. \quad \text{(XIV.124)}$$

The same EOS gives the minimum period of a star of a given mass, proportional as usual to $1/\sqrt{\rho}$, here using the average density of the nonrotating star:

$$P_{\rm min} = (0.96 \pm 0.03)\left[\frac{M_{\rm spherical}/M_\odot}{(R_{\rm spherical}/10\,{\rm km})^3}\right]^{-1/2} {\rm ms} \quad \text{(XIV.125)}$$

Because the minimum period is much shorter than the smallest observed period (1.4 ms), the observational constraint is weak.

The constraints just summarized lead to the diagram below from Lattimer. The EOSs shown there are among those listed in the following table.

TABLE 1

EQUATIONS OF STATE

Symbol	Reference	Approach	Composition
FP	Friedman & Pandharipande (1981)	Variational	*np*
PS	Pandharipande & Smith (1975)	Potential	$n\pi^0$
WFF(1–3)......	Wiringa, Fiks & Fabrocine (1988)	Variational	*np*
AP(1–4)	Akmal & Pandharipande (1997)	Variational	*np*
MS(1–3)........	Müller & Serot (1996)	Field theoretical	*np*
MPA(1–2)......	Müther, Prakash, & Ainsworth (1987)	Dirac-Brueckner HF	*np*
ENG	Engvik et al. (1996)	Dirac-Brueckner HF	*np*
PAL(1–6)	Prakash et al. (1988)	Schematic potential	*np*
GM(1–3)	Glendenning & Moszkowski (1991)	Field theoretical	*npH*
GS(1–2)	Glendenning & Schaffner-Bielich (1999)	Field theoretical	*npK*
PCL(1–2)	Prakash, Cooke, & Lattimer (1995)	Field theoretical	*np*HQ
SQM(1–3)......	Prakash et al. (1995)	Quark matter	Q (*u*, *d*, *s*)

NOTE.—"Approach" refers to the underlying theoretical technique. "Composition" refers to strongly interacting components (n = neutron, p = proton, H = hyperon, K = kaon, Q = quark); all models include leptonic contributions.

As the figure on the next page shows, the listed EOSs with strange quarks in any configuration (strange matter, hyperons, and kaon condensates) were all ruled out by the discovery of a neutron star with mass $2M_\odot$. That has not stopped theorists from producing modified EOSs with hyperons and/or strange quark matter that allow stars that massive, but it does make it more difficult.

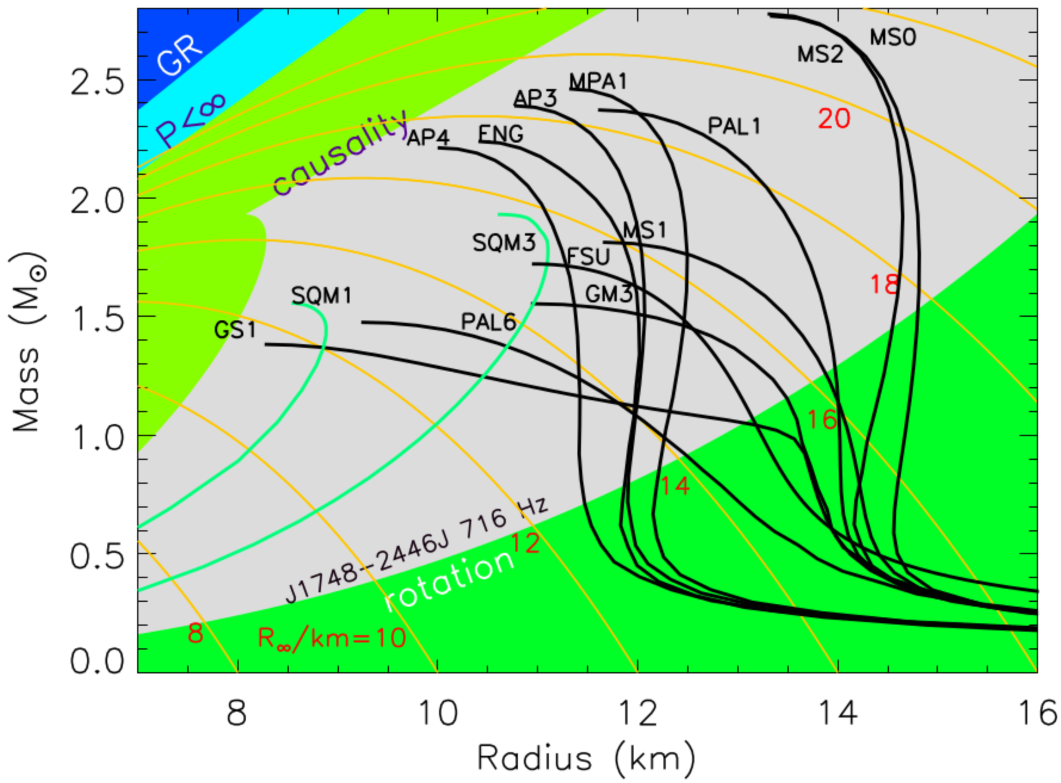


Figure 13: Representative candidate EOSs and constraints.
Each black and green curve shows the masses and radii of the sequence of spherical stars based on a candidate EOS. Of these EOSs, GM3 has hyperons, GS1 has a kaon condensate, and the other solid black lines are sequences based on EOSs with npeμ composition. Sequences of strange quark stars, based on EOS SQM1 and SQM3 (for strange quark matter) are shown as solid green lines. Stars whose mass and radius lie in the upper green or blue regions are forbidden by observation and causality. The upper green region is ruled out by the maximally stiff EOS (the straight boundary) or the maximally soft EOS (the curved part of the boundary). If an EOS were to have stars only in the darker green region in the lower right, it would be unable to yield stars rotating at or above the frequency, 716 Hz, of the fastest observed pulsar. Finally, EOSs whose maximum mass is below 2 $M_\odot$ are ruled out by the observation of neutron stars with $M = 2M_\odot$.

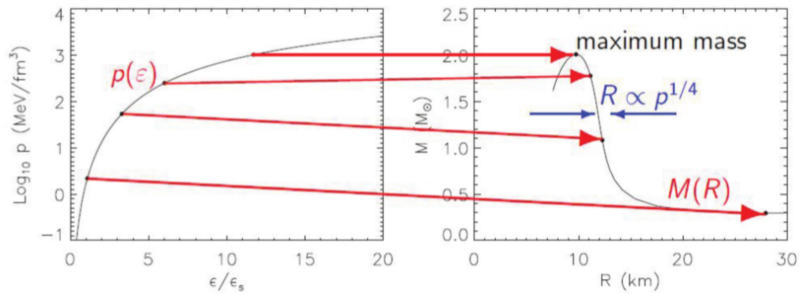


FIGURE 1. Generation of the $M(R)$ curve from a specific EOS $p(\varepsilon)$. $\varepsilon_s \simeq 150$ MeV fm^{-3} is the energy density at the nuclear saturation density $n_s = 0.16$ fm^{-3}. The mass is in solar units.

Figure 14: EOS to M(R) curve
1 fm = 10^{-13} cm. $100\,\mathrm{MeV/fm^3} = 1.7 \times 10^{14}\mathrm{g/cm^3}$.

Fig. 14 illustrates a relation between the neutron-star EOS and the mass vs radius curve $M(R)$ for the sequence of spherical stars based on that EOS. A neutron star of a given mass sees only the EOS below its central density, and arrows in the diagram approximately give the central density (and pressure) of stars with mass and radius on the right. If one can measure the mass-radius curve, one can go backwards and determine the EOS $P(\rho)$.

One can systematize the observational constraints on the neutron-star EOS by introducing a parameterized EOS above nuclear density with a set of parameters large enough to encompass the wide range of candidate EOSs and small enough that the number of parameters is smaller than the number of relevant observations. Typical EOSs behave like polytropes over parts of their range and can be modeled by piecewise-polytropes: That is one approximates the $\log P$ vs $\log \rho_0$ (rest mass) curve by a continuous, picecewise-linear curve. Typical EOSs behave like polytropes over parts of their range and can be modeled by piecewise-polytropes: That is one approximates the $\log P$ vs $\log \rho_0$ (rest mass) curve by a continuous, picecewise-linear curve. One specifies the pressure at a fixed set of densities. (Read *et al.*)

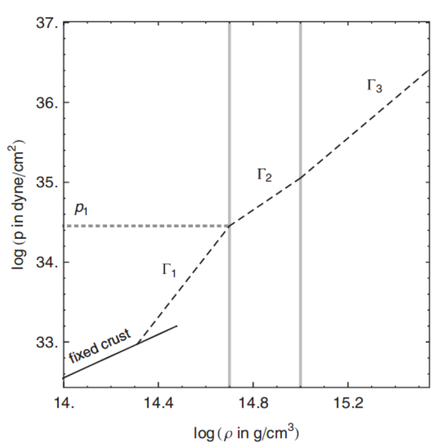


Figure 15: Piecewise polytrope. The log P vs log ρ curve can be approximated by a piecewise-linear curve – a piecewise polytrope.

Two densities are picked out by their relation to neutron-star radius and maximum mass, one at the low end of the core density range, the other at the high end. As Lattimer and Prakash noticed, the radius of a $1.4M_\odot$ neutron star (or one with a given mass of that order) is roughly determined by its pressure at about twice nuclear saturation density, about $2\rho_n = 5.4 \times 10^{14}$ g/cm^3. Inspiral constraints on radius are likely to impose constraints only on the equation of state near or below this density. A radius below 12 km for a $1.4M_\odot$ neutron star corresponds to a pressure at 5×10^{14} g/cm^3 below 3.2×10^{34} dyne, while a measured radius of 11 km corresponds to a pressure about $2/3$ that value.

The high end of the density range is tied to the maximum mass. Stiffer cores support larger masses and yield lower central densities. For equations of state with maximum masses of $2.5M_\odot$ to $2.0M_\odot$, central densities range from about 2×10^{15} to 2.9×10^{15} g/cm^3. The maximum mass is then approximately governed by the pressure at densities of order 7-8$\rho_n \sim 2 \times 10^{15}$ g/cm^3.

Within the next decade, it may be possible to measure the moment of inertia of one of the pulsars (component A) in the binary pulsar PSR J0737-3039. Spin-orbit coupling depends on the moment of inertia of the star, and it leads to precession. In this case, the precession period is about 75 years. The observable correction is the correction to periastron advance. Because the stellar masses of the system are known to high precision, measuring precision in this system, measuring I gives a good measurement of the radius.

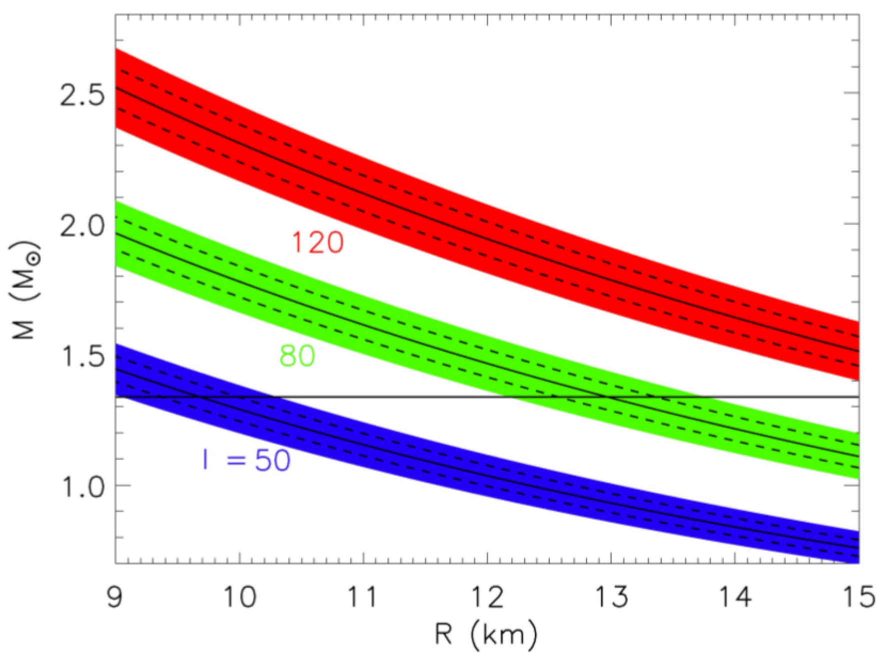


Figure 16: Constraint from a possible future measurement of a moment of inertia. A horizontal line shows the mass of pulsar A in 30737-3039.

Constraints implied by the inspiral and coalescence of two neutron stars in GW710817 will be discussed when we talk about compact binary systems.

Lecture XV Accretion

Let's begins by considering the gravitation potential energy of a proton.

$$E_G(r) = -\frac{GMm_p}{r} \tag{XV.126}$$

So falling from $r = \infty$, this implies

$$\Delta E = E_G(r = \infty) - E_r = \frac{GMm_p}{r}$$

How much energy is this for $M = 1\mathrm{M}_\odot$:

- $r = 10^9$ cm (10000 km) $\rightarrow 133$ keV
- $r = 10^6$ cm (10km) $\rightarrow 133$ MeV
- $r = 2GM/c^2$ (black hole) $\rightarrow 450$ MeV
- Rest mass energy of proton 1 GeV

The virial theorem (from classical mechanics/stat mech) is

$$2K = -U, \tag{XV.127}$$

where K is the kinetic energy and U is the potential. KE for a collection of atoms is their internal energy or temperature k_BT, where k_B is Boltzmann's constant. So the typical temperature of a self gravitating system is

$$k_BT = \frac{GMm_p}{r} \tag{XV.128}$$

For the sun $r = R_\odot \approx 7 \times 10^{10}\,$cm, we have $T \approx 20\,$MK, which is a good estimate for the temperature at the sun's core.

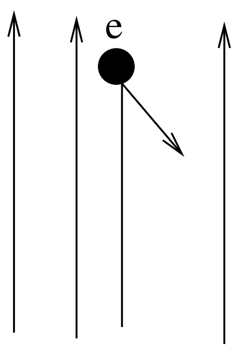
Now suppose material falls toward an object. The energy released per unit time is:

$$L \equiv \dot{E} = \dot{N}\Delta E = \dot{N}\frac{GMm_p}{r} = \frac{GM\dot{M}}{r}, \tag{XV.129}$$

where L is the luminosity.

Up to this point, L appears that it can be arbitrary large, but this is not the case, as the energy release must escape and the escaping energy can exert a pressure. For most cases of interest, the energy released is light and light exerts a pressure.

Let consider the following picture



An electron scatters light so they can absorb momentum. So the force f that light places on it depends on the number of photons hitting it per unit time and the Δp per photon or

$$f = \dot{n}_\gamma \Delta p = \dot{n}\frac{E_\gamma}{c} = \frac{\dot{E}_\gamma}{c} \tag{XV.130}$$

So the thing to figure out is the number of photons scattered per unit time. Let us think of the electron as a solid object so the electrostatic potential energy is equal to the rest mass energy or

$$\frac{e^2}{r_e} = m_e c^2, \tag{XV.131}$$

where r_e is called the classical electron radius. So the cross section is

$$\sigma_T \sim r_e^2 \rightarrow \sigma_T = \frac{8\pi}{3} r_e^2 = 6.6 \times 10^{-25}\,\mathrm{cm}^2, \tag{XV.132}$$

where σ_T is known as the Thomson cross section. The $8\pi/3$ involves a bunch of math, but this OOM estimate got pretty close.

Anyhow, the energy flux in photons is then

$$\dot{E}_\gamma = n_\gamma E_\gamma c \sigma_T = \sigma_T F = \sigma_T \frac{L}{4\pi r^2}, \tag{XV.133}$$

where in the above, I have assumed a spherical radiation field. The force per electron is then:

$$f = \frac{\sigma_T L}{4\pi r^2 c} \tag{XV.134}$$

Let's set this equal to gravity on a *proton*.

$$\frac{\sigma_T L}{4\pi r^2 c} = \frac{GMm_p}{r^2} \rightarrow L_{\mathrm{Edd}} = \frac{4\pi GMm_p c}{\sigma_T} = 1.4 \times 10^{38} \left(\frac{M}{\mathrm{M}_\odot}\right) \mathrm{ergs\,s^{-1}} = 3.6 \times 10^4\, L_\odot \left(\frac{M}{\mathrm{M}_\odot}\right) \tag{XV.135}$$

This is known as the Eddington luminosity. Note that is is a function of M and σ_T only.

When the luminosity approaches the Eddington luminosity, radiation pressure starts to interfere with additional accretion and can actually stop it. The rate of accretion necessary to hit the Eddington luminosity it the Eddington mass accretion rate and is given by

$$L = \frac{GM\dot{M}}{r} = L_{\mathrm{Edd}} \rightarrow \dot{M}_{\mathrm{Edd}} = \frac{4\pi r c m_p}{\sigma_T} = 1.4 \times 10^{-8} \left(\frac{r}{10\,\mathrm{km}}\right) \mathrm{M}_\odot\,\mathrm{yr}^{-1}. \tag{XV.136}$$

Note that $M_{\rm Edd}$ depends on r and σ_T, but not mass.

So above this accretion rate, additional accretion will have to fight through the radiation pressure. This can lead to a stagnation of the accretion rate, thought this does not have to happen. See the homework for a counterexample.

In any case, it is natural to think about the luminosity of falling material into a object in terms of the materials rest mass. This sets an efficiency of accretion

$$L = \frac{GM\dot{M}}{r} = \eta \dot{M} c^2 \rightarrow \eta = \frac{GM}{rc^2} \tag{XV.137}$$

For $r = 2GM/c^2$ for BH's, we have $\eta = 1/2$, which is true for maximally spinning Kerr holes.

For more typically systems like white dwarfs $\eta \sim 10^{-4}$ or NS $\eta \sim 0.1$

XV.2 Spherical Accretion

We begin with the Euler Equations. This include the continuity equation:

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \rho \boldsymbol{v} = 0 \tag{XV.138}$$

and the momentum equation

$$\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} \boldsymbol{v} = -\frac{1}{\rho} \boldsymbol{\nabla} P - \boldsymbol{\nabla} \Phi. \tag{XV.139}$$

Here the symbols has the usual meaning. The pressure P and potential Φ requires additional equations to describe them, i.e., to complete the set of equations. For pressure this is accomplished through an equation of state $P = P(\rho, T)$, where T is the temperature. This just replaces P with T, not a great simplification as you need a description for T. But in many cases, one can argue the the thermodynamics are adiabatic and so $P = \kappa \rho^\gamma$, where κ is the constant and γ is the adiabatic exponent.

Now suppose the (gravitational) force we want to consider is due to a point mass:

$$\boldsymbol{f} = -\frac{GM\rho}{r^2} \rightarrow -\boldsymbol{\nabla}\Phi = -\frac{GM}{r^2}. \tag{XV.140}$$

At this point, we will adopt the first simplification. Let us consider spherically symmetric accretion:

$$\boldsymbol{\nabla} \cdot \rho \boldsymbol{v} \rightarrow \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \rho v_r \quad \text{and} \quad \boldsymbol{\nabla} \rightarrow \frac{\partial}{\partial r} \quad \text{and} \quad \boldsymbol{v} \cdot \boldsymbol{\nabla} \rightarrow v_r \frac{\partial}{\partial r} \tag{XV.141}$$

So in this case, we find

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \rho v_r = 0 \tag{XV.142}$$

$$\frac{\partial v_r}{\partial t} + v_r \frac{\partial}{\partial r} v_r = -\frac{1}{\rho} \frac{\partial}{\partial r} P - \frac{\partial}{\partial r} \Phi. \tag{XV.143}$$

The second simplification that we will apply is steady state where we set the time derivatives $\partial/\partial t \to 0$. This gives:

$$\frac{1}{r^2}\frac{\partial}{\partial r}r^2\rho v_r = 0 \quad \text{and} \quad v_r\frac{\partial}{\partial r}v_r = -\frac{1}{\rho}\frac{\partial}{\partial r}P - \frac{\partial}{\partial r}\Phi. \tag{XV.144}$$

The first equation gives $r^2\rho v_r =$ constant, which is basically the mass flux, i.e.,

$$4\pi r^2\rho v_r = \dot{M} = \text{constant} \tag{XV.145}$$

To finish this off, let's calculate the pressure forces in this situation, using the adopted equation of state $P = \kappa\rho^\gamma$:

$$\frac{\partial}{\partial r}P = \frac{\partial P}{\partial \rho}\frac{\partial}{\partial r}\rho = \frac{\gamma P}{\rho}\frac{\partial}{\partial r}\rho = c_s^2\frac{\partial}{\partial r}\rho, \tag{XV.146}$$

where $c_s = \sqrt{\gamma P/\rho}$ is the sound speed.

Putting this into the spherical momentum equation:

$$v_r\frac{\partial}{\partial r}v_r = -\frac{c_s^2}{\rho}\frac{\partial}{\partial r}\rho - \frac{GM}{r^2} \tag{XV.147}$$

Continuity is

$$\frac{\partial}{\partial r}\rho = -\frac{2}{r}\rho - \frac{\rho}{v_r}\frac{\partial}{\partial r}v_r \tag{XV.148}$$

Using this the eliminate $\frac{\partial}{\partial r}\rho$, we find

$$\frac{1}{2}\frac{\partial}{\partial r}v_r^2 = c_s^2\left[\frac{2}{r} + \frac{1}{v_r}\frac{\partial}{\partial r}v_r\right] - \frac{GM}{r^2} \tag{XV.149}$$

The term $\frac{1}{v_r}\frac{\partial}{\partial r}v_r$ can be written as $\frac{1}{2v_r^2}\frac{\partial}{\partial r}v_r^2$ and so we get

$$\frac{1}{2}\left(1 - \frac{c_s^2}{v_r^2}\right)\frac{\partial}{\partial r}v_r^2 = -\frac{GM}{r^2} + \frac{2c_s^2}{r} \tag{XV.150}$$

Before solving this equation, let consider the case $P = 0 \to c_s = 0$, i.e., the pressureless case.

$$\frac{1}{2}\frac{\partial}{\partial r}v_r^2 = -\frac{GM}{r^2} \to v_r^2 = \frac{2GM}{r}, \tag{XV.151}$$

which is just free-fall. From the continuity equation, we recall

$$\dot{M} = 4\pi r^2\rho v_r = 4\pi r^2\rho\sqrt{\frac{2GM}{r}} \to \rho = \frac{\dot{M}}{4\pi\sqrt{2GM}\; r^{3/2}} \tag{XV.152}$$

Simple enough. Now let suppose you solve the full equations without the restriction of $c_s = 0$. In this case you will notice that when $c_s^2/v_r^2 = 1$, the LHS is zero. Thus the RHS is also zero! This means

$$\frac{GM}{r_s^2} = \frac{2c_s^2}{r_s} \to r_s = \frac{GM}{2c_s^2}, \tag{XV.153}$$

where r_s is called the sonic radius.

You don't have to worry about it if you can avoid it. But this is generally not the case. The general class of solutions is portrayed in Figure 17. This breaks up to a number of cases

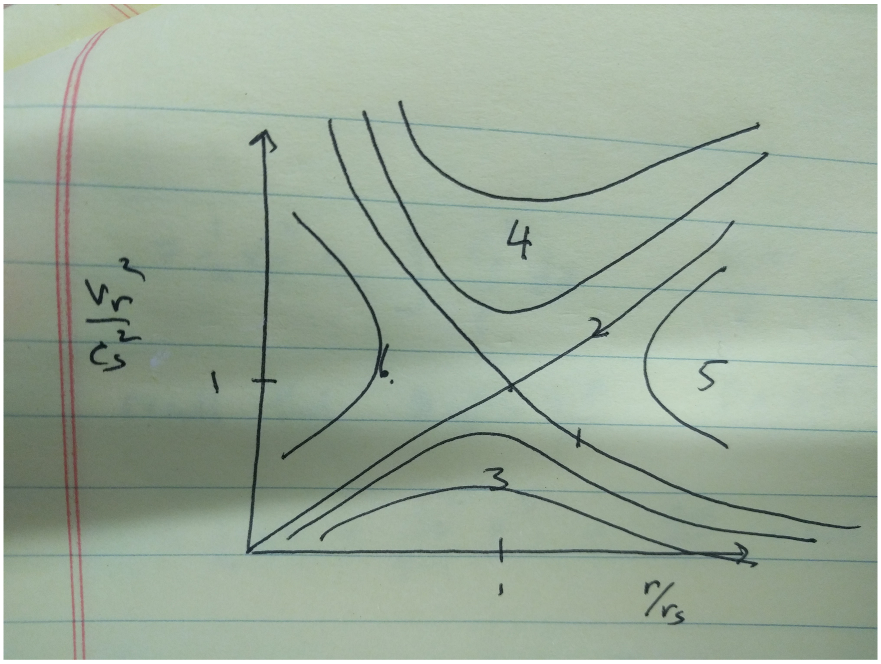


Figure 17: Solutions for spherical accretion.

1. Solution 5,6 have $r < r_s$ or $r > r_s$ so no accretion – killed off immediately
2. Solution 4 has $v_r \gg c_s$ as $r \to \infty$
3. Solution 3 has $v < c_s$ as $r \to 0$ – Settling flow
4. Solution 2 has $v \to 0$ as $r \to 0$ and $v \to \infty$ as $r \to \infty$
5. Solution 1 has $r \to \infty$ as $r \to 0$ and $v \to 0$ as $r \to \infty$

So which solution is the right solution? Well, as $r \to \infty$, we want $v_r \to 0$. This kills off 2 and 4. As $r \to 0$, we want the stuff to accelerate toward the central point, which kills off 3.

Now to make further progress, let's integrate (We're reproducing our conservation of energy equation (III.34), after the time-derivative term is set to zero and conservation of mass, $\nabla \cdot (\rho \mathbf{v}) = \mathbf{0}$, is used):

$$\int v_r \frac{\partial}{\partial r} v_r dr = \int \left(-\frac{c_s^2}{\rho} \frac{\partial}{\partial r} \rho - \frac{GM}{r^2} \right) dr \tag{XV.154}$$

$$\implies \frac{1}{2} v_r^2 + \int \frac{c_s^2}{\rho} d\rho - \frac{GM}{r} = \text{constant} \tag{XV.155}$$

Recall $c_s^2 = \gamma P/\rho$ and $P = \kappa \rho^\gamma$, so this gives

$$\frac{1}{2} v_r^2 + \frac{c_s^2}{\gamma - 1} - \frac{GM}{r} = \text{constant} \tag{XV.156}$$

As $r \to \infty$, $v_r \to 0$ and $\rho \to \rho_0$, so we have the constant $c_{s,0}^2/(\gamma - 1) = c_s^2|_{r=\infty}/(\gamma - 1) = \gamma \kappa \rho_0^{\gamma-1}/(\gamma - 1)$.

Now let's look at what happens at the sonic radius. So at $r = r_s$,

$$\frac{1}{2}c_s^2 + \frac{c_s^2}{\gamma - 1} - \frac{GM}{r_s} = \left(\frac{1}{2} + \frac{1}{\gamma - 1} - 2\right) c_s^2 = c_{s,0}^2/(\gamma - 1) \tag{XV.157}$$

Let's do some algebra to find:

$$c_s^2(r_s) = c_{s,0}^2 \frac{2}{5 - 3\gamma} \tag{XV.158}$$

You may be slightly disturbed by this result. Why might you not be?

The density scales like

$$\rho = \rho_0 \left(\frac{c_s^2}{c_{s,0}^2}\right)^{1/(\gamma-1)}$$

So the mass accretion rate is

$$\begin{aligned}
\dot{M} &= 4\pi\rho r^2 v_r = 4\pi\rho_s r_s^2 c_s && \text{(XV.159)}\\
&= 4\pi\rho_0 \left(\frac{c_s^2}{c_{s,0}^2}\right)^{1/(\gamma-1)} \left(\frac{GM}{c_s^2}\right)^2 c_s && \text{(XV.160)}\\
&= 4\pi\rho_0 \left(\frac{c_s^2}{c_{s,0}^2}\right)^{1/(\gamma-1)-3/2} \frac{G^2M^2}{c_{s,0}^3} && \text{(XV.161)}\\
&= 4\pi\rho_0 \left(\frac{2}{5 - 3\gamma}\right)^{(5-3\gamma)/(2(\gamma-1))} \frac{G^2M^2}{c_{s,0}^3}. && \text{(XV.162)}
\end{aligned}$$

The expression has finite limits for $\gamma = 5/3$ and $\gamma = 1$: In particular, the factor involving γ has the limits[9]

$$\left(\frac{2}{5 - 3\gamma}\right)^{(5-3\gamma)/(2(\gamma-1))} = \begin{cases} 1 & \text{if } \gamma = 5/3 \\ 4.5 & \text{if } \gamma = 1 \end{cases}.$$

You may wonder about when something is $\gamma = 1$. This is an isothermal gas and is frequently used to model the interstellar medium, where the cooling times are fast and the temperature is set by heating from the radiation field of stars.

In any case, for $\gamma = 5/3$, we have

$$\dot{M} = 4\pi\rho_0 \frac{G^2M^2}{c_{s,0}^3}, \tag{XV.163}$$

where the mass accretion rate is set by the density and temperature of the ambient medium through ρ_0 and $c_{s,0}$ respectively. If it was $\gamma = 1$, the result will just change by a factor of 4.

[9]For $\gamma \to 5/3$, write $\epsilon = \gamma - 5/3$. Then $\epsilon^{-K\epsilon} = e^{-K\epsilon \ln \epsilon}$ and, using $\lim_{\epsilon\to 0} \epsilon \ln \epsilon = 0$, we have $\lim\limits_{\epsilon\to 0} \epsilon^{-K\epsilon} = 1$. Similarly, for $\gamma \to 1$, use $\epsilon = \gamma - 1$. Then $\dfrac{5 - 3\gamma}{2} = 1 - \dfrac{3}{2}\epsilon$. We have $(1 - 3\epsilon/2)^{1/\epsilon} = \exp[\epsilon^{-1}\ln(1 - 3\epsilon/2)]$ and $\lim\limits_{\epsilon\to 0} \epsilon^{-1}\ln(1 - 3\epsilon/2)] = -\dfrac{3}{2}$, implying $\lim\limits_{\gamma\to 1}\left(\dfrac{2}{5 - 3\gamma}\right)^{(5-3\gamma)/[2(\gamma-1)]} = e^{3/2} \approx 4.5$.

This result can be derived from OOM estimates exactly for $\gamma = 5/3$. For instance, consider a mass embedded in an external medium with the above density and temperature. The distance at which the material feels the gravitational force of the object would be $c_s^2 = GM/r$. So let's say at this radius, the material suddenly falls into the mass at some speed. What would that speed be? Well the only speed scale in this problem $c_{s,0}$. If it falls onto the object spherically then we have

$$\dot{M} \sim 4\pi\rho_0 r^2 c_{s,0} = 4\pi\rho_0 \frac{G^2M^2}{c_{s,0}^3} \tag{XV.164}$$

which is the exactly the result derived above. Note this result is independent of γ, which is not the case in the exact result. But hey, this is what it means for an OOM estimate.

Incidentally, this result is known as the Bondi-Hoyle accretion rate and in most of astrophysics, the OOM estimate is used. Just as an exercise let us consider the accretion rate of a solar mass point mass in the ambient ISM, where $n = 1\,\mathrm{cm}^{-3}$ and $T = 10^4$ K $\to c_s = 10\,\mathrm{km\,s}^{-1}$. This gives:

$$\dot{M} = 4\pi n m_p \frac{(GM)^2}{c_s^3} \approx 6\times 10^{-15}\left(\frac{M}{1\mathrm{M}_\odot}\right)^2\left(\frac{n}{1\,\mathrm{cm}^{-3}}\right)\left(\frac{c_s}{10\,\mathrm{km\,s}^{-1}}\right)^{-3}\mathrm{M}_\odot\,\mathrm{yr}^{-1}, \tag{XV.165}$$

which corresponds to a luminosity of $L = \eta\dot{M}c^2 \approx 8\times 10^{31}\,\mathrm{ergs\,s}^{-1}$ for $\eta = 0.1$.

We should note that the speed scale in the problem can be reset – for instance, one can consider a particle moving through the ambient medium with the replacement $c_s \to \sqrt{c_s^2 + v^2}$. But we will leave the exploration of this as a HW problem.

XV.3 Accretion Disks

Thus far, we have only considered spherical accretion and this got quite a bit of the way. Now we will consider a disk. Why a disk you ask? Well if you have some material falling in from ∞, it will have some angular momentum. Now rotating stars are nearly spherical when the pressure support is large compared to the rotational support: In other words, when rotational energy $\ll$ internal energy (thermal energy or total Fermi energy in the degenerate case). As one spins up a star, it becomes more oblate, and the opposite limit, when pressure is negligible compared to rotation, is a disk, with each fluid element moving in a Keplerian orbit.

We'll look at that limit and at the correction to it when the pressure is small but nonzero. As one heats up a disk, the ratio of thermal energy to rotational energy increases and the disk becomes thicker. We will soon be working to first order in the ratio H/R of the thickness of the disk to its radius (or to the characteristic length R for which $dP/dr \sim P/R$).

Let's first consider a particle moving toward a central potential with impact parameter b as shown in Figure 18. The energy of the particle with mass m is

$$E = \frac{1}{2}mv^2 - \frac{GMm}{r} = \frac{1}{2}m\left(v_r^2 + r^2\dot{\phi}^2\right) - \frac{GMm}{r} \tag{XV.166}$$

The key difference is the inclusion of the second term in the kinetic energy. To use this conservation law, we must include the additional angular momentum conservation law (for now):

$$l = \mathrm{constant} = r^2\dot{\phi} \tag{XV.167}$$

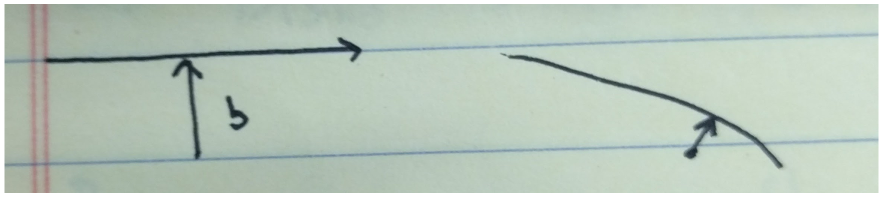


Figure 18

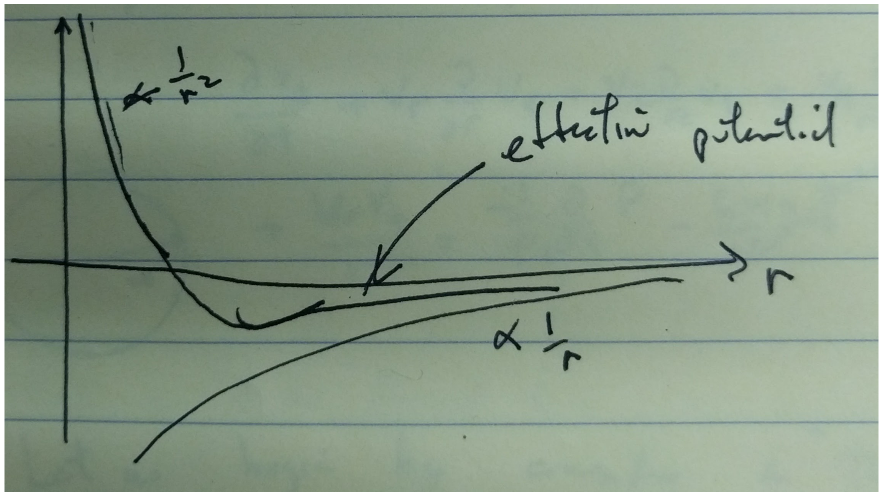


Figure 19

Putting this together, we have

$$E = \frac{1}{2}mv_r^2 + \frac{1}{2}\frac{ml^2}{r^2} - \frac{GMm}{r} = \frac{1}{2}mv_r^2 + m\left(\frac{l^2}{2r^2} - \frac{GM}{r}\right) = \frac{1}{2}mv_r^2 + mU_{\text{eff}}(r) \quad \text{(XV.168)}$$

The second and third term only depends on r and thus can be thought of as a new type of potential, i.e., the effective potential, U_{eft}. Let's draw this potential as in Figure 19. What we should notice here is that there is a minimum in the effective potential

Let's find the minimum; we set $dU_{\text{eft}}/dr = 0$ to find $r_{\text{min}} = l^2/GM$ or $l^2 = GMr_{\text{min}}$, which is the condition for circular orbits. Thus for a fixed l, there is a minimum $E = U_{\text{eft}}(r_{\text{min}})$ that is allowed. Thus, if you can lose energy via shocks, radiation, etc, then you will come to a circular orbit. So if I throw some material with some net angular momentum toward an object and this stuff loses energy, it is inevitable that a disk (material in circular orbits) will form at some radius.

To make further progress, we must "lose" angular momentum. Toward that end, let us consider the Euler equations in cylindrical coordinate. Continuity becomes

$$\frac{\partial}{\partial t}\rho + \boldsymbol{\nabla}\cdot\rho\boldsymbol{v} = 0 = \frac{\partial}{\partial t}\rho + \frac{1}{r}\frac{\partial}{\partial r}r\rho v_r + \frac{\partial}{\partial z}\rho v_z \quad \text{(XV.169)}$$

and the momentum equation:

$$\frac{\partial}{\partial t}\boldsymbol{v} + \boldsymbol{v}\cdot\boldsymbol{\nabla}\boldsymbol{v} = -\frac{1}{\rho}\boldsymbol{\nabla}P - \boldsymbol{\nabla}\Phi$$

becomes

$$\begin{aligned}
\frac{\partial}{\partial t}v_r + \left(v_r\frac{\partial}{\partial r} + \frac{v_\phi}{r}\frac{\partial}{\partial \phi} + v_z\frac{\partial}{\partial z}\right)v_r - \frac{v_\phi^2}{r} &= -\frac{1}{\rho}\frac{\partial}{\partial r}P - \frac{\partial}{\partial r}\Phi \quad \text{(XV.170)}\\
\frac{\partial}{\partial t}v_\phi + \left(v_r\frac{\partial}{\partial r} + \frac{v_\phi}{r}\frac{\partial}{\partial \phi} + v_z\frac{\partial}{\partial z}\right)v_\phi + \frac{v_\phi v_r}{r} &= -\frac{1}{\rho r}\frac{\partial}{\partial \phi}P - \frac{1}{r}\frac{\partial}{\partial \phi}\Phi \quad \text{(XV.171)}\\
\frac{\partial}{\partial t}v_z + \left(v_r\frac{\partial}{\partial r} + \frac{v_\phi}{r}\frac{\partial}{\partial \phi} + v_z\frac{\partial}{\partial z}\right)v_z &= -\frac{1}{\rho}\frac{\partial}{\partial z}P - \frac{\partial}{\partial z}\Phi \quad \text{(XV.172)}
\end{aligned}$$

Note here that r is the cylindrical r not the spherical r and the v_ϕ^2/r and $v_r v_\phi/r$ comes from the derivative of $\frac{\partial}{\partial\phi}\hat{\phi} = -\hat{r}$ and $\frac{\partial}{\partial\phi}\hat{r} = \hat{\phi}$ in the ϕ and r momentum equation. This is a rather complicated set of equations to deal with, but becomes much simpler when we look for solutions (1) with $v_z = 0$ as it is in a rotating star and (2) with $H/R \ll 1$. With $v_z = 0$, the z-momentum equation (z-component of the Euler equation) simplifies to

$$0 = -\frac{1}{\rho}\frac{\partial}{\partial z}P + \frac{\partial}{\partial z}\Phi = -\frac{1}{\rho}\frac{\partial}{\partial z}P - \frac{GM}{\mathtt{r}^2}\hat{\mathtt{r}}\cdot\hat{z}, \quad \text{(XV.173)}$$

where $\mathtt{r}$ is the *spherical* radius. Note that $\hat{r}\cdot\hat{z} = \sin\theta \approx z/r$, where that last r is cylindrical. For "thin" disk the difference between the spherical and cylindrical r are similar. This gives

$$0 = -\frac{1}{\rho}\frac{\partial}{\partial z}P - \frac{GM}{r^2}\frac{z}{r} \quad \text{(XV.174)}$$

Now we will use cylindrical symmetry, which implies that $\frac{\partial}{\partial\phi} = 0$. So we have for our sets of equations

$$\begin{aligned}
\frac{\partial}{\partial t}\rho + \frac{1}{r}\frac{\partial}{\partial r}r\rho v_r &= 0 \quad \text{(XV.175)}\\
\frac{\partial}{\partial t}v_r + v_r\frac{\partial}{\partial r}v_r - \frac{v_\phi^2}{r} &= -\frac{1}{\rho}\frac{\partial}{\partial r}P - \frac{\partial}{\partial r}\Phi \quad \text{(XV.176)}\\
\frac{\partial}{\partial t}v_\phi + v_r\frac{\partial}{\partial r}v_\phi + \frac{v_\phi v_r}{r} &= 0 \quad \text{(XV.177)}\\
-\frac{1}{\rho}\frac{\partial}{\partial z}P - \frac{GM}{r^2}\frac{z}{r} &= 0 \quad \text{(XV.178)}
\end{aligned}$$

The equations above are for a thin, inviscid disk. How thin is thin? Well consider the last equation in the limit where $z \ll r$. Let $z = H$, where H is the scale height of the disk (typical height of the disk). So we have

$$-\frac{1}{\rho}\frac{\partial}{\partial z}P - \frac{GM}{r^3}\frac{z}{r} \to \frac{P}{\rho H} = \frac{GMH}{r^3} = \frac{c_s^2}{H} = v_{\rm orb}^2\frac{H}{r^2} \to c_s^2 = v_{\rm orb}^2\left(\frac{H}{r}\right)^2, \quad \text{(XV.179)}$$

where $v_{\rm orb} = GM/r$ is the orbital speed. So the condition that $z \ll r \to H \ll r \to c_s \ll v_{\rm orb}$. Another way of thinking about this is $c_s^2 = k_BT \ll v_{\rm orb}^2 = k_BT_{\rm vir}$, i.e., $T \ll T_{\rm vir}$, i.e., the gas temperature is much smaller than the virial temperature.

With this in mind, lets consider steady state as we did last time. So we have

$$\begin{aligned} \frac{1}{r}\frac{\partial}{\partial r} r\rho v_r &= 0 && \text{(XV.180)} \\ v_r \frac{\partial}{\partial r} v_r &= -\frac{GM}{r^2} + \frac{v_\phi^2}{r} && \text{(XV.181)} \\ v_r \frac{\partial}{\partial r} v_\phi + \frac{v_\phi v_r}{r} &= 0 && \text{(XV.182)} \\ 0 &= -\frac{1}{\rho}\frac{\partial}{\partial z} P - \frac{GM}{r^2}\frac{z}{r} && \text{(XV.183)} \end{aligned}$$

An alert reader will notice that $\rho^{-1} dP/dr$ disappeared. This is no accident as $\rho^{-1} dP/dr \sim P/\rho r = c_s^2/r \ll v_{\rm orb}^2/r = GM/r^2$, much smaller that the competing terms.

One solution to the above set is $v_r = 0$. In this case, we have a complete solution: $v_\phi^2 = GM/r$, i.e., the Keplerian disk, with the fluid moving in circular orbits. On the other hand, if $v_r \neq 0$, then no steady state solution is possible as one of the equations explicitly prevents a solultion close to the Keplerian disk for small v^r (can you see which one?). This just means you have to consider time variations or extra physics to describe an accreting disk. (Answer to the question is in the footnote below[10]). In a Keplerian disk, orbits at different radii have different angular velocity. When one includes viscosity, two things happen: The viscous dissipation heats the fluid and, as a ring of fluid loses energy, it loses angular momentum, moving inward and thus acquiring a nonzero radial velocity. Because the thermal energy of a thin disk is of order e^2, it is natural to look for a solution in which the viscous force of order order e^2 times the gravitational force. We begin by formally introducing shear viscosity and then giving a separate argument that thin disk viscosity is of that order.

We return to the Euler equations, adding an extra term

$$\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} \boldsymbol{v} = -\frac{1}{\rho}\boldsymbol{\nabla} P - \boldsymbol{\nabla}\Phi + \frac{1}{\rho}\boldsymbol{\nabla} \cdot \boldsymbol{T}, \qquad \text{(XV.184)}$$

where $\boldsymbol{T}$ is the viscous stress tensor. The off-diagonal terms in the stress tensor will allow flows to exchange momentum with each other via contact and will not require them to physically collide with one anther.

In particular, we will look at shearing flows with nonzero viscosity. A good introduction to viscosity is Feynman's second chapter on fluids, Chap. 41 of the second volume (feynmanlectures.caltech.edu/II_41.html).[11] The stress tensor associated with viscosity has the form $T_{ij} = \eta(\nabla_i v_j + \nabla_j v_i)$, when $\nabla \cdot v = 0$, as is the case for a rotating disk or for a star in which rings at different cylindrical radii rotate with different angular velocities. When $\nabla \cdot v \neq$, one subtracts off the divergence:

$$T_{ij} = \eta(\nabla_i v_j + \nabla_j v_i - \frac{2}{3}\delta_{ij}\nabla_k v_k),$$

[10] When $v_r \neq 0$, the φ component (XV.182) of the Euler equation has the form $\frac{v_r}{r}\partial_r(r v_\phi) = 0$, implying $v_\phi = C/r$, and contradicting the $v_\phi = \sqrt{GM/r} + O(e)$ behavior of a thin disk.

[11] See also Section 13.7 of Blandford-Thorne.

making T_{ij} tracefree. In a gas, shear viscosity is dominated not by the chemical bonds that determine the viscosity of ordinary liquids but by the momentum carried by the particles. Viscosity *increases* with an increase in the mean free path of particles, because a larger mean-free path allows particles to transfer momentum between fluid elements having larger differences in their fluid velocities. The viscosity of each particle species is then limited by the dominant interaction that limits its mean free path. (The $\nabla \cdot v$ term that is removed from the shear tensor measures the change in volume, not the shear; dissipation due to a change in volume is called bulk viscosity. It comes, for example, from induced nuclear reactions as a fluid element expands or contracts, and it has a coefficient that is ordinarily unrelated to shear viscosity.)

Here, for our thin disk, the important term is $T_{xy} = \eta \partial v_x / \partial y$ in cartesian coordinates, where η is the dynamical viscosity. Shear flows are particularly important because the orbital velocity scales like $v_{\rm orb} \propto r^{-1/2}$, so at any radii the local fluid flow in disk experiences some shear. Noting the form of the stress tensor is of the form $T_{ij} \propto \partial v_i / \partial x_j$ and for a thin disk the dominant velocity is $v_\phi = v_{\rm orb}(r)$, the only component of the stress tensor that we need to concern ourselves with is the

$$T_{\phi r} = \eta r \frac{\partial}{\partial r} \Omega$$

component.

With this in mind, the ϕ momentum equation becomes

$$\frac{\partial}{\partial t} v_\phi + v_r \frac{\partial}{\partial r} v_\phi + \frac{v_\phi v_r}{r} = \frac{1}{r\rho} \left(\frac{\partial}{\partial r} r T_{\phi r} + T_{\phi r} \right) \tag{XV.185}$$

where the last term in the middle equation comes from an r-derivative of $\hat{\phi}$. We group terms together to find

$$\frac{1}{r} \left(\frac{\partial}{\partial t} + v_r \frac{\partial}{\partial r} \right) r v_\phi = \frac{1}{r} \frac{d}{dt} l = \frac{1}{r^2 \rho} \frac{\partial}{\partial r} r^2 T_{\phi r} = \frac{1}{r^2 \rho} \frac{\partial}{\partial r} r^3 \eta \frac{\partial}{\partial r} \Omega, \tag{XV.186}$$

where $l = r v_\phi$ is the specific angular momentum and $\frac{d}{dt} \equiv \frac{\partial}{\partial t} + v_r \frac{\partial}{\partial r}$ is the complete time derivative. Written another way the above equation becomes:

$$\frac{d}{dt} l = \frac{1}{r\rho} \frac{\partial}{\partial r} r^3 \eta \frac{\partial}{\partial r} \Omega. \tag{XV.187}$$

This equation is simply an evolution equation for the angular momentum. If the viscosity is zero, i.e., $\eta = 0$, then we have a conservation law for angular momentum. But the fact that there is a source term on the RHS suggest that angular momentum is in fact not *locally* conserved, but can evolve. We note that the above looks like a diffusion equation, something that will be important in a second. Let us OOM the equation above

$$\frac{l}{t} \sim \frac{1}{\rho} \frac{1}{r^3} \eta r^3 \Omega = \frac{\eta l}{\rho r^2} \to t_{\rm visc} \sim \frac{\rho}{\eta} r^2 = \frac{r^2}{\nu} \tag{XV.188}$$

where $\nu = \eta / \rho$ is the kinetimatic viscosity and has unit of cm^2/s, aka a diffusion coefficient.

The determination of this ν for astrophysical systems was a key problem and remains one today, though we have a better idea of how this might work. Before we go on let's discuss briefly how big viscosity might be. For molecular viscosity, the fact that it is a diffusion coefficent allows us to estimate it in the standard way:

$$\nu \sim \lambda v_{\rm th}$$

where $v_{\rm th} = c_s$ is the thermal velocity and λ is the mean free path. The combination of λ, which is short and $v_{\rm th}$, which is slow tends to give very slow diffusive processes. This gives very weak viscous forces.

How weak is weak? Well one must compare this the other forces in the systems, the so-called inertial forces, which come into play in the hydrodynamic equations as the $\boldsymbol{v}\cdot\boldsymbol{v}$ terms. Because the size of the system is r and $v_{\rm orb}$ is the largest velocity, the inertial forces have an associate acceleration that is $v_{\rm orb}^2/r$. Now the viscous force is $\boldsymbol{\nabla}\cdot\boldsymbol{T} \sim \nu r^{-2} v_\phi = \lambda v_{\rm th} v_\phi r^{-2}$. The ratio of these two is known as the Reynold's number

$$\mathrm{Re} = \frac{\text{inertial forces}}{\text{viscous forces}} \sim \frac{v_\phi^2/r}{\lambda v_{\rm th} v_\phi/r^2} \sim \frac{r}{\lambda}\frac{v_{\rm orb}}{v_{\rm th}} \tag{XV.189}$$

This ratio is typically really large $\sim 10^{14}$ though I'll leave it to you to convince yourself of that. Prior to 1990, the usual way out was to note that such large Reynolds numbers inevitable lead to turbulent flow in terrestial environments. This is not clear if it is the case in Keplerian disks as the angular momentum tends to keep everything in check. Nevertheless one can try to make this assumption. If this leads to turbulence, then one can think of a turbulent viscosity in the same manner as molecular viscosity, i.e.,

$$\nu \sim \text{length scale} \times \text{velocity scale} \tag{XV.190}$$

In a disk, there are two velocity scales c_s and $v_{\rm orb}$. If one thinks about local turbulence, supersonic flow quickly damps out, so the typical velocity scale for turbulence is $< c_s$. Now the length scale is set by the size of the largest eddies, which is limited by the size of the systems and here again there are two scales H and r. Again if one thinks about isotropic turbulence, then scales above H are no longer isotropic. So we conclude:

$$\nu = \alpha H c_s = \alpha \frac{H^2}{r^2} v_{\rm orb}, \tag{XV.191}$$

where $\alpha \sim 0.1$ is a numerical constant than encapsulates our ignorance and we recalled that $H/r = c_s/v_{\rm orb}$. This gives an all important viscous time of

$$t_{\rm visc} = \frac{r^2}{\alpha v_{\rm orb} H^2/r^2} = \alpha^{-1}\left(\frac{r}{H}\right)^2 t_{\rm orb} \tag{XV.192}$$

XV.4 Steady Disks

Armed with this α prescription, we now turn to developing a steady disk model

$$\begin{aligned}
\frac{\partial}{\partial t}\rho + \frac{1}{r}\frac{\partial}{\partial r} r\rho v_r &= \frac{d}{dt}\rho + \frac{\rho}{r}\frac{\partial}{\partial r} r v_r = 0 && \text{(XV.193)}\\
\frac{\partial}{\partial t}v_r + v_r\frac{\partial}{\partial r}v_r - \frac{v_\phi^2}{r} &= \frac{d}{dt}v_r - \frac{v_\phi^2}{r} = -\frac{1}{\rho}\frac{\partial}{\partial r}P - \frac{GM}{r^2} && \text{(XV.194)}\\
\frac{\partial}{\partial t}v_\phi + v_r\frac{\partial}{\partial r}v_\phi + \frac{v_\phi v_r}{r} &= \frac{d}{dt}v_\phi + \frac{v_\phi v_r}{r} = \text{(sources)} && \text{(XV.195)}\\
\frac{\partial}{\partial t}v_z + v_r\frac{\partial}{\partial r}v_z &= \frac{d}{dt}v_z = -\frac{1}{\rho}\frac{\partial}{\partial z}P - \frac{GM}{r^2}\frac{z}{r}, && \text{(XV.196)}
\end{aligned}$$

This is different from the previous set that we used because we found that the previous set didn't work. Namely for $v_r \neq 0$, we cannot have a steady state equation for v_ϕ if the source for the third equation is 0 (and if the solution is close to a Keplerian disk for small v^r). Now we have computed the source, which we found compactly written as:

$$\frac{d}{dt} r v_\phi = \frac{1}{r\rho}\frac{\partial}{\partial r} r^3 \eta \frac{\partial}{\partial r}\Omega. \quad \text{(XV.197)}$$

It turns out that steady-state can be simplified further in the limit where v_r is small. So we will make the approximation $v_r \approx 0$ and $\partial/\partial t \approx 0$. This gives $d/dt \approx 0$, which is the reason why we wrote the set of equation using the total time derivative. So the equations are now:

$$\begin{aligned}
\frac{d}{dt}\rho + \frac{\rho}{r}\frac{\partial}{\partial r} r v_r &= 0 && \text{(XV.198)}\\
-\frac{v_\phi^2}{r} &= -\frac{1}{\rho}\frac{\partial}{\partial r}P - \frac{GM}{r^2} && \text{(XV.199)}\\
\frac{d}{dt} r v_\phi &= \frac{1}{r\rho}\frac{\partial}{\partial r} r^3 \rho\nu \frac{\partial}{\partial r}\Omega && \text{(XV.200)}\\
0 &= -\frac{1}{\rho}\frac{\partial}{\partial z}P - \frac{GM}{r^2}\frac{z}{r}, && \text{(XV.201)}
\end{aligned}$$

where $\nu = \alpha H c_s$. An alert reader will recognize that first equation is not completely consistent with this set. We will ignore this inconsistency for now because we need this result in a second.

Now if the pressure gradient (in radius) is small compared to gravity, then

$$v_r \text{is small} \rightarrow \frac{v_\phi^2}{r} = \frac{GM}{r^2}, \quad \text{(XV.202)}$$

i.e., Keplerian rotation. Now the z equation decouples so we can ignore this equation for now. So we only need to solve everything using the first and third equations. Expand out the third equation:

$$\frac{\partial}{\partial t} r^2\Omega + v_r \frac{\partial}{\partial r} r^2\Omega = \frac{1}{r\rho}\frac{\partial}{\partial r} r^3 \rho\nu\frac{\partial}{\partial r}\Omega, \quad \text{(XV.203)}$$

Multiply by ρ to get:

$$\frac{\partial}{\partial t}\rho r^2\Omega + \frac{1}{r}\frac{\partial}{\partial r} r v_r \rho r^2 \Omega = \frac{1}{r}\frac{\partial}{\partial r} r^3 \rho\nu \frac{\partial}{\partial r}\Omega, \tag{XV.204}$$

where we have used the first equation in this substitution.

Now ρ is a function of z, so I will integrate this out. Technically, v_r and Ω are also functions of z, but they are weak functions, so we will ignore this dependence. Define

$$\Sigma = \int \rho dz, \tag{XV.205}$$

and integrate the continuity and angular momentum equation in dz

$$\begin{aligned} \frac{\partial}{\partial t}\Sigma + \frac{1}{r}\frac{\partial}{\partial r} r\Sigma v_r &= 0 && \text{(XV.206)} \\ \frac{\partial}{\partial t}\Sigma r^2\Omega + \frac{1}{r}\frac{\partial}{\partial r} r^3 v_r \Sigma\Omega &= \frac{1}{r}\frac{\partial}{\partial r} r^3 \Sigma\nu \frac{\partial}{\partial r}\Omega, && \text{(XV.207)} \end{aligned}$$

Using steady state now, we have $\frac{\partial}{\partial t} = 0$

$$\begin{aligned} \frac{1}{r}\frac{\partial}{\partial r} r\Sigma v_r &= 0 && \text{(XV.208)} \\ \frac{1}{r}\frac{\partial}{\partial r} r^3 v_r \Sigma\Omega &= \frac{1}{r}\frac{\partial}{\partial r} r^3 \Sigma\nu \frac{\partial}{\partial r}\Omega. && \text{(XV.209)} \end{aligned}$$

The first equation implies that $r\Sigma v_r =$ constant, which we recognize as the mass accretion rate $\dot{M} = 2\pi r\Sigma v_r$. For the second equation, we multiply by r on both sides and integrate in dr to eliminate the leading $\frac{\partial}{\partial r}$, which gives

$$r^3 v_r \Sigma\Omega = r^3 \Sigma\nu \frac{\partial}{\partial r}\Omega + \frac{C}{2\pi}, \tag{XV.210}$$

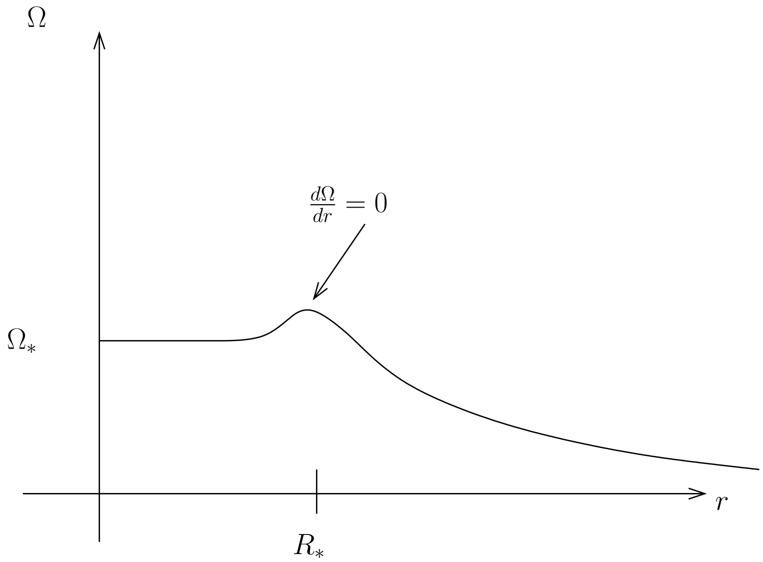
where C is some integration constant that we will need to set. Staring at the first term on the right had side, we define

$$\mathcal{G} \equiv 2\pi r \nu \Sigma r^2 \frac{\partial}{\partial r}\Omega, \tag{XV.211}$$

as the local viscous torque. In any case, the steady state angular momentum equation can be simplified to

$$v_r \Sigma\Omega = \Sigma\nu \frac{\partial}{\partial r}\Omega + \frac{C}{2\pi r^3}. \tag{XV.212}$$

We have to set C now via boundary conditions. To do so, let's think about what the right BC might look like. Suppose you have an accretion disk that comes down to the surface of a star at R_* and the star's intrinsic rotation rate is Ω_*



Far away from the star, the material is in Keplerian orbit so $\Omega \propto r^{-3/2}$. Deep in the star, the material is rotating at Ω_*. So somewhere in between there must be a peak in Ω, which gives $d\Omega/dr = 0$. Let say that this point is close to the surface of the star R_*. This allows us derive the BC. From the preceding equation as the first term on the RHS is zero:

$$v_r(R_*)\Sigma(R_*)\Omega(R_*) = \frac{C}{2\pi R_*^3} \to C = -\dot{M}\sqrt{GM_*R_*}. \qquad \text{(XV.213)}$$

This boundary condition is know as the torque free boundary condition because there is no torque on the inner edge of the accretion disk, i.e., $\mathcal{G} = 0$ near the surface of the star since $d\Omega/dr = 0$. Plugging this result back in, we find the following

$$2\pi r^3 v_r \Sigma\Omega = 2\pi r^3 \Sigma\nu \frac{\partial}{\partial r}\Omega - \dot{M}\sqrt{GM_*R_*} \qquad \text{(XV.214)}$$

$$\dot{M}\sqrt{GM_*r} = 3\pi\sqrt{GM_*r}\nu\Sigma + \dot{M}\sqrt{GM_*R_*} \to \nu\Sigma = \frac{\dot{M}}{3\pi}\left(1 - \sqrt{\frac{R_*}{r}}\right) \qquad \text{(XV.215)}$$

Under viscous dissipation the energy dissipation rate is

$$\frac{d\epsilon}{dt} = \frac{1}{2}\rho\nu\sum_{i,j}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)^2 = \frac{1}{2}\rho\nu r^2\left(\frac{\partial}{\partial r}\Omega\right)^2, \qquad \text{(XV.216)}$$

where ϵ is the energy dissipation rate per unit volume. As we argue previously, the only term that matters is the $i, j = \phi, r$ term. Additionally if we integrate over z, we get the flux, i.e., dissipation per unit area

$$F = \int \frac{d\epsilon}{dt}dz = \frac{1}{2}\Sigma\nu r^2\left(\frac{\partial}{\partial r}\Omega\right)^2. \qquad \text{(XV.217)}$$

Plugging in our result for $\nu\Sigma$, we find

$$F = \frac{1}{2}\frac{\dot{M}}{3\pi}\left(1 - \sqrt{\frac{R_*}{r}}\right)^{1/2} r^2\left(\frac{3}{2}\frac{\Omega}{r}\right)^2 = \frac{3}{4\pi}\frac{GM\dot{M}}{r^3}\left(1 - \sqrt{\frac{R_*}{r}}\right)^{1/2} \to T_e \propto r^{-3/4} \qquad \text{(XV.218)}$$

This flux needs to be integrated over the area of the disk to get the total luminosity:

$$L = \int_{r_*}^{r} F 2\pi r dr \approx \frac{GM\dot{M}}{2R_*}, \tag{XV.219}$$

assuming $r \gg R_*$. This is in line with the expections of the virial theorem that $T = -2U$ so $\Delta e = GM/2R_*$.

Armed with this, we can figure out how a disk will look like if it emitted like a blackbody – which depending on the system is a good or poor assumption. So we set

$$2\sigma T_e^4 = F = \frac{3}{4\pi}\frac{GM\dot{M}}{r^3}\left(1 - \sqrt{\frac{R_*}{r}}\right)^{1/2}, \tag{XV.220}$$

where the leading 2 comes from the fact that the disk emits through the top and bottom. Now the spectrum of a blackbody with effective temperature T_e is

$$I_\nu = \frac{2h\nu^3}{c^2(\exp(h\nu/k_B T_e) - 1)}. \tag{XV.221}$$

Noting that T_e is a function of r in a disk, the observed flux that someone sees a distance d away is

$$F_\nu = \frac{\cos i}{d^2}\int 2\pi r dr I_\nu = \frac{4\pi \cos i h\nu^3}{c^2 d^2}\int \frac{r dr}{\exp(h\nu/k_B T_e) - 1} \tag{XV.222}$$

where i is the inclination angle $i = 90$ is face on. To do this integral, let make note that $T_e \propto r^{-3/4} \to r \propto T^{-4/3}$ and thus $dr \sim - = - = T^{-1/3} d(1/T)$. This gives

$$F_\nu \sim \int \frac{\nu^3 T^{-4/3} T^{-1/3} d(1/T)}{\exp(h\nu/k_B T_e) - 1}, \tag{XV.223}$$

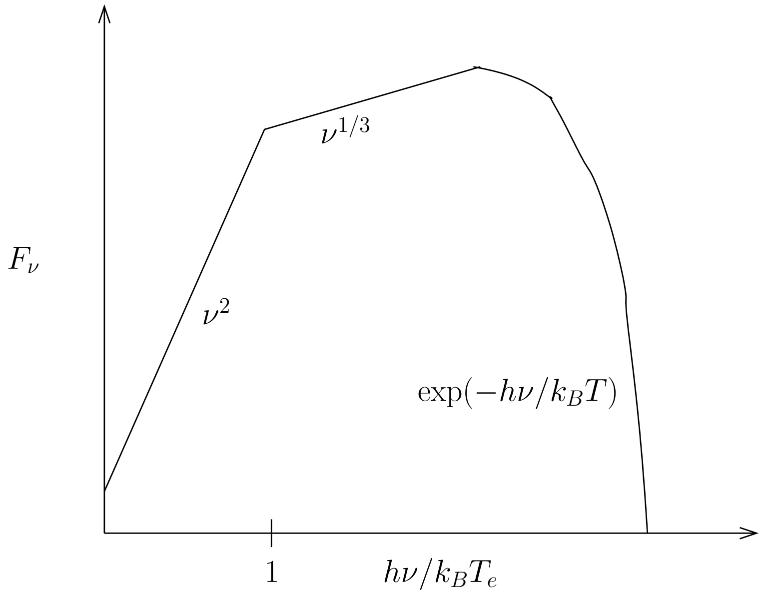
Now let $x = h\nu/k_B T \to 1/T = k_B x/h\nu, d(1/T) = k_B dx/h\nu$. This gives

$$F_\nu \sim \int \frac{\nu^3 \nu^{-5/3} x^{5/3} \nu^{-1} dx}{\exp(x) - 1} = \nu^{1/3}\int \frac{x^{5/3} dx}{\exp(x) - 1}. \tag{XV.224}$$

At this point three possibilities are available

$$F_\nu \propto \begin{cases} \nu^2 & h\nu \ll k_B T_e \to x \ll 1 \\ \nu^{1/3} & h\nu \sim k_B T_e \to x \sim 1 \\ \exp(-h\nu/k_B T_e) & h\nu \gg k_B T_e \to x \gg 1 \end{cases} \tag{XV.225}$$

We can draw that this looks like as follows:



This spectrum is known as the multicolor blackbody. Note that this spectra is just assumes that

1. $\dot{M}$ is constant throughout the disk – it is a steady state disk.

2. dissipation is entirely local, i.e., at every point the disk is in roughly Keplerian rotation.

It is important that this spectra does not depend on the details on how dissipation occurs, just that it does and is roughly partitioned by the mass of the disk – equal dissipation per gram. So this profile is fairly free of the assumptions and microphysics of dissipation, an attractive situation to be in. For this reason this is one of the key predictions of accretion disk theory and has been more or less confirmed from (some) astrophysical observations.

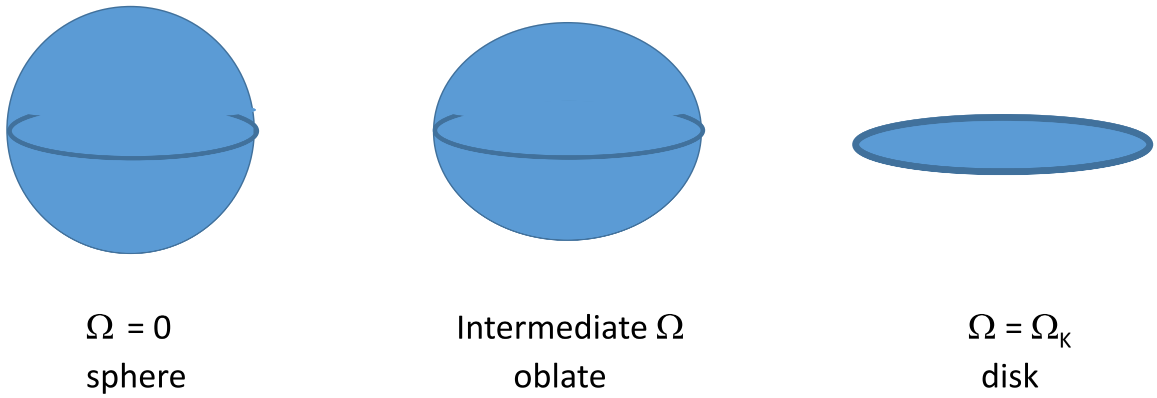
XV.4.1 Derivation from Perturbation Theory

This section derives the thin disk equations as perturbation of a Keplerian disk at first order in the viscosity. It is helpful to think first of a spherical star and a Keplerian disk as the two extremes of a family of rotating stars. When $\Omega = 0$, the star is supported against gravity entirely by pressure, and it is spherical. As Ω increases, the star grows increasing oblate, supported by both rotation and pressure:

$$\rho\nabla\Phi = -\nabla P + \rho\Omega^2 r. \tag{XV.226}$$

To reach a disk as the final configuration, the pressure must decrease to zero along the sequence. Think of gas whose temperature drops to zero, with the gas sparse enough that the particles are not degenerate. In the thin disk we are considering, a small viscosity dissipates energy, slightly heating the disk and so leading to a small nonzero pressure. We will look at perturbations of the Keplerian disk arising from adding a small viscosity.

One can describe perturbation theory as follows. Consider a family of solutions $P(e), \rho(e), \mathbf{v}(e)$ to the mass conservation equation and the Euler equation with viscosity (XV.184) (the Navier-Stokes



equation). For $e = 0$, the solution is the Kepleran disk:

$$P(0,t,\mathbf{r}) = 0, \quad \rho(0,t,\mathbf{r}) = \Sigma(r)\delta(z), \quad \mathbf{v}(0,t,\mathbf{r}) = r\Omega(r), \text{ with } \Omega(r) = \sqrt{\frac{GM}{r^3}}. \quad \text{(XV.227)}$$

To make precise the statement that we work to first order in viscosity, one takes the the coefficent of viscosity ν to be proportional to e, vanishing at $e = 0$:

$$\nu = e\overset{1}{\nu}. \quad \text{(XV.228)}$$

We can expand the solution in powers of e, writing for each quantity Q

$$Q(e) = Q(0) + e\overset{1}{Q} + O(e^2), \text{ where } \overset{1}{Q} := \left.\frac{\partial Q}{\partial e}\right|_{e=0}. \quad \text{(XV.229)}$$

The usual physicist's notation is

$$Q = Q_0 + \delta Q; \quad \text{(XV.230)}$$

here $\delta Q = e\overset{1}{Q} = e\left.\frac{\partial Q}{\partial e}\right|_{e=0}$. If Q represents all of the fluid variables, then for each e, the exact solution $Q(e)$ satisfies a set of equations of the form

$$E(e, Q(e)) = 0, \quad \text{(XV.231)}$$

and the first-order perturbation $\overset{1}{Q}$ satisfies the first-order equation

$$\left.\frac{\partial}{\partial e}E(e,Q(e))\right|_{e=0} = 0, \text{ or, equivalently } \delta E = 0. \quad \text{(XV.232)}$$

A finite disk is a stationary axisymmetric exact solution $P(e,t,\mathbf{r}),\ \rho(e,t,\mathbf{r}),\ \mathbf{v}(e,t,\mathbf{r})$ to the set of equations (XV.193)-(XV.196)

$$
\begin{aligned}
\frac{\partial}{\partial r}(r\rho v_r) &= 0 && \text{(XV.233)}\\
\rho\left(v_r\frac{\partial}{\partial r}v_r - \frac{v_\phi^2}{r}\right) &= -\frac{\partial}{\partial r}P - \frac{GM\rho}{\mathtt{r}^2}\frac{r}{\mathtt{r}} && \text{(XV.234)}\\
\rho v_r\frac{\partial}{\partial r}(rv_\phi) &= \frac{1}{r}\frac{\partial}{\partial r}\left(r^3\rho\nu\frac{\partial}{\partial r}\Omega\right) && \text{(XV.235)}\\
\rho v_r\frac{\partial}{\partial r}v_z &= -\frac{\partial}{\partial z}P - \frac{GM\rho}{\mathtt{r}^2}\frac{z}{\mathtt{r}}. && \text{(XV.236)}
\end{aligned}
$$

At $e = 0$, the mass conservation equation and the ϕ and z components of the Euler equation vanish when $v_r = 0$. With $\rho_0 = \Sigma(r)\delta(z)$, the remaining equation implies for Ω the Keplerian angular velocity, and it leaves the surface density profile $\Sigma(r)$ arbitrary.

$$
-\rho_0\Omega^2 r = \frac{GM\rho_0}{r^2} \qquad \text{(XV.237)}
$$

The first-order equations (XV.232) are

$$
\begin{aligned}
\frac{\partial}{\partial r}(r\Sigma\overset{1}{v}_r) &= 0 && \text{(XV.238)}\\
-\overset{1}{\rho}\Omega^2 r - 2\rho_0\Omega\overset{1}{v}_\phi &= -\frac{\partial}{\partial r}\overset{1}{P} - \frac{GM\overset{1}{\rho}}{r^2} && \text{(XV.239)}\\
\Sigma\overset{1}{v}_r\frac{\partial}{\partial r}(r^2\Omega) &= \frac{1}{r}\frac{\partial}{\partial r}\left(r^3\sigma\overset{1}{\nu}\frac{\partial}{\partial r}\Omega\right) && \text{(XV.240)}\\
0 &= -\frac{\partial}{\partial z}\overset{1}{P} - \frac{GM\overset{1}{\rho}z}{r^3}. && \text{(XV.241)}
\end{aligned}
$$

As before, using the first of these equations to bring $r\Sigma\overset{1}{v}_r$ inside ∂_r in the third equation gives

$$
\frac{1}{r}\frac{\partial}{\partial r}\left(r^3\overset{1}{v}_r\Sigma\Omega\right) = \frac{1}{r\rho}\frac{\partial}{\partial r}\left(r^3\sigma\overset{1}{\nu}\frac{\partial}{\partial r}\Omega\right) \qquad \text{(XV.242)}
$$

The first and third equations, with $\overset{1}{v}_r$ replaced by $v_r = e\overset{1}{v}_r$ are just equations (XV.208) and (XV.209). One needs the additional physical assumption that the disk extends to the star's surface, which rotates more slowly that the Keplerian speed, to infer that $\partial_r\Omega = 0$ close to the surface of the star and thereby get the boundary condition (XV.208).

XV.5 Local Structure of Thin Disks

In the previous lecture, we saw that the bulk properties of the disks depends only on their mass accretion rate. It turns out that the simple assumptions the determine the bulk structure also determines their local structure as we will see here.

Because these disks are thin, we can solve for their radial and vertical structure independently. Let begin with the vertical momentum equation, which is just vertical hydrostatic balance:

$$\frac{1}{\rho}\frac{\partial}{\partial z}P = -\frac{GMz}{r^3} \tag{XV.243}$$

Define H as the scale height, we can approximate the above equation via

$$-\frac{P}{\rho H} = -\frac{c_s^2}{H} = \frac{GMH}{r^3}, \tag{XV.244}$$

where I used $P \sim \rho c_s^2$. This gives:

$$\frac{c_s^2}{v_{\rm orb}^2} = \frac{H^2}{r^2} \quad \text{or} \quad H = \frac{c_s}{\Omega} \tag{XV.245}$$

We also note that

$$\Sigma = \int \rho dz \approx \rho H \to \rho = \Sigma/H. \tag{XV.246}$$

This gives ρ in terms of Σ and H. And we know H in terms of c_s. To get c_s, we will assume an ideal gas equation of state $P = \rho k_B T/\mu m_p$, where $\mu = A/Z + 1$ is mean molecular weight. Now the relevant pressure is for the midplane temperature T_m so:

$$c_s^2 = \frac{k_B T_m}{\mu m_p} \tag{XV.247}$$

To get this midplane temperature we have to work backwards. We will get the flux, which is determined by local dissipation, then work back to get the midplane temperature to support that flux. So recall the local flux

$$F = \sigma T_e^4 = \frac{3}{8\pi}\frac{GM\dot{M}}{r^3}\left(1 - \sqrt{\frac{r_*}{r}}\right), \tag{XV.248}$$

where we have divided by a factor of 2 to account for the flux leaving through the top and bottom. In radiative equilibrium, the flux equation is

$$F = -\frac{4\sigma}{3\kappa\rho}\frac{\partial}{\partial z}T^4 \sim \frac{4\sigma}{3\kappa\rho}\frac{T^4}{H} = \frac{4\sigma T^4}{3\kappa\Sigma}, \tag{XV.249}$$

where κ is the opacity and is a function of $\kappa(\rho, T)$ or $\kappa(\Sigma, H, T)$. We approximate $\frac{\partial}{\partial z} = 1/H$. This allows us to connect the midplane temperature to the flux. In particular:

$$F = \frac{4\sigma T^4}{3\kappa\Sigma} = \frac{3}{8\pi}\frac{GM\dot{M}}{r^3}\left(1 - \sqrt{\frac{r_*}{r}}\right) \tag{XV.250}$$

All we need now is Σ which we get from

$$\nu\Sigma = \frac{\dot{M}}{3\pi}\left(1 - \sqrt{\frac{r_*}{r}}\right), \tag{XV.251}$$

and the kinetimatic viscosity is

$$\nu = \alpha c_s H \tag{XV.252}$$

Equations (XV.245), (XV.246), (XV.247), (XV.250),(XV.251), and (XV.252) are six equations with seven unknowns ν, c_s, T_m H, Σ, ρ, and κ. To complete the set, we need a prescription for $\kappa(\rho, T)$ and then we can solve this set of seven algebraic equations. For instance, let me assume a Kramers (free-free) opacity:

$$\kappa = 5 \times 10^{24} \rho T_m^{-3.5}\,\mathrm{cm}^2\,\mathrm{g}^{-1}. \tag{XV.253}$$

You can then show (in your HW):

$$\Sigma = 5.2\alpha^{-4/5}\dot{M}_{16}^{7/10} M_1^{1/4} r_{10}^{-3/4} f^{14/5}\,\mathrm{g}\,\mathrm{cm}^{-2}, \tag{XV.254}$$

where $M_1 = M/\mathrm{M}_\odot$, $r_{10} = r/10^{10}\,\mathrm{cm}$, $\dot{M}_{16} = \dot{M}/10^{16}\,\mathrm{g}\,\mathrm{s}^{-1}$. The complete set is in your HW. This solution is known at the Shakura-Sunyaev alpha disk solution and is one of the great results in astrophysics.

XV.6 Radiative Inefficient Accretion Flows

Thus far we have studied the radiative efficient accretion flow. We will note that the solutions that we found are power-law in radius, ignoring the $(1 - \sqrt{r_*/r})$ term. So the question is if there are other solutions that are available. Let us return to the Navier-Stokes equations:

$$\begin{aligned}
\frac{\partial}{\partial t}\rho + \frac{1}{r}\frac{\partial}{\partial r} r\rho v_r &= 0 && \text{(XV.255)}\\
\frac{\partial}{\partial t} v_r + v_r\frac{\partial}{\partial r} v_r - r\Omega^2 &= -\frac{1}{\rho}\frac{\partial}{\partial r}P - \frac{GM}{r^2} && \text{(XV.256)}\\
\frac{\partial}{\partial t} r v_\phi + v_r \frac{\partial}{\partial r} r^2\Omega &= \frac{1}{r\rho}\frac{\partial}{\partial r} r^3\rho\nu\frac{\partial}{\partial r}\Omega && \text{(XV.257)}\\
0 &= -\frac{1}{\rho}\frac{\partial}{\partial z}P - \frac{GM}{r^2}\frac{z}{r}, && \text{(XV.258)}
\end{aligned}$$

As usual the last equation gives me $H = c_s/\Omega_K$, where $\Omega_K = \sqrt{GM/r^3}$ is the Keplerian rotation rate. I will also use $\nu = \alpha c_s^2/\Omega_K$. To this set I will add an energy equation

$$\rho\frac{\partial}{\partial t}e + \rho v_r\left(\frac{\partial}{\partial r}e - \frac{P}{\rho^2}\frac{\partial}{\partial r}\rho\right) = \rho\nu r^2\left(\frac{\partial}{\partial r}\Omega\right)^2 - \text{(cooling)}, \tag{XV.259}$$

where $e \approx c_s^2$ is the internal energy, the second term on the LHS contains pdV work, and the right hand side contains the dissipation term that we discussed last time. Now at this point a few thing should bother you. First there is a difference between Ω and Ω_K. Second, the energy equation has this unspecified cooling term there. We have ignored the cooling term up to this point as we have set it to exactly balance the heating term. So it seems strange to include it now.

However, let us consider the possibility that it doesn't cool effectively or just partially. Let us rewrite the energy equation as

$$\rho\frac{\partial}{\partial t}e + \rho v_r\left(\frac{\partial}{\partial r}e - \frac{P}{\rho^2}\frac{\partial}{\partial r}\rho\right) = f\rho\nu r^2\left(\frac{\partial}{\partial r}\Omega\right)^2, \tag{XV.260}$$

where f is some constant between 0 and 1. If it is 1, then all the heat that is produced goes into heating – none is radiate away. If it is 0, the all of it is radiated away. We now use the steady state approximation and drop the vertical equation:

$$\frac{1}{r}\frac{\partial}{\partial r}r\rho v_r = 0 \quad \text{(XV.261)}$$

$$v_r\frac{\partial}{\partial r}v_r - r\Omega^2 = -\frac{1}{\rho}\frac{\partial}{\partial r}P - \frac{GM}{r^2} \quad \text{(XV.262)}$$

$$v_r\frac{\partial}{\partial r}r^2\Omega = \frac{1}{r\rho}\frac{\partial}{\partial r}r^3\rho\frac{\alpha c_s^2}{\Omega_K}\frac{\partial}{\partial r}\Omega \quad \text{(XV.263)}$$

$$v_r\frac{\partial}{\partial r}\rho c_s^2 = f\rho\frac{\alpha c_s^2}{\Omega_K}r^2\left(\frac{\partial}{\partial r}\Omega\right)^2. \quad \text{(XV.264)}$$

Let me define $\tilde{r} = r/r_0$, where r_0 is some characteristic radius. Let me also assume power law solutions for each

$$\rho = \rho_0\tilde{r}^a, \quad v_r = v_{r,0}\tilde{r}^b, \quad \Omega = \Omega_0\tilde{r}^c, \quad c_s^2 = c_{s,0}^2\tilde{r}^d. \quad \text{(XV.265)}$$

and $P = \rho c_s^2$. Plugging in these power law solutions, we find

$$2 + a + b + d/2 + 1/2 = 0 \quad \text{(XV.266)}$$

$$2b - 1 = 1 + 2c = d - 1 = -2 \quad \text{(XV.267)}$$

$$b - 1 + 2 + c = d + c + 3/2 \quad \text{(XV.268)}$$

$$b - 1 + d = d + 3/2 + 2 - 2 + 2c \quad \text{(XV.269)}$$

A solution to this set is

$$a = -3/2, \quad ,b = -1/2, \quad c = -3/2, \quad \text{and} \quad d = -1 \quad \text{(XV.270)}$$

or

$$\rho = \rho_0\tilde{r}^{-3/2}, \quad v_r = v_{r,0}\tilde{r}^{-1/2}, \quad \Omega = \Omega_0\tilde{r}^{-3/2}, \quad c_s^2 = c_{s,0}^2\tilde{r}^{-1}. \quad \text{(XV.271)}$$

Now let's OOM some of the normalizations: looking at the energy equation, we can show

$$v_r\frac{\partial}{\partial r}\rho c_s^2 = f\rho\frac{\alpha c_s^2}{\Omega_K}r^2\left(\frac{\partial}{\partial r}\Omega\right)^2 \rightarrow \frac{v_{r,0}\rho_0 c_{s,0}^2}{r_0} = f\alpha\frac{\rho_0 c_{s,0}^2\Omega_0^2}{\Omega_{K,0}} \rightarrow v_{r,0} = \alpha f\frac{\Omega_0}{\Omega_{K,0}}r_0\Omega_0, \quad \text{(XV.272)}$$

This implies that the radial infall velocity scales likes f. For $f = 0$ (efficient cooling), v_r is small as it is in radiative efficient accretion disks, but if f is near unity, then the radial infall velocity is comparable to the rotational velocity, with a factor of α. Let look at the angular momentum equation:

$$v_{r,0}r_0\Omega_0 = \frac{\alpha c_{s,0}^2}{\Omega_{K,0}}\Omega_0 \rightarrow c_{s,0}^2 = fr_0^2\Omega_0^2 \quad \text{(XV.273)}$$

Now this one is interesting. The sound speed, i.e., internal energy is similar to the rotation velocity, i.e., gas is near the virial temperature for $f \neq 0$. Finally let's look at the radial momentum equation:

$$-r_0\Omega_0^2 = -\frac{c_{s,0}^2}{r_0} - \frac{GM}{r_0^2} \rightarrow \Omega_0^2 = \frac{GM}{(1+f)r_0^3} \quad \text{(XV.274)}$$

Now don't take the $1 + f$ too seriously as we have dropped factors of order unity, But the rotation rate is going to be different from the Keplerian rate due to the nontrivial contribution of the gas pressure. The structure of this solution make up what is called a radiative inefficient accretion flow or RIAF. This is increasing important in astrophysics as they have many important properties for high energy systems. In particular

1. Inefficient cooling implies gas near the virial temperature.
2. Larger radial velocities $v_r \sim \alpha c_s$
3. Sub-Keplerian angular velocities
4. The binding energy of the gas is near zero – possibility for outflows.

This last fact is perhap the most important aspect of RIAF that you should be concerned with because these flows mean jets and mechanical power which is a powerful way by which these flows might be observed. In addition just because these things are radiatively inefficient does not mean that they don't radiate. Instead the hot gas can radiate via synchrotron emission or inverse Compton scattering – They radiate nonthermally, which implies hard photons.

Interestingly, x-ray binary systems have been observed to migrate between a radiatively efficient thin disk state and a radiative inefficient state. Because it undergoes the transition in one system, we can learn alot about these systems. Here, there is a definitive link between the type of outflows and the state of the disk. In Figure 20, we show how this works. On the x-axis we show the x-ray hardness which is a proxy for radiative efficient vs inefficient. Blackbody spectrum are the softest so they are associated with thin disks. Hard spectra are associated with thick disks. On the y-axis is the luminosity, which is a proxy for the accretion rate. What is not shown is that jets appear when the disk is on the right side of the plot (hard state, generally lower accretion rate) whereas these jets disappear on the left, replaced by disk winds (a more recent observed phenomenon).

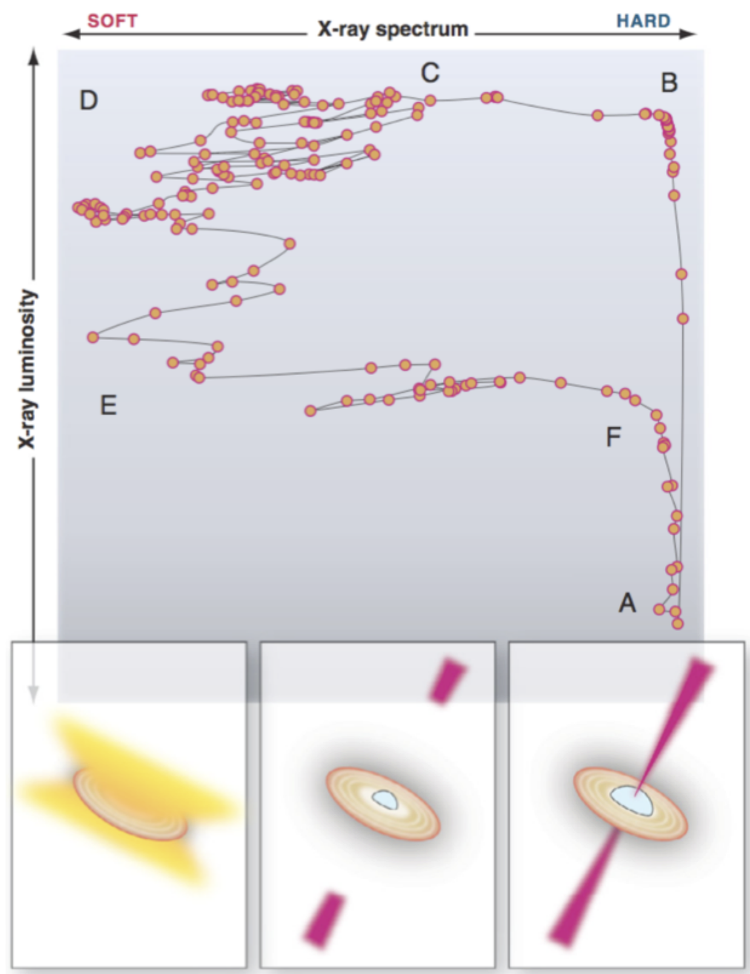


Figure 20: Accretion states and outflows

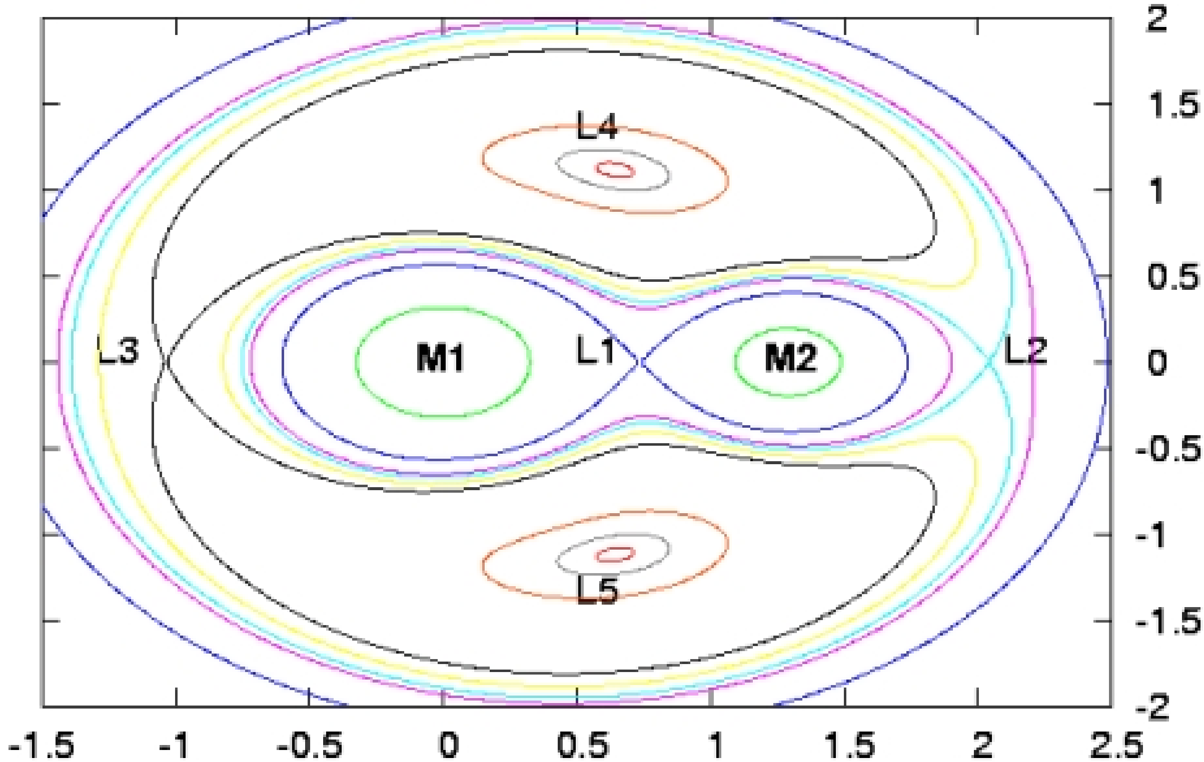
Lecture XVI Interacting Binaries

Interacting means accretion (mass transfer).

XVI.2 Gravity and the Roche Potential

Consider two stars orbiting their COM. They have angular velocity $\omega = v_1/a_1 = v_2/a_2$. We will work in a coordinate system that follows this rotation, where the gravitational force will be balanced by a centrifugal force $F_c = m\omega^2 r$.

Instead of forces consider the potential energy. From gravity this is $U_g = -GM_1M_2/a$. We can integrate the centrifugal force to find a centrifugal potential, with $U_c = -m\omega^2 r^2/2$ for a mass m a distance r from the origin.



The full potential for a test mass in the plane of the orbit is:

$$U = -G\left(\frac{M_1 m}{s_1} + \frac{M_2 m}{s_2}\right) + \frac{1}{2}m\omega^2 r^2 \tag{XVI.275}$$

where s_1 and s_2 are the distances from the test mass to the stars. Or, we can find:

$$\Phi = -G\left(\frac{M_1}{s_1} + \frac{M_2}{s_2}\right) + \frac{1}{2}\omega^2 r^2 \tag{XVI.276}$$

We can solve for s_1 and s_2 via geometry (θ is the angle from the COM to the test mass):

$$s_1^2 = a_1^2 + r^2 + 2a_1 r\cos\theta \tag{XVI.277}$$

$$s_2^2 = a_2^2 - r^2 + 2a_2 r\cos\theta \tag{XVI.278}$$

And we can get ω from Kepler:

$$\omega^2 = \frac{G(M_1+M_2)}{a^3} \tag{XVI.279}$$

With this potential, we can evaluate the forces in 2D ($\nabla\Phi$) to find the Lagrange points where there is no force. There are three Lagrange points on the axis between the stars: L_1 between then, and L_2 and L_3 on either side. These are equilibrium points, but they are unstable since they are local maxima of Φ.

The inner point L_1 is the most important for this discussion. When one star expands, the material fills equipotential surfaces. As the radius of the star approaches the L_1 point matter can overfill its potential well and fall onto the other star. This is mass transfer. The potential well is known as the "Roche Lobe". We have

$$r_L = \frac{R_L}{a} = 0.5 - 0.227\log_{10} q \tag{XVI.280}$$

with $q = M_2/M_1$ is the mass ratio. The other point has $1/q$, so the sign on the log is flipped. There are various useful approximations for r_L from Eggleton (1983, ApJ, 268, 368) and Paczyński (1967, Acta Astrononica, 17, 287).

If mass is transferred, we say a star is experiencing "Roche Lobe overflow."

XVI.2.1 Accretion Disks

Convert bulk motion into heat through viscosity, leading to matter traveling inward in the disk. For a start, assume that all of that energy is radiated as a uniform blackbody disk in steady-state. In a ring at radius r, the matter has $E = -GMm/2r$ (Virial theorem). As it moves inward, E gets more negative, and this energy is radiated:

$$dE = \frac{dE}{dr}dr = \frac{d}{dr}\left(-G\frac{Mm}{2r}\right)dr = G\frac{M\dot{M}t}{2r^2}dr \tag{XVI.281}$$

with $m = \dot{M}t$ matter entering (and leaving) the disk. And this will be radiated:

$$dLt = dE = G\frac{M\dot{M}t}{2r^2}dr \tag{XVI.282}$$

Assume blackbody, with $dL = 4\pi r\sigma T(r)^4 dr$ (top and bottom, so $A = 2\times 2\pi r dr$). We can solve:

$$T = \left(\frac{GM\dot{M}}{8\pi\sigma R^3}\right)^{1/4}\left(\frac{R}{r}\right)^{3/4} \tag{XVI.283}$$

This says that the inner edge will be the hottest, at $T_{\rm disk} = (GM\dot{M}/8\pi\sigma R^3)^{1/4}$, although a better fit to data says that the inner edge is at roughly half this. Beyond the inner edge, the temperature falls as $T(r) \propto r^{-3/4}$. Integrating, we find:

$$L_{\rm disk} = G\frac{M\dot{M}}{2R} \tag{XVI.284}$$

which is 50% of the total amount of energy that could be tapped. So 50% can be radiated while the gas accretes, which leaves 50% to be radiated upon accretion.

Looking at the luminosity and temperature of the accretion flow, we can determine whether the object is a WD or a NS. For instance, we might find $T_{\rm max} = 3 \times 10^4$ K for a WD, but $T_{\rm max} = 7 \times 10^6$ K for a NS.

XVI.3 Period-Density relation

Star is losing mass through L_1 point. So it fills the whole Roche lobe. Size of Roche lobe is:

$$\frac{R_2}{a} \approx \frac{2}{3^{4/3}}\left(\frac{M_2}{M_1+M_2}\right)^{1/3} \tag{XVI.285}$$

(approximate relation from Paczynski). Cube both sides:

$$\frac{3^4 R_2^3}{8M_2} = \frac{a^3}{M_1+M_2} \tag{XVI.286}$$

But Kepler's third law is:

$$\frac{a^3}{G(M_1+M_2)} = \frac{P^2}{4\pi^2} \tag{XVI.287}$$

So we have:

$$\frac{3^4 R_2^3}{8M_2} = \frac{GP^2}{4\pi^2} \tag{XVI.288}$$

We can relate this to $\bar{\rho} = M_2/(4\pi R_2^3/3)$, with:

$$\bar{\rho} \approx 110\, P_{\rm hr}^{-2}\, {\rm g\, cm^{-3}} \tag{XVI.289}$$

This is the period mean-density relation. We can turn this into a period-mass relation by using a mass-radius relation. e.g., for main-sequence we know that $R \propto M$, which says:

$$M_2 \approx 0.11\, M_\odot P_{\rm hr} \tag{XVI.290}$$

XVI.4 Effect of Mass Transfer

Does the orbit get wider or narrower?

Assume conservative mass transfer: mass and L are fixed. So $M_1 + M_2 = M$, and $\dot{M}_1 = -\dot{M}_2$ (this need not be the case). And:

$$L = \mu\sqrt{GMa} = M_1 M_2 \sqrt{\frac{Ga}{M}} \tag{XVI.291}$$

is also constant. We can take a total derivative of the above to find:

$$\frac{\dot{a}}{a} = 2\frac{\dot{L}}{L} - 2\left(1 - \frac{M_2}{M_1}\right)\frac{\dot{M}_2}{M_2} \tag{XVI.292}$$

assuming that star 2 is the donor ($\dot{M}_2 < 0$) and that $\dot{L} = 0$, we find that:

$$\frac{\dot{a}}{a} = -2\dot{M}_2 \frac{M_1 - M_2}{M_1 M_2} \tag{XVI.293}$$

So if $M_1 > M_2$ (transfer from less to more massive) then $\dot{a} > 0$ and the orbit gets wider. if $M_1 < M_2$ (transfer from more to less massive) then $\dot{a} < 0$ and the orbit gets smaller. As that happens, ω increases. However, this treatment ignores possible losses of L. What could that be? Simplest is GR:

$$\dot{L}_{GR} = -\frac{32}{5}\frac{G^{7/2}}{c^5}\frac{M_1^2 M_2^2 (M_1 + M_2)^{1/2}}{a^{7/2}} \tag{XVI.294}$$

With this as a minimum, it often drives systems to closer orbits even if $M_1 > M_2$.

Is this mass transfer stable? Will the movement of mass from one star to the other proceed slowly (on the nuclear evolutionary timescale of the star, $\sim 10^{10}$ yr for the Sun) or faster (thermal timescale $\sim 10^7$ yr, dynamical timescale ~ 90 min)?

Consider how the star and the Roche lobe adjust to the change in mass. The star loses a parcel of mass, and it then adjusts its radius. First it restores hydrostatic equilibirum on a dynamical timescale (so fast it's adiabatic: no energy can enter or leave). Then adjusts thermal equilibrium on thermal timescale. If it becomes bigger compared to the change in the Roche lobe, we get unstable transfer and it runs away on the relevant timescale. If it is stable, it will continue slowly and only transfer matter as the star expands on the nuclear timescale.

If transfer is stable, then the radius of the donor will essentially track the Roche lobe radius as they both evolve slowly. Or, $R_2 = R_L = ar_L$. So we can re-write $\dot{a}/a$ in terms of these.

Assume that the donor star (M_2) has $R_2 \propto M_2^{\zeta}$, where $\zeta \equiv d\ln R_2/d\ln M_2$. For a normal star this would be ≈ 1, or $-1/3$ for a WD or NS. Would actually need to be careful to consider changes in radius on the appropriate timescales. Can also find ζ_L for the Roche lobe r_L. Use the slightly better Paczyński formula:

$$r_L \approx 0.46\left(\frac{M_2}{M_1 + M_2}\right)^{1/3} \tag{XVI.295}$$

We find (assuming $dM_1 = -dM_2$) $\zeta_L = 1/3$. So the change in a will be:

$$\frac{\dot{a}}{a} = \frac{\dot{M}_2}{M_2}(\zeta - \zeta_L) \tag{XVI.296}$$

Equate this with $\dot{a}/a$ from above, and solve:

$$\frac{\zeta - \zeta_L}{2} = \frac{M_2}{M_1} - 1 \tag{XVI.297}$$

If the LHS is < 0, then mass transfer will be dynamically stable: the star's radius will shrink relative to R_L, and we will need to wait for nuclear evolution to make it larger. This requires:

$$\frac{M_2}{M_1} < \frac{5}{6} + \frac{\zeta}{2} \tag{XVI.298}$$

So for a WD or degenerate object, $\zeta \approx -1/3$, and we must have $M_2/M_1 < 2/3$ for stability.

Lecture XVII Magnetic Accretion

So far we have ignored how the material gets to the star. For a WD/NS with $B = 0$, it will just fall. But what if $B > 0$?

Consider a dipole magnetic field with dipole moment $\mu \sim B_0 R^3$. So the strength of the field will vary as:

$$B \sim \frac{\mu}{r^3} \tag{XVII.299}$$

Far away it is negligible. But close to the star it will be strong enough to interrupt the accreting gas. The magnetic field has a pressure:

$$P_B = \frac{B^2}{8\pi} \tag{XVII.300}$$

which increases like r^6 as you approach the star. It will channel the material. It can't stop it overall, but the material (which we assume is ionized) is forced to follow the magnetic field lines. It end up moving along them and only hitting the star at the magnetic poles.

The field interrupts the flow where the magnetic pressure is equal to the gas pressure + ram pressure. But since the flow is highly supersonic, we can neglect gas pressure and say that r_M is where:

$$\frac{B^2}{8\pi} = \frac{\rho v^2}{2} \tag{XVII.301}$$

If the flow is spherically symmetric, then we have v^2 close to the free-fall value of $2GM^2/r$ and $4\pi r^2 \rho v = \dot{M}$. So we can solve for r_M and find:

$$r_M = 5.1 \times 10^8 \,\mathrm{cm} \dot{M}_{16}^{-2/7} \mu_{30}^{4/7} M^{-1/7} \tag{XVII.302}$$

with μ expressed in $10^{30}\,\mathrm{G\,cm}^3$ (a WD with $B = 10^4\,\mathrm{G}$ and $R = 5 \times 10^8$ cm or a NS with $B = 10^{12}\,\mathrm{G}$ and $R = 10^6$ cm).

This is pretty rough, and in general things will be non-spherical etc. But P_B increases so steeply with radius that gas will not penetrate much beyond r_M before it gets caught. This radius is known as the Alvén radius. Clearly this will happen well outside a NS, and if B is strong outside a WD too.

A common sign of channeled accretion is X-ray hotspots which cause pulsations when the star rotates (this requires misalignment). How big will the spots be? Consider a NS with a disk with an angle α between the magnetic axis and the disk. In polar coordinates (r, θ) the dipole geometry says that a field line will collor $r = C \sin^2\theta$, with C a constant labeling individual field lines. At the disk we have $r = r_M$ and $\theta = \alpha$, so $C = r_m / \sin^2\alpha$. At the star we have $r = R$ and $\theta = \beta$. We can solve:

$$\sin^2\beta = \frac{R}{C} = \frac{R}{r_M}\sin^2\alpha \tag{XVII.303}$$

Only the part of the star with $\theta < \beta$ will be hit by accreted material. The area of this cap is (as a fraction of the total area of the star):

$$f \sim \frac{\pi R^2 \sin^2\beta}{4\pi R^2} = \frac{R \sin^2\alpha}{4 r_M} \tag{XVII.304}$$

which we multiply by two for the opposite pole. Typically this is 0.01%–10%.

What happens to the angular momentum of the star? The material in the disk will have a Keplerian orbit with angular velocity $\Omega_K(r) = \sqrt{GM/r^3}$. It will hit the star's field lines, which rotate with Ω_0. If:

$$\Omega_0 > \Omega_K(r_M) \tag{XVII.305}$$

then the star will be rotating faster than the disk when it encounters the field lines. The material will hit a centrifugal barrier and be flung away (known as "propeller accretion"). We parameterize this with the "fastness parameter":

$$\omega_0 = \frac{\Omega_0}{\Omega_K(r_M)} \tag{XVII.306}$$

and require $\omega_0 < 1$ for actual accretion.

Assuming that it can penetrate this barrier, the material can still drag on the star. P (spin period) is typically $1\,\mathrm{s}$–$10^3\,\mathrm{s}$ for accreting systems (not for rotation-powered pulsars). But P is getting shorter slowly, spinning up. If $\omega_0 \ll 1$ we can treat it simply. Basically the star gains angular momentum at a rate:

$$\dot{L} \sim \dot{M} r_M^2 \Omega_K(r_M) = I\dot{\Omega}_0 \tag{XVII.307}$$

We can then work out a relation between the spin-up rate and the star's properties.

Eventually we get $\omega_0 \sim 1$ and the star is in near-equilibrium. Here we see periods of spin-up and spin-down. We can equivalently express this by asking whether the corotation radius:

$$R_\Omega = \left(\frac{GMP^2}{4\pi^2}\right)^{1/3} \tag{XVII.308}$$

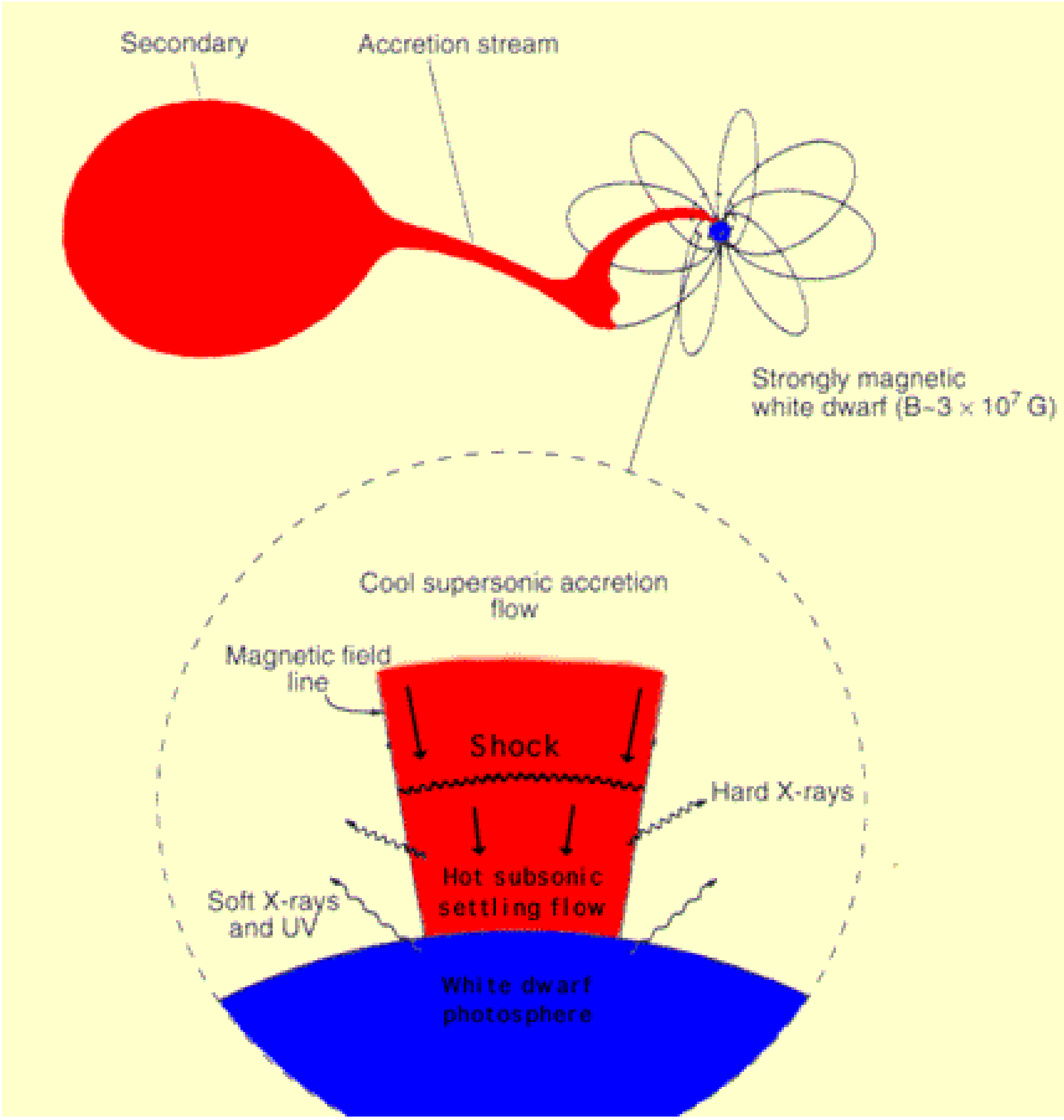
is bigger or smaller than r_M. For $\omega_0 < 1$ that is $R_\Omega > r_M$. We end up with an equilibrium period where these are equal.

Lecture XVIII WD Binaries

Many different names, based on phenomenology. Generic term is Cataclysmic Variable (CV). Usually WD + main sequence star. Pretty common. Outbursts can repeat. P_b is hours usually. Can see double-peaked emission lines from disk that tell us the Keplerian velocities.

Covers a range of related phenomena:

- Classical nova: very large outbursts. Accreted material gets hot enough for unstable fusion (H burning via CNO). $> 10^4$ increase in L.
- Recurrent nova: same process but we see them more than once. Can relate release of energy to $\dot{M}$, recurrence time, and efficiency of fusion.
- Dwarf nova: Changes in L (factor of 10 to 100) are changes in $\dot{M}$, due to complicated physics in the disk. Ionization instability in the disk: neutral gas has low opacity, can cool easily, so low T, low viscosity, lots material in disk. Ionized gas has high viscosity so dumps the material onto the star that has accumulated.
- AM CVn: like CV, but accretion from low-mass (bigger) WD (usually He-core) onto higher-mass (smaller) WD
- Polar/AM Her: strongly magnetized CV. Accretor has $B > 10^7\,$G, so flow is channeled and rotation of WD is locked to orbit (donor is also locked). Emission is polarized (interaction with B), which gives the name. Strong X-ray emission from shocks in accretion column.



- Intermediate polar (DQ Her): magnetic field is intermediate between CV and Polar (10^{6-7} G). Strong X-rays from accretion column, and polarized optical emission.

The last are magnetic variations (channel the accretion flow).

These are related by having accretion from a low-mass donor onto a WD. Phenomena differ by changes in L over time, and by other things (B or no, nature of donor). How do we know this? Basic properties:

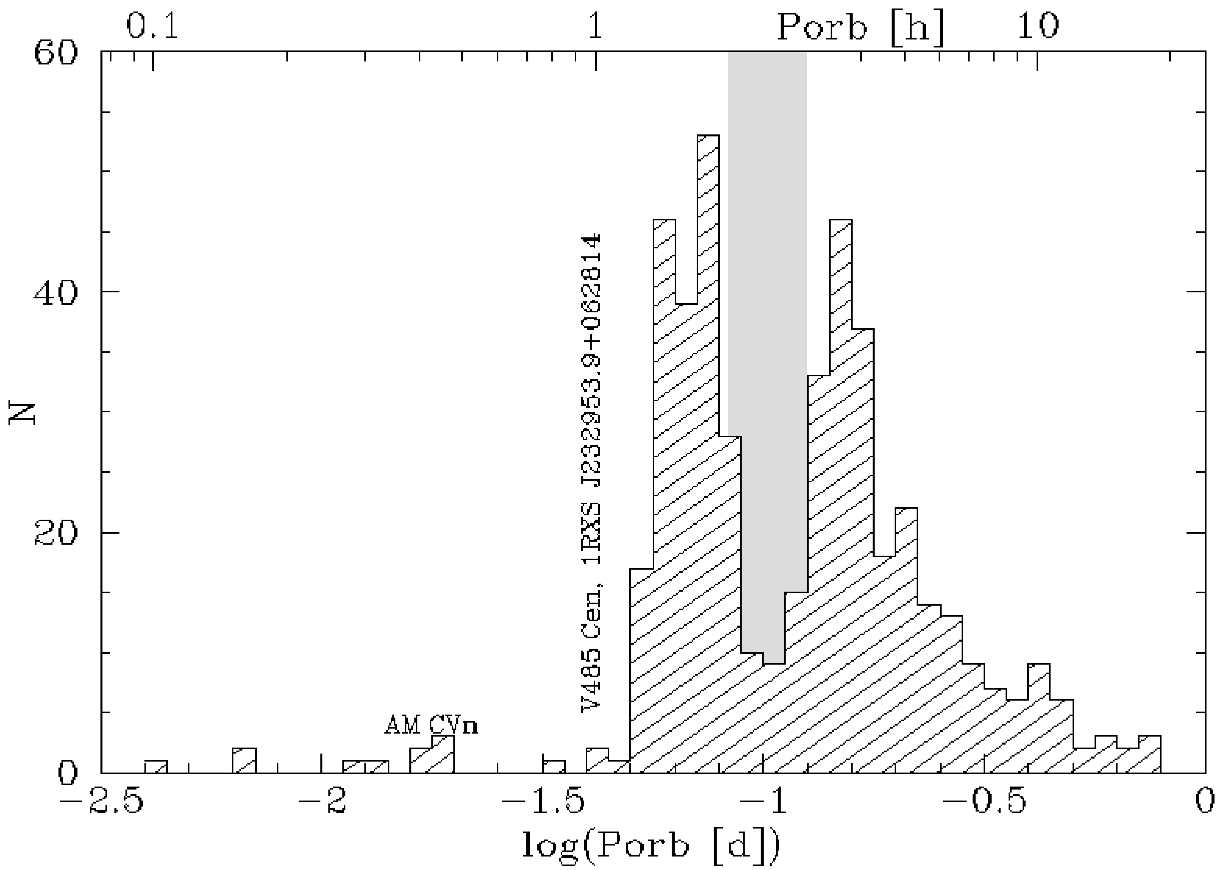
- binaries with periods 23 min–5 d, most 78 min–12 hr (with a gap at 1.5 hr–3.25 hr): can see this from photometric modulation and/or spectroscopy

- Mass of accretor (primary) is $\approx 0.9\,M_\odot$. We get this from Kepler's third law. Other star we know is less massive, like a main-sequence star.

If $P < 1$ hr, we know $M_2 < 0.1\,M_\odot$ from P-M relation.

During accretion, we can use eclipsing systems where the companion blocks part of the disk to map out the disk temperature profile. When accretion is high enough, we see an optically thick disk with something similar to the $T \propto r^{-3/4}$ relation we expect. Also see broadened, double-peaked line that moves around with mass of WD: circular accretion onto the WD. Basically all of the emission that we see (at least at wavelengths of UV and longer) comes from the disk: disk area is $\gg$ WD area and it dominates.

Origin of period gap uncertain. For $P_b > 3$ hr, magnetic braking dominates loss of angular momentum: magnetized stellar wind from low-mass donor star goes out to infinity. As it moves out it slows down rotation (conservation of angular momentum) and the magnetic field slows down the star too. This also carries away orbital angular momentum since the donor star is assumed to be locked to the binary orbit. So there is an additional sink term for angular momentum.



[CV period distribution from `http://www.caha.es/newsletter/news03b/gaensicke/`, based on the sample from Ritter & Kolb (2003).]

Following Verbunt & Zwaan (1981), take a functional form for the equatorial velocity on the age of a star. This allows us to infer the rotation rate of a single star as a function of age, telling us about how braking of the star's rotation would proceed in the absence of a binary. So:

$$v_e = f \times 10^{14} t^{-0.5}\,\mathrm{cm\,s^{-1}} \tag{XVIII.309}$$

where $f \approx 1$. The angular momentum of rotation for the star is:

$$L_2 = k_2 M_2 R_2^2 \omega \tag{XVIII.310}$$

where $k_2 \approx 0.1$ is a constant. So:

$$\frac{dL_2}{dt} = -0.5 \times 10^{-28} f^{-2} k_2 M_2 R_2^4 \omega^3 \tag{XVIII.311}$$

We also have from Kepler $\omega^2 = GM/a^3$. This assumes efficient tidal coupling so that rotation is locked to the orbit. For a low-mass main-sequence star

$$\frac{R_2}{R_\odot} \approx \frac{M_2}{M_\odot} \tag{XVIII.312}$$

and the secondary always fills the Roche lobe, $R_2 = R_L$. Using the Paczyński formula for R_L/a again, we get:

$$\frac{dL_2}{dt} = -1.8 \times 10^{37} f^{-2} k_2 \left(\frac{M_2}{M_\odot}\right)^2 \tag{XVIII.313}$$

Note that all dependence on ω has gone, since we assumed a mass-radius relation (not really true for stars that are undergoing rapid mass-loss) and that $R_2 = R_L$.

But this can be compared with the $\dot{L}_{\mathrm{GR}}$ that we had up before:

$$\frac{dL_{\mathrm{GR}}}{dt} = -1.1 \times 10^{34} \frac{m_1^2}{m^{2/3} m_2^{1/3}} \tag{XVIII.314}$$

where $m_1 = M_1/M_\odot$ etc. So the effects of magnetic braking can dominate GR by orders of magnitude.

We also have

$$L_{\mathrm{orb}} = \frac{M_1 M_2}{M} a^3 \omega \tag{XVIII.315}$$

which gives

$$\frac{dL_{\mathrm{orb}}}{dt} = 3.0 \times 10^{51} \left(\frac{m_2}{m}\right)^{1/3} (4m - 7m_2)\dot{m}_2 \tag{XVIII.316}$$

which relates the change of angular momentum to the change in mass (it is assumed that the overall mass-loss is less efficient than the angular momentum loss, so that the total mass is roughly constant). This change in angular momentum can be related to the change in rotation:

$$\dot{L}_{\mathrm{orb}} = \dot{L}_2 \tag{XVIII.317}$$

so we get:

$$\frac{d\mu}{d\tau} = 1.9 \times 10^{-7} f^{-2} k_2 \frac{\mu^{5/3}}{7\mu - 4}\,\mathrm{yr}^{-1} \tag{XVIII.318}$$

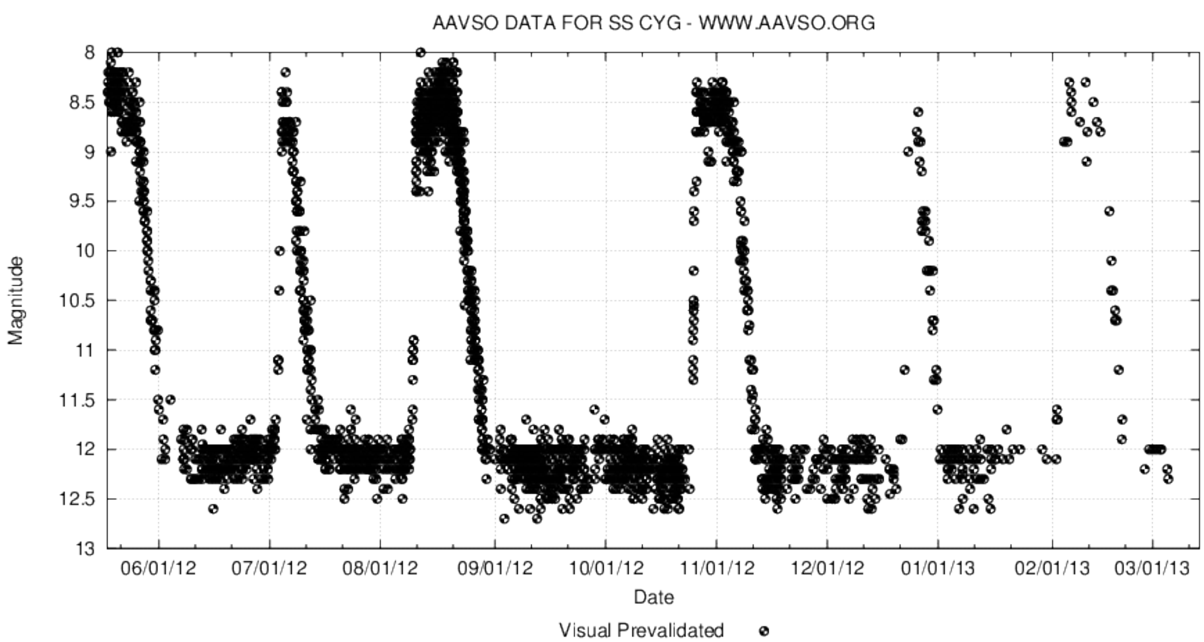
where $\mu \equiv M_2/M$ and τ is in years. We know that $(7\mu - 4) < 0$ because we are assuming mass transfer from a low-mass donor to a more massive accretor, i.e., $\mu < 0.5$. So:

$$\dot{m}_2 = m\frac{d\mu}{d\tau}\, M_\odot\,\mathrm{yr}^{-1} \tag{XVIII.319}$$

Below < 2 hr, GR dominates loss of angular momentum. From Knigge et al. (2011): "The standard model accomplishes this by assuming that when the donor star in a CV becomes fully convective – which happens at around $P_b \approx 3$ hr – magnetic braking will abruptly shut off. At this point in its evolution, the donor star has been driven slightly out of thermal equilibrium and is therefore somewhat oversized for its mass. When the sudden cessation of magnetic braking reduces the mass-loss rate, the secondary contracts, causing it to lose contact with the Roche lobe altogether. The system then evolves towards shorter periods as a detached binary, driven only by GR. The Roche lobe eventually catches up with the donor again at $P_b \approx 2$ hrs. At this point mass transfer restarts, and the system reemerges as an active CV at the bottom of the period gap."

Also expect period minimum near 80 min: donor is partially degenerate, so grows in response to mass loss. This causes increase in size of orbit (and period). Response of donor depends on $\tau_{\dot{M}} \sim M/\dot{M}$ compared to $\tau_{KH} \sim GM^2/LR$. If $\tau_{\dot{M}} \gg \tau_{KH}$, mass loss is slow, and the donor stays in equilibrium (adjusts on nuclear timescale and looks like a normal main sequence star, with $\zeta \approx 1$)). If $\tau_{\dot{M}} \ll \tau_{KH}$, then mass loss is fast and the donor adjust adiabatically, with $\zeta = \zeta_{\rm ad} \approx -1/3$ (grows). But many cases have these timescales comparable, so the evolution is sensitive to the details.

XVIII.2 Dwarf Novae



Dwarf nova SS Cygni from amateur observers (AAVSO; `https://onwardtotheedge.wordpress.com/2013/03/`).

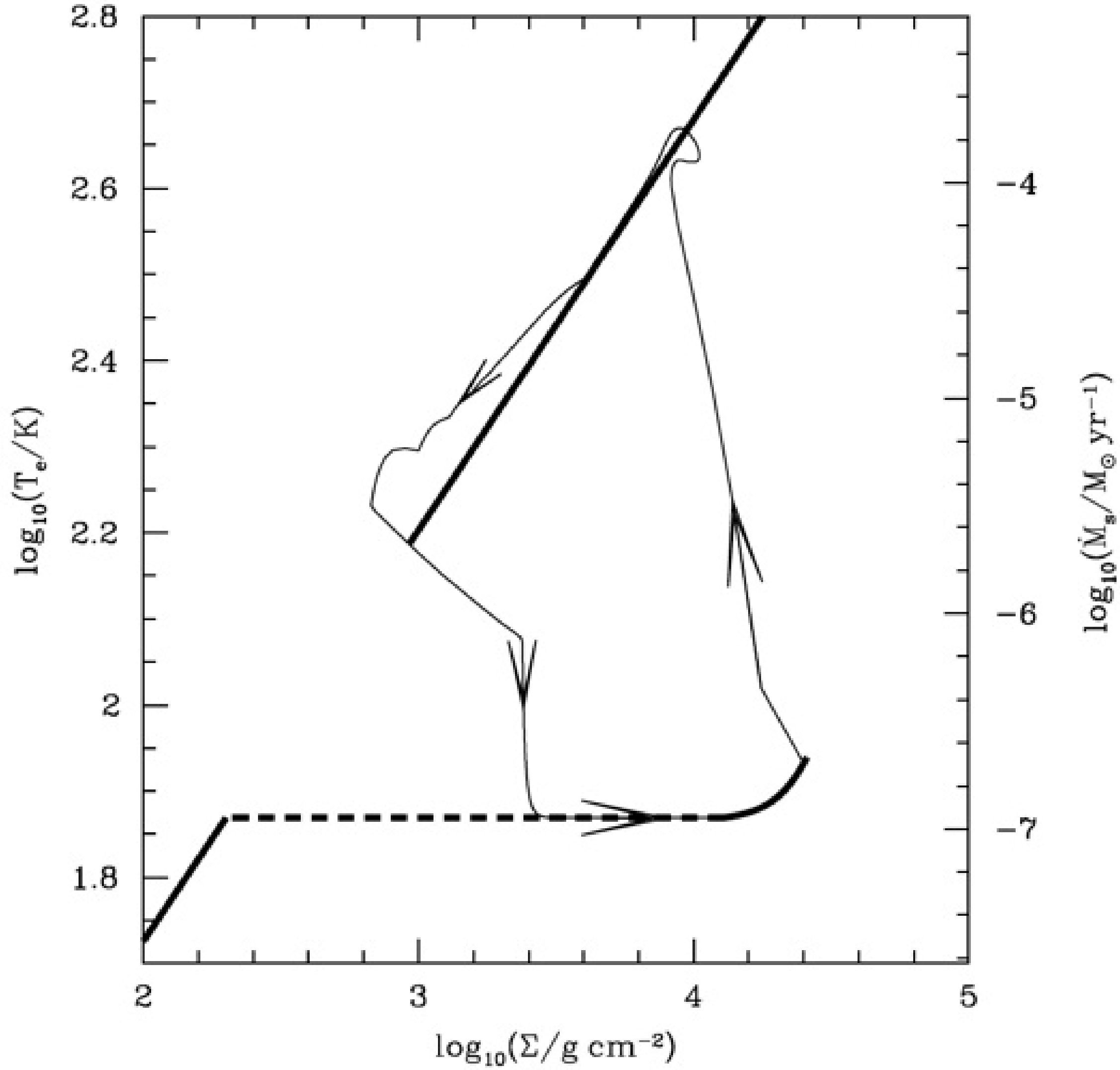
Basic phenomenon: star brightens by 2–5 mag (factor of 10–100) over the course of $\sim$ day, lasts for $\sim$ week, repeats after $\sim$ months. What is this?

Since L is dominated by disk, with $L \propto \dot{M}$, we are seeing enhancement in $\dot{M}$. Why? Is this actually a change in the amount of matter donated by the companion? Or just how much reaches the WD?

We think the latter. In the "low" state, more mass is supplied to the disk than reaches the companion. This is because viscosity ν is low, so material doesn't dissipate energy and it remains at large radii. Eventually the material heats up sufficiently that we hit hydrogen ionization at 10^4 K. We transition:

- Neutral H:
  - low κ
  - efficient cooling
  - low T
  - low ν
  - mass accumulates
- Ionized H:
  - high κ
  - inefficient cooling
  - high T
  - high ν
  - mass dumped through disk onto WD

This mechanism operates as limit cycle: once enough matter has moved through the disk it cools and returns to neutral.



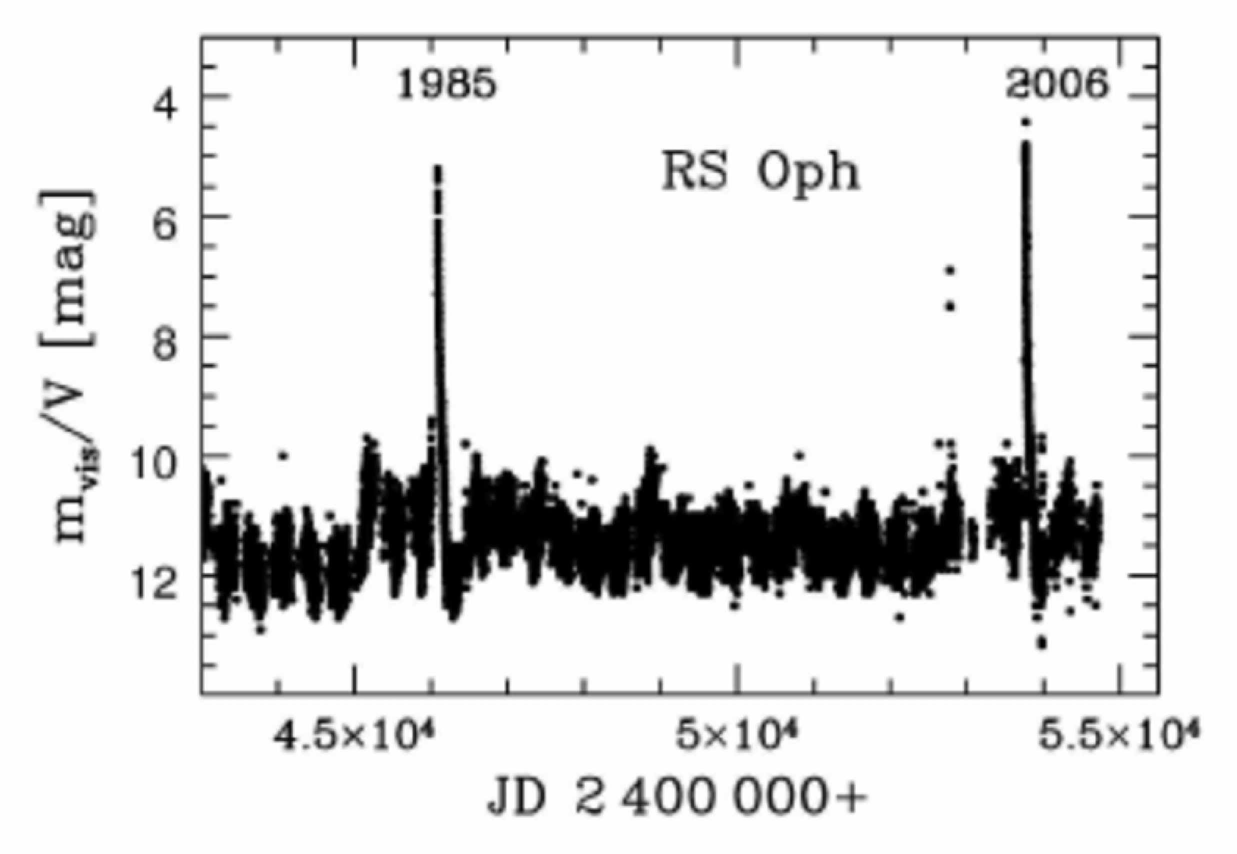
Example from Martin & Lubow (2011, 2013).

This tries to match an unstable external $\dot{M}$ with varying actual $\dot{M}$. We only see this in a limited range of $\dot{M}$ ($< 10^{-11}\,M_\odot\,\mathrm{yr}^{-1}$): if it is too high then H is always ionized.

Note that this is over-simplified since the instability is formally local (i.e., applies to only one radius) but it must become global through some cascade.

XVIII.3 Classical Nova

Thermonuclear explosion on the surface of a WD. If we see it $\sim$ once it is "classical." If we see more then it is "recurrent." So this is an observer-dependent classification rather than something intrinsic, and it is related to $\dot{M}$.



Recurrent nova RS Oph from AAVSO (from Mikolajewska et al. 2011).

Basic picture: brighten by ≈ 10 mag (factor of 10^4) over few days. May peak at $10^5\,L_\odot$ and emit 10^{45} erg over ~ 100 day. See ejected material ($10^{-5}-10^{-4}\,M_\odot$) with $v = 100-1000\,\mathrm{km\,s^{-1}}$ and a large amount of elements which were "burned" in CNO cycle at high T. See 30/year in Milky Way (compare to 1/100 yr for SNe).

In quiescence, see accretion luminosity that implies $\dot{M} \sim 10^{-9}\,M_\odot\,\mathrm{yr}^{-1}$.

If we have accretion onto a WD, material will hit with high velocity and get warm. This will keep piling up. Base of accreted layer will be hotter and hotter. It is also degenerate, and mixes a bit with the C/O elements beneath it. When it is few $\times 10^6$ K, becomes unstable to CNO burning. But it is degenerate. $L_{\rm CNO} \propto T^{17}$, so it becomes very unstable. But hotter does not make it less dense (usual stellar thermostat) since $P = P(\rho)$ only in degenerate material. Stops when it gets to 10^8 K, at which point becomes non-degenerate and rate plateaus.

When $L \sim L_{\rm Edd}$, the outer layers of the star lift off and expel into space (hydrodynamic ejection). Most of the material isn't actually burned. That would be 10^{47} erg — enough to shine for century. Instead propels about 10% of the layer into space and the rest cools down.

After ejection, hydrostatic burning phase with $L \sim L_{\rm Edd}$. The layer continues to expand slowly, cooling but keeping L constant.

Then it stops burning, and the rest of the shell falls back onto the WD. With $\dot{M} = 10^{-9}\,M_\odot\,\mathrm{yr}^{-1}$, will take 10^4 yr to build up enough mass. This recurrence time will change depending on the actual conditions, leading to recurrent novae.

After a nova, loss of mass will make the orbit expand a little so that accretion stops as the donor loses contact with the Roche lobe.

Eject a shell of mass $\Delta M \sim 10^{-4}\, M_\odot$. After this the orbit widens by:

$$\frac{\Delta a}{a} \sim \frac{\Delta M}{M} \tag{XVIII.320}$$

Accretion will stop until gravitational waves bring it back into contact, on a timescale:

$$t_{\text{detach}} = t_{\text{GW}} \frac{\Delta a}{a} \sim t_{\text{GW}} \frac{\Delta M}{M} \tag{XVIII.321}$$

Which gives $t_{\text{detach}} \sim 100\,\text{yr}$ for typical parameters. Compare to the time between novae:

$$t_{\text{recur}} \sim \frac{\Delta M}{\dot{M}} \tag{XVIII.322}$$

which will be $\sim 10^4$ yr for typical parameters. So we don't expect to find them in the non-accreting state: only $\sim 1\%$ of them will be there.

Lecture XIX Neutron Star Binaries

- Low-mass X-ray binaries: NS + main-sequence
- High-mass X-ray binary (HMXB): high-mass star.

Both have NS (usually).

X-ray bursts: unstable H/He burning in accreted layer. Again can relate release of energy to $\dot{M}$, recurrence time, and efficiency of fusion. Can use to measure NS properties in two ways: cooling of surface following impulsive heating, and radius expansion bursts.

These last are long bursts that can lift up a puffy photosphere over the NS ($L \sim L_{\rm Edd}$). Can decompose X-ray spectrum into change in R and change in T. We know how these should behave ($L \sim R^2 T^4 =$ constant), and can use their limits (especially when we know distances) to get radii.

We see quiescent luminosity which is $L_q = \eta G M \dot{M}/R$. This accumulates for some time t, at which point $\Delta m = dot M t$ mass has built up. This can burn stably or unstably. Then, at a column density $y \sim 2 \times 10^8\,{\rm g\,cm^{-2}}$, the pressure at the base is $P = gy \sim 4 \times 10^{22}\,{\rm dyne\,cm^{-2}}$. It ignites and releases $E_b = \epsilon \Delta m c^2$ energy during a burst, with ϵ determined by nuclear physics. So:

$$E_q = L_q t = \eta \frac{G M \dot{M}}{R} t = \eta \frac{GM}{R} \Delta m \tag{XIX.323}$$

is released during quiescence, and we can define:

$$\alpha \equiv \frac{E_q}{E_b} = \frac{\eta \frac{GM}{R} \Delta m}{\epsilon \Delta m c^2} = \frac{\eta}{\epsilon} \frac{GM}{Rc^2} \tag{XIX.324}$$

as the ratio of quiescent to burst energy. This should be determined by the properties of the NS and some physics. Typically we see $\alpha = 10 - 100$, since burning gives ~MeV per nucleon and accretion gives 100 MeV per nucleon (look at the energetics above).

Example: GS 1826−238. $t \approx 5.76$ hr. Persistent flux $F_x \approx 2 \times 10^{-9}\,{\rm erg/s/cm^2}$. As F_x (i.e., $\dot{M}$) increases, t decreases, with $t \propto \dot{M}^{-1}$. Gives $\alpha = 40$ for mixed H/He burning.

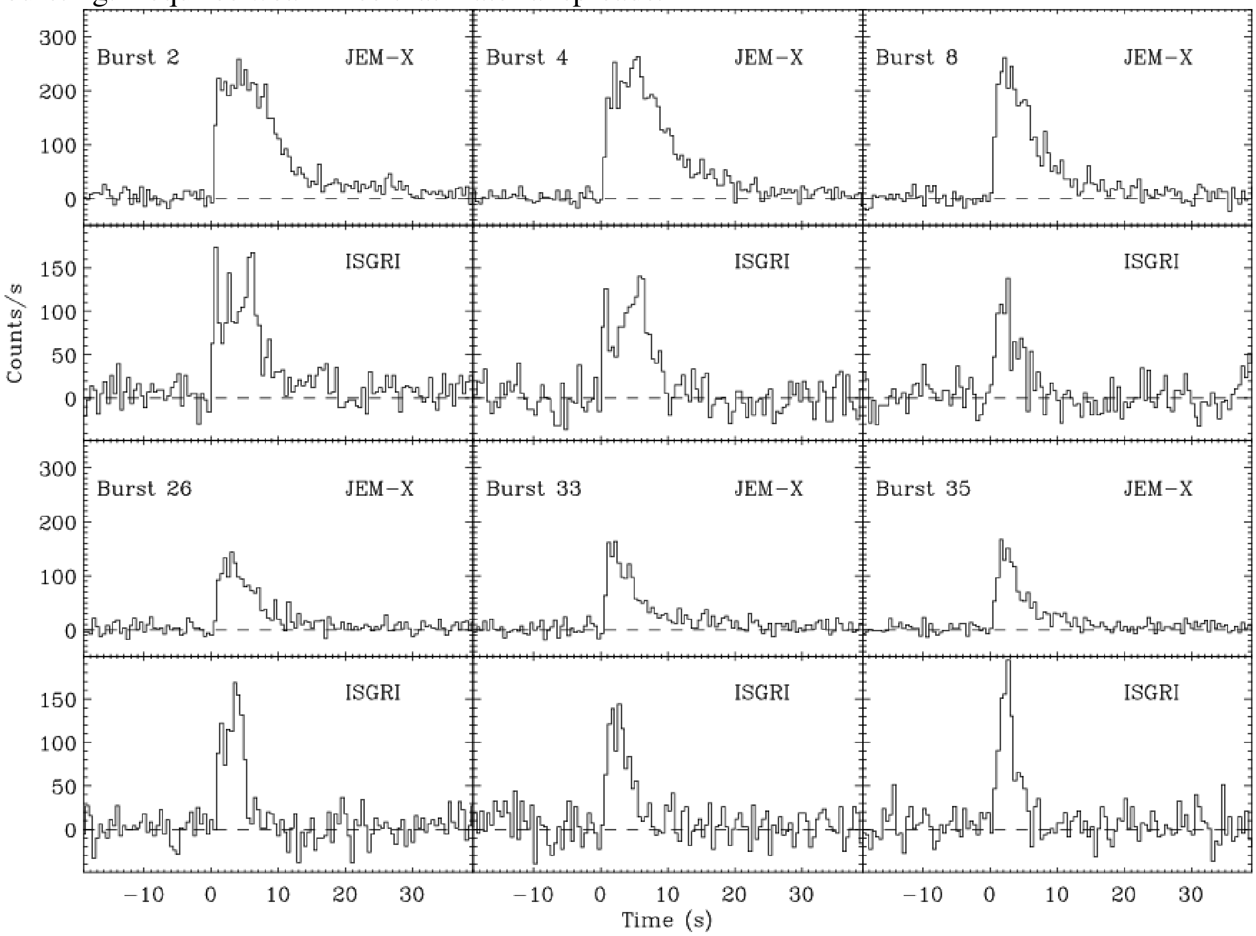
- X-ray pulsar: high B ($\sim 10^{12}$ G) channels accretion onto poles. Leads to pulses, equilibrium period of $\sim$ s
- LMXB: low B ($< 10^9$ G), so accretion is onto all of NS. Know of ~ 100 in MW, and ~ 15 in globular clusters

Complication: formation of NS comes from evolution of a massive star, which takes up a lot of space. Generally need something to bring NS progenitor in closer with companion. "Common envlope" will do this: two cores in one envelope, is very dissipative (brings them together) and ejects the envelope. Then later companion will evolve and make LMXB.

Periods similar to CVs: dominated by the size of the companion. Generally hours → minutes.

XIX.2 Type I X-ray Burst

Explosive burning related to the amount of accreted material. Rise is factor of ~ 10 in 1–10 s. Time between bursts is ~ 1000 times the duration of a burst, so quiescent energy output is $\sim 100\times$ bursting. Requires weak B so that material spreads.



Type I X-ray bursts from 4U 1728−34, from Falagna et al. (2006).

"The H/He fuel for type I bursts is accreted from the binary companion and accumulates on the surface of the neutron star, forming a layer several meters thick. The accreted material is compressed and heated hydrostatically, and if the temperature is sufficiently high, any hydrogen present burns steadily into helium via the 'hot' (β-limited) carbon-nitrogen-oxygen (CNO) process. After between $\sim$1 and several tens of hours, the temperature and density at the base of the layer become high enough that the fuel ignites, burning unstably and spreading rapidly to consume all the available fuel on the star in a matter of seconds" (Galloway et al. 2008). Burning/bursting behavior changes depending on $\dot{M}$. Instability is the same as for AGB stars — requires burning to be more temperature sensitive than cooling. He burning can be more unstable since it has no β reactions.

- Mixed hydrogen and helium burning triggered by thermally unstable hydrogen ignition for $\dot{M} < 2 \times 10^{-10}\, M_\odot\,\mathrm{yr}^{-1}$
- Pure helium shell ignition for next regime following completion of hydrogen burning.

- Mixed hydrogen and helium burning triggered by thermally unstable helium ignition for $\dot{M} > 4 \times 10^{-10}\, M_\odot\,\mathrm{yr}^{-1}$

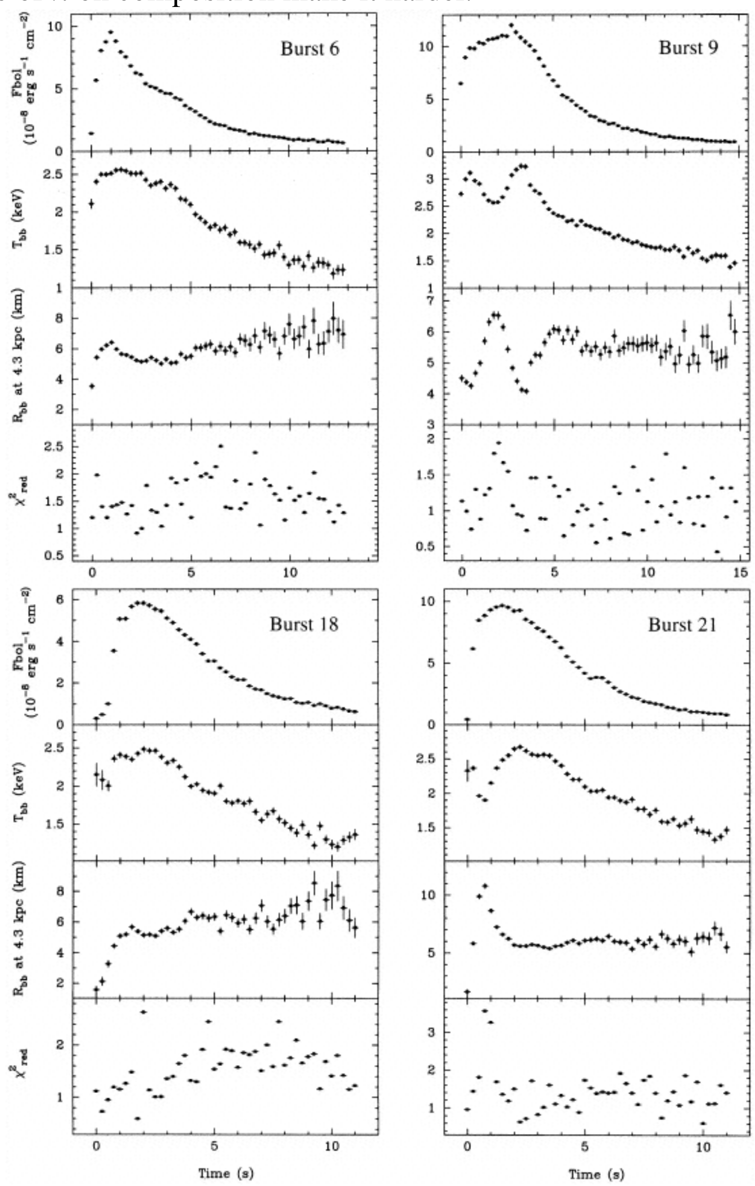
Some sources show oscillations at hundreds of Hz during burst: evidence for fast spin. But exact interpretation of oscillations is not clear. msec spin periods tie in to role as progenitors of MSPs.

XIX.2.1 Radius Expansion Bursts

Often from pure He, since can be fastest. Takes 5–10 s. Limited by:

$$L_{\mathrm{Edd}} = \frac{4\pi c G M}{\kappa}\left(1 - \frac{2GM}{Rc^2}\right)^{-1/2} = 4\pi R^2 \sigma T_{\mathrm{Eff}}^4 \qquad \text{(XIX.325)}$$

In principle this works like a standard candle, getting constraint on M (and M/R). In reality things like the dependence of κ on composition make it harder.



Radius expanion bursts from 4U 1728−34 (van Straaten et al. 2001).

Lecture XX High Mass X-ray Binaries

NS or BH in a binary with a high-mass star ($> 5\,M_\odot$). See donors that are Be star (B with emission lines): lines are from circumstellar material around the star (not the BH/NS). Also see HMXBs with supergiant donors.

Be/X-ray binaries: usually have wide eccentric orbit and disk around Be star. Accretion is tied to NS passing through disk (small duty cycle).

Supergiants typically have high mass-loss rates from winds ($\dot{M} \sim 10^{-6}\,M_\odot\,\mathrm{yr}^{-1}$), roughly spherical. Some of this gets captured by NS or BH and accretes, sort of like Bondi Hoyle. Remember that this rate is $> \dot{M}_{\rm Edd}$ for a $1\,M_\odot$ object, so not all can be accreted. Relative velocity of wind onto NS/BH is $\sqrt{v_{\rm orbit}^2 + v_{\rm wind}^2}$. Capture radius is $2GM/v^2$. With a circular orbit at a, we get:

$$L_X \approx \frac{\eta \dot{M}}{4}\left(\frac{2GM}{a}\right)^2 v_{\rm wind}^{-4} \qquad \text{(XX.326)}$$

Much less efficient than RLO, since we only capture a small amount of the material. But the mass-loss from the donor is so high (and the donor itself is very bright) that we can see many (~ 100) of these.

Lecture XXI Common Envelope Envolution

When two stars get close, they may orbit inside a single shared envelope that is made of the material of one or both stars. This is a short live phase that is believe to be critical for the formation of a number of systems including double neutron stars, type Ia progenitors, low mass x-ray binaries, etc. In essence the idea is that the ejection of the common envelope allows the formation of these tight binaries.

In Figure 21, we illustrate some examples where CEE is important.

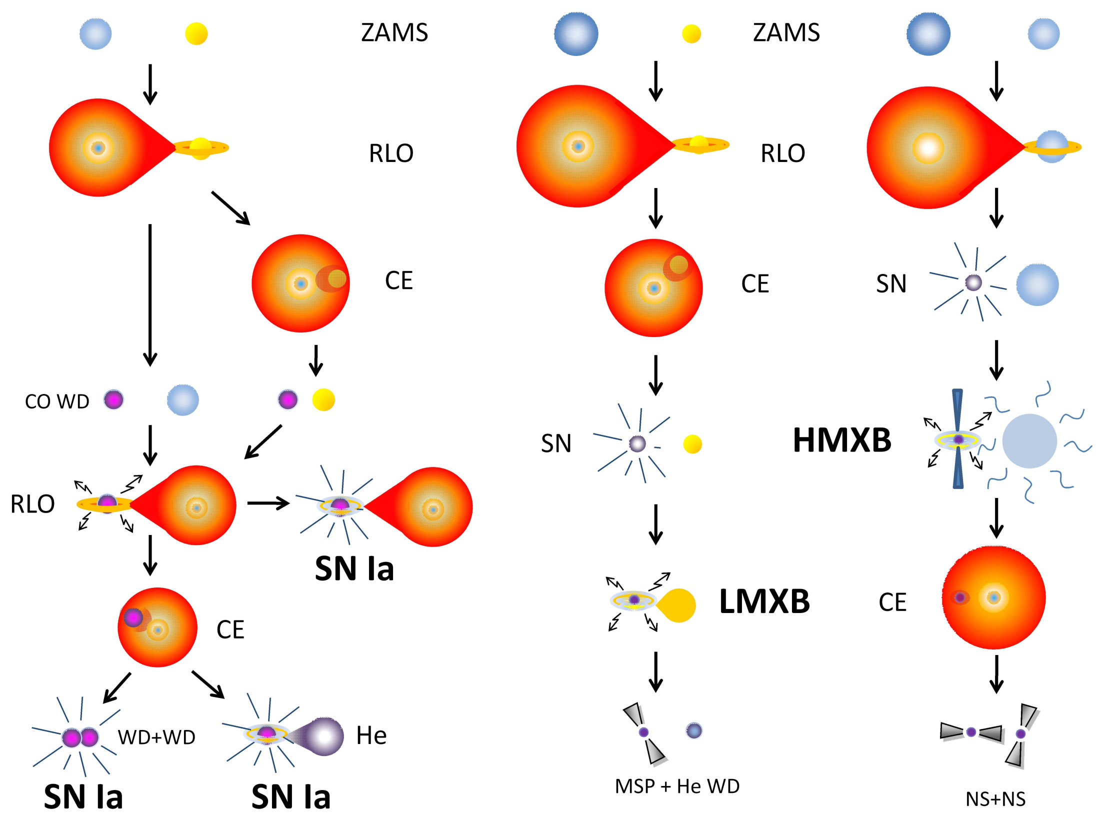
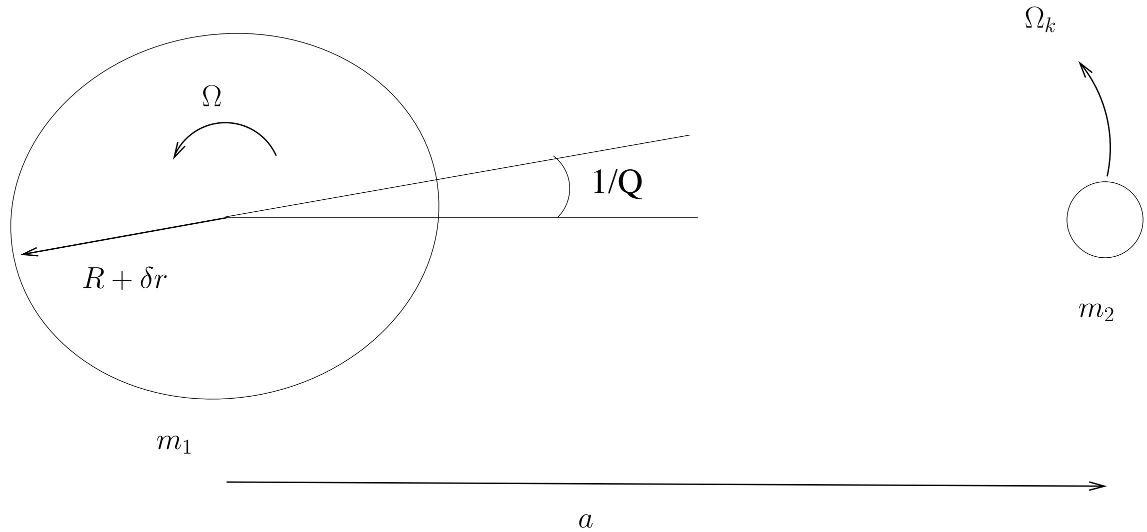


Figure 21

XXI.2 Initial Conditions

Let us begin with how CE is thought to occur. Suppose you have two stars $M_1 > M_2$ that are "close" to each other. We shall define "close" in a second. Because of how stars evolve, 1 will evolved up the red giant branch first increasing its radii by factors of up to 1000. As it evolves up the RG branch it will become tidally locked with orbital period of the system. Let see how this works.

To understand this let us come up with a simple picture of tides. Consider the picture below:



The equilibrium tide picture assumes that there is an angle between the tidal bulge and the phase of the orbit that is determined by the dissipation of the tide. This depends on the quality factor of the oscillation called the tidal Q, which is exactly analogous to the Q-factor in SHO. This angle is $1/Q$. Interestingly, if the oscillator is perfect, the phase angle is 90 degrees – exactly as it is in a driven SHO. The tide raises a bulge on the surface of the star or planet that we can estimate from the modification of the equipotential surface.

$$\frac{\delta r}{R} = \frac{\delta \Phi}{\Phi} = \frac{F_{\rm tid} R}{G M_1^2/R} = \frac{G M_2 M_1 R^2/a^3}{G M_1^2/R} = q\left(\frac{R}{a}\right)^3 \tag{XXI.327}$$

The tidal torque is then

$$\tau = F_{\rm tid}\delta r = M_1 Q^{-1}\frac{G M_2 R^2}{a^3} q\left(\frac{R}{a}\right)^3 \tag{XXI.328}$$

We can then set the tidal torque to be spin angular momentum of star 1

$$I\frac{d\Omega}{dt} = I\Omega t_{\rm tid}^{-1} \sim M_1 R^2 \Omega t_{\rm tid}^{-1} = \tau = M_1 \frac{G M_2 R^2}{a^3}\frac{q}{Q}\left(\frac{R}{a}\right)^3. \tag{XXI.329}$$

Solving for $t_{\rm tid}$, we find

$$t_{\rm tid} = \Omega_k^{-1}\frac{\Omega_k}{\Omega} Q q^{-2}\left(\frac{a}{R}\right)^3 \tag{XXI.330}$$

As star 1 goes up the RG branch, R increases up the to Roche radius $R_L \approx (M_2/M_1)^{1/3}a$. But well before it reaches the Roche radius, $t_{\rm tid}$ decreases such that eventually, it is low enough to ensure the tidal synchronization time is sufficient short. Hence, even if stars are not in initial corotation, the star that goes up the RG branch will enter co-rotation.

XXI.3 Stages of CEE

Simple 1-d models CEE evolution can describe the system as in Figure 22 These phases are:

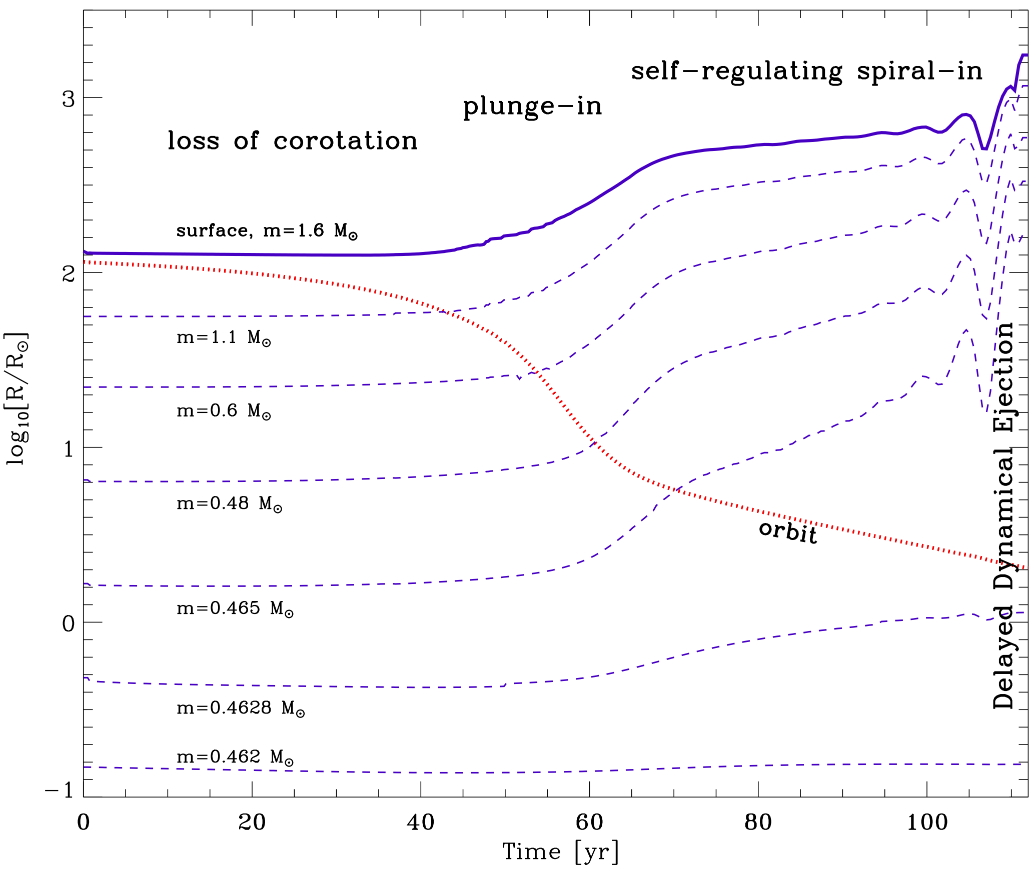

self−regulating spiral−in
plunge−in
loss of corotation
surface, m=1.6 M⊙
m=1.1 M⊙
m=0.6 M⊙
m=0.48 M⊙
m=0.465 M⊙
m=0.4628 M⊙
m=0.462 M⊙
orbit
Delayed Dynamical Ejection
log10[R/R⊙]
Time [yr]
3
2
1
0
−1
0
20
40
60
80
100

Figure 22

1. Loss of corotation - initially it is assumed the system, which are close to each other is tidally locked. But during the spiral-in, a loss of corotation will occur if the spiral-in process occurs faster than restoration by tidal damping.
2. Plunge-in and termination - one star plunges into the other star and rapidly deposits orbital energy in the envelope which is ejected. This is a rapid dynamical phase and is easily studied numerically.
3. Self-regulating spiral in - The envelope may expand to such an extent that the plunge is slowed down. At this point, the envelope is huge and low density, and so to allow for further spiral in, the envelope must cool to allow further deposition of heat by the spiraling-in star. This occurs on the thermal time.
4. End of self-regulating phase.
5. End CE - Just because the envelope is gone does not mean the end of CE is over. Remaining material can still affect the remnants.

Play movie

XXI.4 Models of CEE

At its core the evolution of a system during CEE is likely governed by

- Conservation of Energy
- Conservation of Angular Momentum

Here we will make the assumption that the orbital energy goes into unbinding a envelope with no loss and likewise for the orbital angular momentum. This is insufficient because one must know how the energy is partitioned, but let's ignore this complication for now.

It turns out that models of CEE is surprisingly crude. There are two methods of understanding the problem, either we model the energy or we model the angular momentum.

XXI.4.1 Energy Formalism

Consider a giant star of mass m_1 that enters CEE via a second compact star m_2. At the end of CEE, 1 has sheds its envelope which had initial binding energy $E_{\rm bind}$. If all this energy came from the orbit

$$E_{\rm bind} = -\frac{Gm_1m_2}{2a_i} + \frac{Gm_{1,c}m_2}{2a_f}, \tag{XXI.331}$$

where $m_{1,c} = m_1 - m_{\rm env}$ is the mass 1's remaining core. Now we need $E_{\rm bind}$, but the equation above assumes perfect efficiency, which is likely not the case. To hide our ignorance, we will multiply the RHS by $\alpha_{\rm CE} \leq 1$:

$$E_{\rm bind} = \alpha_{\rm CE}\left(-\frac{Gm_1m_2}{2a_i} + \frac{Gm_{1,c}m_2}{2a_f}\right), \tag{XXI.332}$$

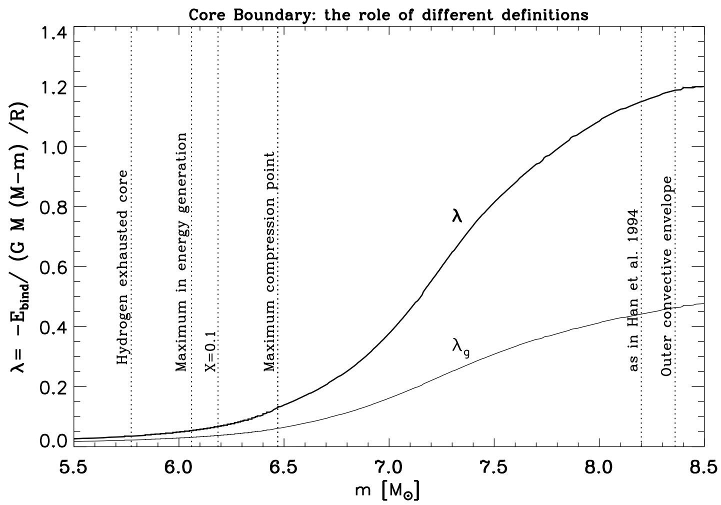


Figure 23

Now the binding energy can be estimated by $E_{\rm bind} \sim Gm_1 m_{1,\rm env}/R_1$. But this envelope is distributed in some way. So to parameterized our ignorance again, we introduce λ, which depends on the structure of the envelope.

$$E_{\rm bind} = \frac{Gm_1 m_{1,\rm env}}{\lambda R_1} = \alpha_{\rm CE}\left(-\frac{Gm_1 m_2}{2a_i} + \frac{Gm_{1,c}m_2}{2a_f}\right), \tag{XXI.333}$$

This can be solved for a_f in terms of the other variables as:

$$\frac{a_f}{a_i} = \frac{m_{1,c}}{m_1}\left[1 + \frac{2a_i}{\alpha_{\rm CE}\lambda R_1}\frac{m_2}{m_{1,\rm env}}\right], \tag{XXI.334}$$

Usually, the 1 star has to fills it Roche lobe, which is

$$\frac{R_{1,L}}{a_i} = 0.49q^{1/3} \tag{XXI.335}$$

where $q = m_1/m_2$ is the mass ratio.

We should be careful now as the energy budget is not completely accounted for – we only include orbital energy. There is also the issue of where the core ends and the envelope begins. There is some justification for our approximations. First, it is thought that CE evolution occurs more rapidly than the thermal time – so we can assume that the material is adiabatic. Second the material is much lower density than the in-falling star, so this may limit the amount of accretion and hence accretion energy. In addition, there is large contrast between the envelope and the core both in terms of density and composition – so there is some clue as to how to define the envelope.

The condition that star 1 is Roche filling demanding loss of orbital energy puts a surprising constraint on CE evolution. In particular we can also demand that the angular momentum of the system be conserved. Hence the total angular momentum of a binary is

$$J = \mu\sqrt{Gm_{\rm tot}a(1-e^2)}, \tag{XXI.336}$$

where $\mu = m_1 m_2/m_{\rm tot}$ is the reduced mass, $m_{\rm tot} = m_1 + m_2$ is the total mass, and e is the eccentricity. The orbital energy is $E_{\rm orb} = -Gm_1m_2/2a$ and thus we can show:

$$\frac{J_f}{J_i} = \left(\frac{m_{1,c}}{m_1}\right)^{3/2}\sqrt{\frac{(m_1+m_2)(1-e_f^2)}{(m_{1,c}+m_2)(1-e_i^2)}}\sqrt{\frac{E_{\rm orb,i}}{E_{\rm orb,f}}}, \tag{XXI.337}$$

Typically, we expect $e_f \sim e_i \approx 0$ and $m_{1,c} < m_1$, i.e., envelope mass loss, this means that for any $|E_f| > |E_i|$ demands that $J_f < J_i$. This means that energy loss requires a loss of angular momentum and hence the orbital separation will sink. This is obvious in hindsight. It turns out that the converse statement is not true.

We can do a example of CE evolution as follows: Consider the system 0135-052 which is a double WD system. The period of this system is $P = 1.5$ days and $m_1 = 0.47$ solar masses while the second WD is $m_2 = 0.52$ solar masses. Kepler's law tells us that $a = 4 \times 10^{11}$ cm. Let's begin with a story of how this system evolved. The larger white dwarf evolved up the red giant branch first and made m_2 during this phase some degree of CE likely took placed, but we will ignore this fact for now. This leaves the systems with a white dwarf (m_2) and a star $m_i > m_2$. Now the second star evolved up the giant branch and m_i and m_2 enter second CE phase which leaves the system as it appears today. It is the second CE phase that we wish to consider now.

As discussed above the CE phase begins when the giant fills its Roche lobe and begins mass transfer to the lower mass companion. As discussed, this situation is unstable and rapidly leads to a spiral in of the white dwarf which expells the envelope. It turns out that we know quite few things about the progenitor of m_1. In particular, we know its core mass (assuming that the white dwarf is the core) and hence its separation when it filled the Roche lobe.

The hence part is due to a special property of red giants – that is their radii is completely determined by the core mass. To understand why, you need to know a few things. First the surface temperature is constant for all red giants – determined by H^- opacity, which is super temperature sensitive and their luminosity is determined by CNO burning.

For the first part, $H-$ opacity is temperature sensitive – if the temperature it too low, it becomes transparent – cooling is much faster and radiates from deeper in the star and the envelope constracts and heats up. If the temperature is too high, cooling is slowed, the envelope expands and cools off. For the second part, the important thing to keep in mind it that the nuclear energy production rate is extremely temperature sensitive – $\epsilon \sim T^{15}$.

A red giant is structured as follows: You have a degenerate core and a diffuse envelope with shell burning occuring in the narrow region between the core and the envelope. Although the core is degenerate, we will ignore it variation with core mass. The burning shell's temperature is roughly the local virial temperature

$$kT \sim \frac{GM_c}{R_c} \sim M_c \tag{XXI.338}$$

Since the energy generation rate is $\epsilon \sim T^{15}$, we have for the luminosity

$$L \sim \epsilon \sim M_c^{15} \tag{XXI.339}$$

So the luminosity is extremely temperature sensitive. For fixed effective temperature, we find

$$R \sim M_c^{15/2}. \tag{XXI.340}$$

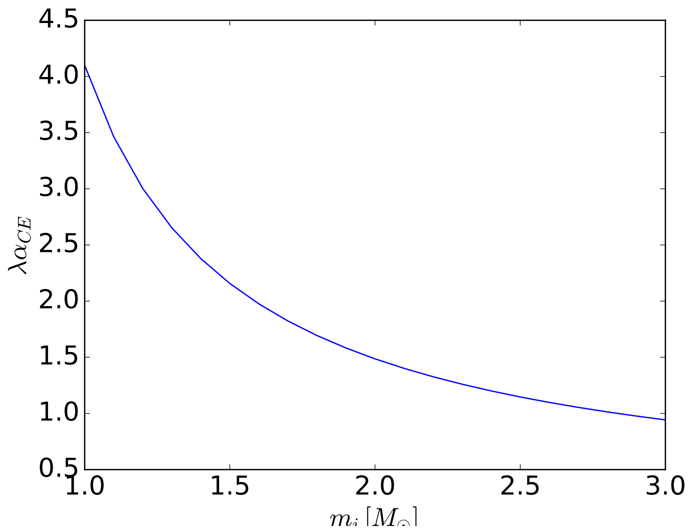


Figure 24

Numerical calculation bear this out though not with this strong power. A good estimate is

$$R_g = 3000 \left(\frac{M_c}{1M_\odot}\right)^4 R_\odot \tag{XXI.341}$$

So given what we have above, we can estimate $a_i = 2R_g$, i.e., setting it equal to the Roche radius and ignoring the weak dependence on q. For $m_1 = 0.47$, we have

$$a_i = 6000 \left(\frac{m_1}{1M_\odot}\right)^4 R_\odot \approx 2 \times 10^{13}\,\mathrm{cm} \tag{XXI.342}$$

Now solving equation (XXI.334) for $\lambda\alpha_{\rm CE}$, we find

$$\lambda\alpha_{\rm CE} = \frac{4}{m_1/m_i - a/a_i}\frac{m_1 m_2}{(m_i - m_1)m_i} \tag{XXI.343}$$

A plot of this as a function of m_i is shown in Figure 24. Typical values of $\lambda \approx 0.5$, so this means that $\alpha \sim 2 - 8$. This should bother you at some level as it implies that it takes more orbital energy that is available to unbind the emvelope. Some of this can be alleviated by considering other sources of energy (thermal, etc) as shown earlier – but in any case it appears the the process of unbinding an envelope demands very high efficiency.

XXI.4.2 The Angular Momentum Formalism

Using angular momentum as opposed to energy might avoid some the issues that we outlined above (and create others). Toward that end, let us consider one way of using angular momentum, which is known as the γ formalism:

$$\frac{J_i - J_f}{J_i} = \gamma \frac{m_{1,\rm env}}{m_1 + m_2}, \tag{XXI.344}$$

where J_i and J_f are the initial and final orbital angular momentums of the system, where γ is a constant that parameterizes the amount of angular momentum loss relative to the mass loss. Using J as defined above, we can find the ratios of the final and initial orbital separations:

$$\frac{a_f}{a_i} = \frac{m_1^2}{m_{1,c}^2}\frac{m_{1,c}+m_2}{m_1+m_2}\left(1-\gamma\frac{m_1-m_{1,c}}{m_1+m_2}\right)^2 \tag{XXI.345}$$

We can place some constraints on what γ might be. Typically angular momentum is loss at the same rate as mass. It is difficult to lose mass at a greater rate than angular momentum so this means that $\gamma > 1$. On the other hand, we can assume that $J_f = 0$ – things don't get more efficient than this. This gives:

$$1 = \gamma\frac{1-m_{1,c}/m_1}{1+q} \rightarrow \gamma = \frac{1+q}{1-m_{1,c}/m_1}. \tag{XXI.346}$$

For some typically numbers like $m_{1,c} = m_2 = 0.25m_1$, this gives $\gamma \approx 1.7$ so the constraint on $1 < \gamma < 1.7$. In fact, one can use this method to determine CEE as we did for the energy formalism and in these cases $\gamma \sim 1.5 - 1.7$.

XXI.5 Full 3-D models

Increasingly, a number of groups are starting to use full 3-d models to simulation CEE. There are two method to compute the fluid equations. Recall

$$\frac{\partial\rho}{\partial t} + v_x\frac{\partial\rho}{\partial x} + \rho\frac{\partial v_x}{\partial x} = \frac{d\rho}{dt} + \rho\frac{\partial v_x}{\partial x} = 0 \tag{XXI.347}$$

$$\frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + \frac{1}{\rho}\frac{\partial P}{\partial x} - \frac{\partial\Phi}{\partial x} = \frac{dv_x}{dt} + \frac{1}{\rho}\frac{\partial P}{\partial x} - \frac{\partial\Phi}{\partial x} = 0, \tag{XXI.348}$$

where $d/dt = \partial/\partial t + (dx/dt)\partial/\partial x = \partial/\partial t + v_x\partial/\partial x$. The difference between the left hand side and right hand side is that on the left, we write $f(t,x)$ for the left and $f(t,x(t))$ for the right. The left hand side is call the Eulerian picture and envision a mesh on which we plut fluid quantities and stuff flows in and out of it. The right hand side is called the Lagrangian picture and here each fluid quantity moves around with the fluid and is affected by its neighbors as it moves around. As a result, the right hand side for the momentum equation is exactly the definition of Newton's second law $F = ma$.

In any case the Eulerian picture is computed with a grid code while the Lagrangian picture is computed with what is called a smooth particle hydrodynamics code or SPH for short. A number of groups have done some computation on these systems in last five years. In general these groups have found that the efficiency of envelope ejections is far less efficient that is required by simple models of CEE evolution and only a small fraction of the envelope (typical $0 - 25\%$) is ejected. This strongly suggests that additional physics is required – suggestive of an additional energy source.

XXI.6 Link to Observations

Up until recently, CEE was not observed, but this changed recently. The strongest candidate for a CEE event is V1309 Sco, which outburst in 2008 where it got $> 100x$ brighter. At the same time, it got cooler – which given the physics of radiation $L = \sigma T_e^4 R^2$, implies that the object got substantially bigger. In addition, it was observed to be a contact binary with a period of 1.4 days and for six year prior to 2008, the period decreased at a steady rate of 1.2% a year.

The outburst on V1309 Sco is also similar to a observed class of transients known as "red novae" that was discovered by PTF, which get substantially brighter, but remain cool. These also show expansion velocities of $100 - 1000$ km/s, which is broadly consistent with a stellar merger (and the low end is consistent with CEE). But these are early days where detailed models for these systems have not yet been developed.

Lecture XXII Black Hole XRBs

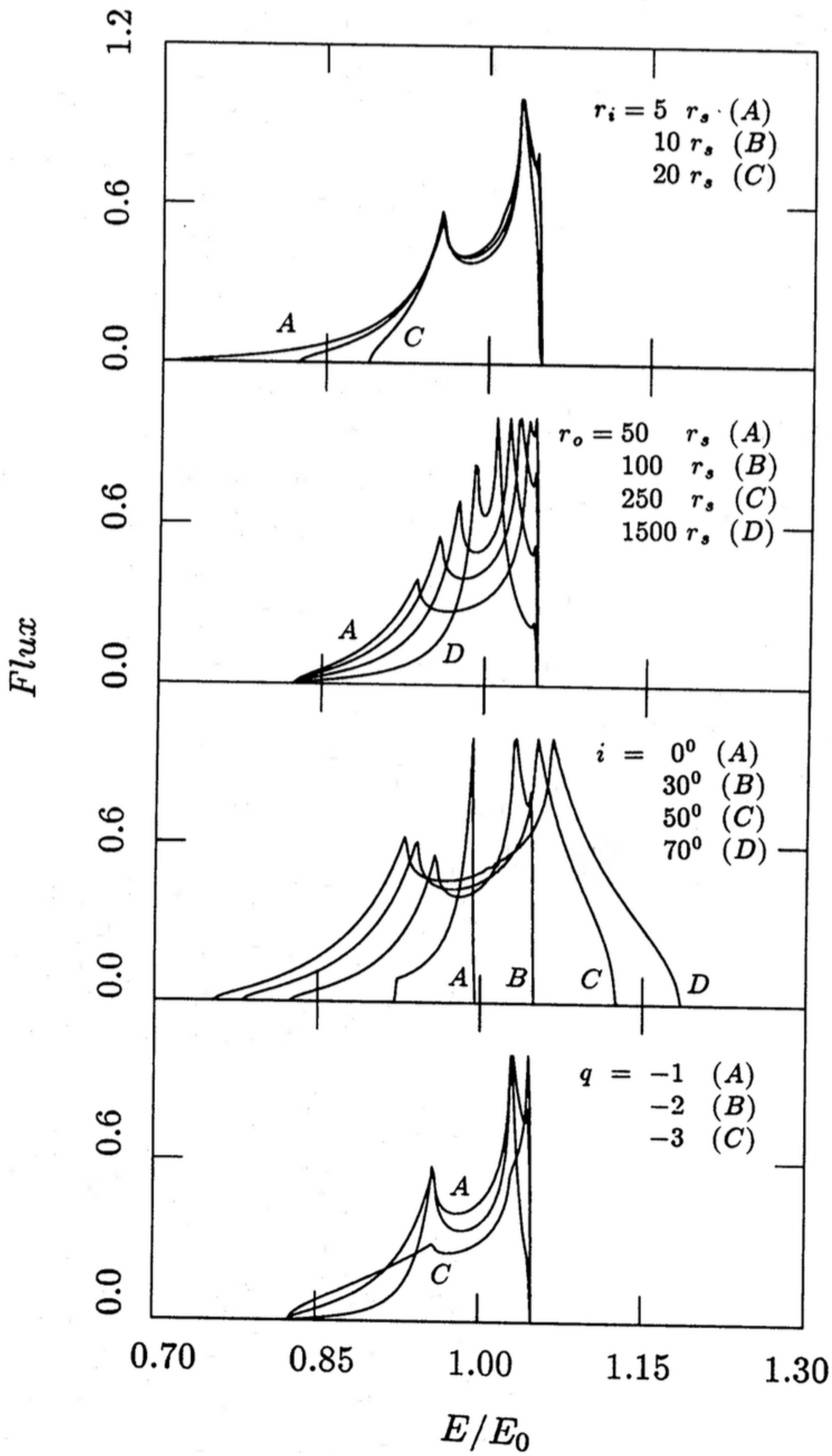
XXII.2 BH Binaries

Often still LMXB. We know that there is a BH because the orbit tells us (making assumptions about the inclination, etc.). If $M_{2,min} > 3\,M_\odot$ or more, then we are in BH range. We know of > 20 such systems.

No surface, so no fusion. Only the L from the disk, with the inner edge possible at the ISCO.

Measure BH mass and spin additionally through accreting systems. Fit for disk $T(r)$, and try to get maximum T at minimum r. If we can relate this to ISCO can determine M and L for BH. Find spins can be quite large, up to $L \sim M$ in appropriate units.

Another method uses the profile of Fe Kα line. This is an atomic transition at ~ 6 keV from inner-shell electrons. Like in CVs with H emission, we see emission from Fe in a disk that is broad/double peaked from motion. Importantly, motion is relativistic so have to deal with beaming, gravitational redshift, and frame dragging. These effects distort the line shape, giving information about the BH mass, spin, inclination (Fabian et al. 1989).

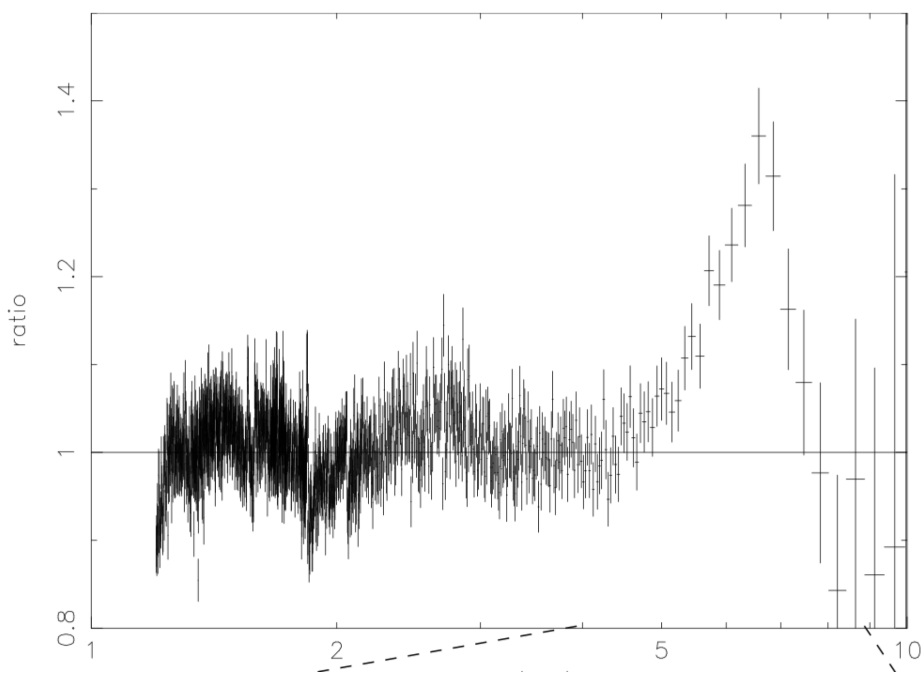


Fe Kα predictions from Fabian et al. (1989). The models change inner radius r_i and outer radius r_o, both in units of the Schwarzschild radius R_S. They also change the inclination i and radial dependence of the emissivity, $\propto r^q$. "When not specified the other parameters are fixed at: $r_i = 10R_S$, $r_o = 100R_S$ $i = 30°$ and $q = -2$. Note that the blue horn is always brighter than the red one and a net redshift only occurs for low inclinations." Some aspects of the shape are easy to understand:

- Fractional Doppler width $z_d \sim \sqrt{R_S/2r}\sin i$
- Gravitational redshift $z_g =\sim R_S/2r$
- Blue horn is higher from aberration, time dilation, and blueshift

Again, often find high spins. Can also apply this to supermassive black holes.

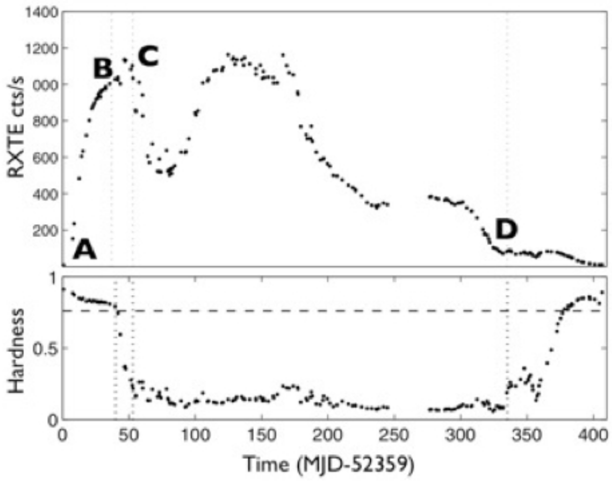
Neither approach is perfect

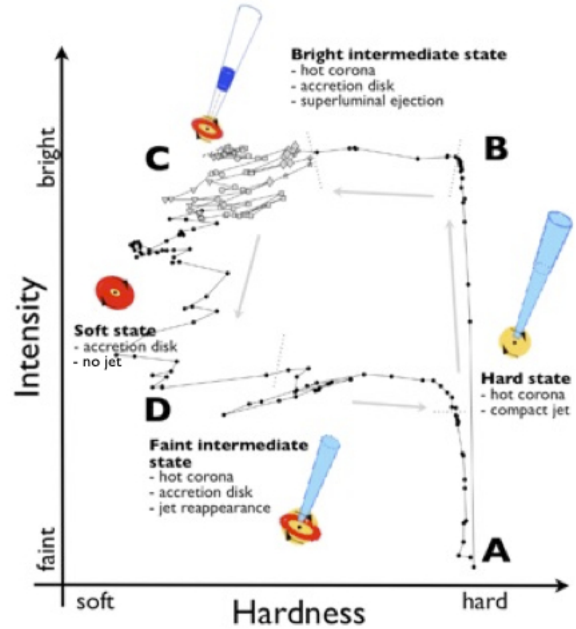


Fe Kα line from the binary GX 339$-$ observed with *Chandra* (Miller et al. 2004).

Most show significant variability (changes in $\dot{M}$). Lots of different states:

- Thermal: disk is $> 75\%$ of total emission, little or no QPOs (quasi-periodic oscillations)
- Hard: disk is $< 20\%$, X-rays have "hard" power-law
- Soft: low disk, X-rays are "softer", source is much brighter





Data on the binary GX 339−4 from *RXTE* from Belloni et al. (2005) (left), along with a hardness-intensity tiagram from `http://ipag.osug.fr/ANR-CHAOS/project.html`. In this diagram "Soft state" corresponds to our "Thermal state," while "Bright intermediate state" corresponds to our "Soft high state."

XXII.3 Thermal State

Emission dominated by $T(r)$ of optically-thick disk. A single source can transition around a range of temperatures/luminosities, but generally follows $L \sim T^4$.

XXII.4 Hard State

See radio jets when system changes to hard state. Jets emit synchrotron radiation.

XXII.5 Soft High State

Dominated by comptonization of radiation above the disk?

XXII.6 QPOs

Most QPOs have frequencies $0.1 - 30\,\mathrm{Hz}$. Likely some mode in the disk interfering with rotation of compact object. High frequency ($\sim 500\,\mathrm{Hz}$) QPOs seen in a few sources. Possibly related to (harmonic of) orbit at ISCO:

$$\nu_{\mathrm{ISCO}} = 2.7 \times 10^3 \left(\frac{M}{M_\odot}\right)^{-1} \mathrm{Hz} \tag{XXII.349}$$

so if fundamental, imply $M = 5 - 15\,M_{\odot}$.

XXII.7 Microquasar

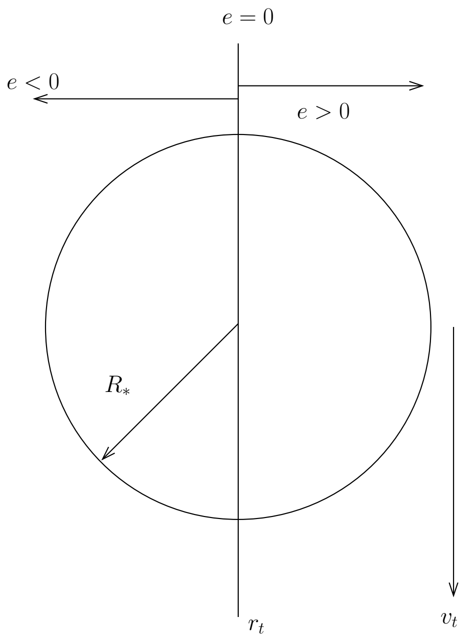
A stellar analog of supermassive BH with relativistic jets. See apparent superluminal motion of the jets: in reality this is just from Doppler boosting.

Lecture XXIII Tidal Disruption Events

Supermassive black holes in the range between $10^5 - 10^9\, M_\odot$ are believed to exist in the centers of all galaxies. While their formation and evolution are still topics of debate, their presence is not. If a star comes too close to a the supermassive black hole, it may be disrupted. The tidal disruption radius is Hills radius or

$$r_t \lesssim r_{t,0} = \left(\frac{M_{\rm BH}}{M_*}\right)^{1/3} R_* = 100 \left(\frac{M_{\rm BH}/M_*}{10^6}\right)^{1/3} R_* \qquad \text{(XXIII.350)}$$

Now suppose that this star comes in on a highly eccentric orbit. This is not unexpected as many stars are in near radial orbits in the centers of galaxies. In such a case, the total specific energy of the orbit is nearly $e = 0$, i.e., nearly unbound. At the moment of disruption, we can assume that it is moving at some fixed velocity $v_t \approx \sqrt{2GM_{\rm BH}/r_t}$. Now if we assume that the entire star is going to move ballistically, we can now calculate how the star disrupts.



First, let's calculate the change in energy as we move across the star. The minimum energy is

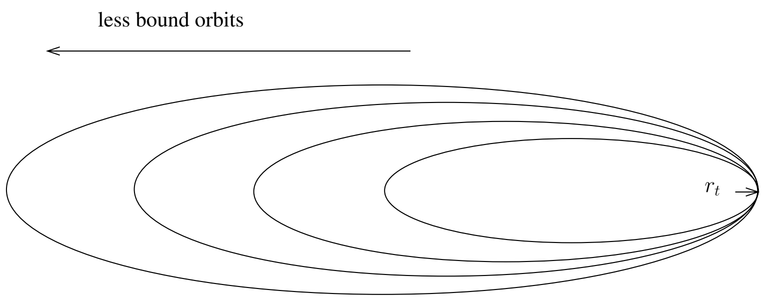
$$e_{\rm min} = \frac{1}{2}v_t^2 - \frac{GM_{\rm BH}}{r_t - R_*} \approx -\frac{GM_{\rm BH}R_*}{r_t^2}. \qquad \text{(XXIII.351)}$$

Likewise the maximum energy is

$$e_{\rm max} = \frac{1}{2}v_t^2 - \frac{GM_{\rm BH}}{r_t + R_*} \approx +\frac{GM_{\rm BH}R_*}{r_t^2}. \qquad \text{(XXIII.352)}$$

This leads to our first result. Half the star is bound. Half the star is unbound. The energy of the bound part of the star goes from $e = e_{\rm min}$ to $e = 0$ and moves out into ballistic orbits. For Keplerian orbits, the period is entirely determined by their energy so we can determine

$$-\frac{GM_{\rm BH}}{2a} = e \quad \text{and} \quad P = 2\pi\sqrt{\frac{a^3}{GM_{\rm BH}}} \rightarrow P = 2\pi GM_{\rm BH}\sqrt{\frac{1}{8|e|^3}} \qquad \text{(XXIII.353)}$$



For $e = e_{\rm min}$, we have

$$P_{\rm min} = 2\pi G M_{\rm BH}\sqrt{\frac{1}{8e_{\rm min}^3}} = \frac{\pi}{\sqrt{2}}\sqrt{\frac{r_t^3}{GM_{\rm BH}}}\left(\frac{r_t}{R_*}\right)^{3/2} \quad \text{(XXIII.354)}$$

$$\approx 41\left(\frac{r_t}{r_{t,0}}\right)^3\left(\frac{R_*}{1R_\odot}\right)^{3/2}\left(\frac{M_*}{1\,M_\odot}\right)^{-1/2}\left(\frac{M_{\rm BH}/M_*}{10^6}\right)^{1/2}\ \text{days} \quad \text{(XXIII.355)}$$

So after disruption, this is the time it takes for the first material from the disrupted star to return to pericenter at $r = r_t$. Other material takes longer. We can estimate the rate at which material returns as

$$\dot{M} = \frac{dM}{dt} = \frac{dM}{de}\frac{de}{dt} \quad \text{(XXIII.356)}$$

Now, the time it takes to return is P so we can identify $t = P$ and use the result of equation (XXIII.353), which gives

$$|e| \propto t^{-2/3} \quad \text{(XXIII.357)}$$

In addition, we will make an assumption that the mass gets equally divided up between energies, i.e., $dM/de =$ constant. This assumption is ab initio, but later numerical simulations suggest that is it ok. In any case we find

$$\dot{M} = \frac{dM}{de}\frac{de}{dt} = \frac{M_*}{t_{\rm min}}\left(\frac{t}{t_{\rm min}}\right)^{-5/3} \quad \text{(XXIII.358)}$$

$$\approx 9\left(\frac{r_t}{r_{t,0}}\right)^2\left(\frac{R_*}{1R_\odot}\right)\left(\frac{M_*}{1\,M_\odot}\right)^{-1/3}\left(\frac{M_{\rm BH}/M_*}{10^6}\right)^{1/3}\left(\frac{t}{41\ \text{days}}\right)^{-5/3} M_\odot\,\text{yr}^{-1} \quad \text{(XXIII.359)}$$

As this material falls back, it is expect that it results in rapid circularization and formation of a disk at $a = 2r_t$ (Can you show why?) This disk will accrete onto the central SMBH at initially super-Eddington rates, falling back to sub-Eddington rates after some time. Super-Eddington is interesting because several things can happen. First, it material may not effectively cool, so that you might have a RIAF situation. This can lead to jets or outflows.

In any case the tidal disruption of a star can lead to a number of areas from which emission is produced an possibly observed. These are shown in Figure 25 and include:

- accretion disk – this can produce jets in the super-Eddington phase
- reprocessing of radiation from accretion

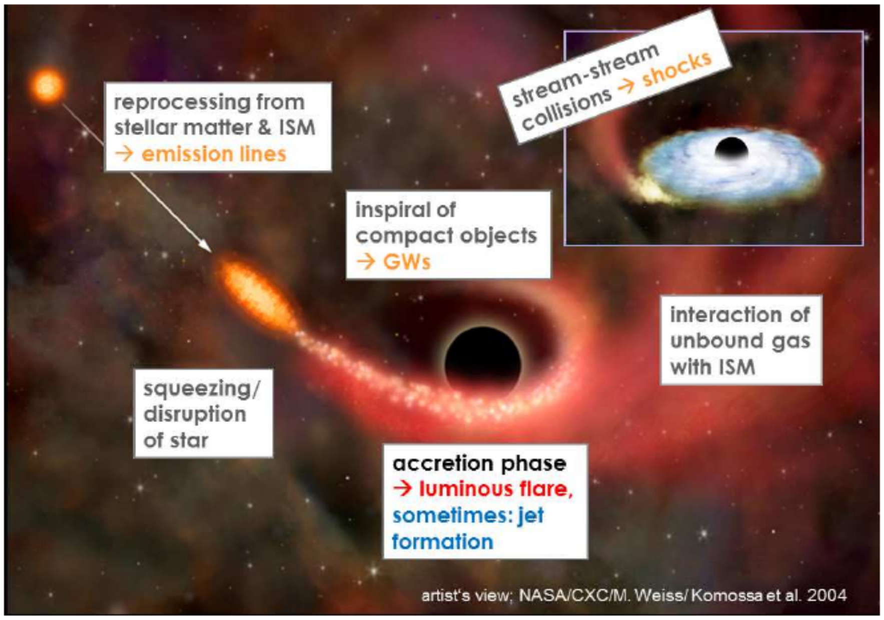


Figure 25: Regions of TDE emission

- shocks on the disk from returning material and on the surrounding ISM.

In any case, the radiation that might be seen include: x-rays, optical, and radio – i.e., long term jets and shock evolution. ROSAT first detected a bunch of TDE's with luminosities that approach that is required in a TDE, albeit a bit lower at 10^{43} ergs/s. A summary is shown in Figure 26 where the peak was shifted to $t = 0$. Note that in the x-rays the emission follows the $t^{-5/3}$ law as expected from our OOM calculations.

In addition to the direct x-ray detection, optical and UV detection is also possible. Here transient finders have found a number of TDEs that have been followed up with SWIFT and larger optical telescopes. Here, the combination of UV and optical observation constrains the temperature of the photosphere to be about $1 - 3 \times 10^4$ K, much lower than the 10^5 K that is observed in x-ray bright sources. Coincidentally, these sources usually don't show strong x-ray emission. In addition, it is strange that the photosphere are all at constant effective temperature and it appears that the blackbody radius is changing. The optical TDEs also seem to have higher peak luminosities of $\approx 10^{44}$ ergs/s.

Finally, perhaps the most exciting observation is the discovery of jetted TDEs, where the TDE leads to a formation of a jet. The first such source as SwiftJ1644+57, which looked like a GRB at first, but the emission took too long to fade. The emission was about $L_x \approx 10^{44} - 10^{45}$ erg/s with flares that reached 10^{48} ergs/s. This source was also associated with strong radio emission – also suggestive of a jet. Interestingly, these jets source appear to be the minority of TDE events as deep radio follow up of other TDE candidates found no radio emission.

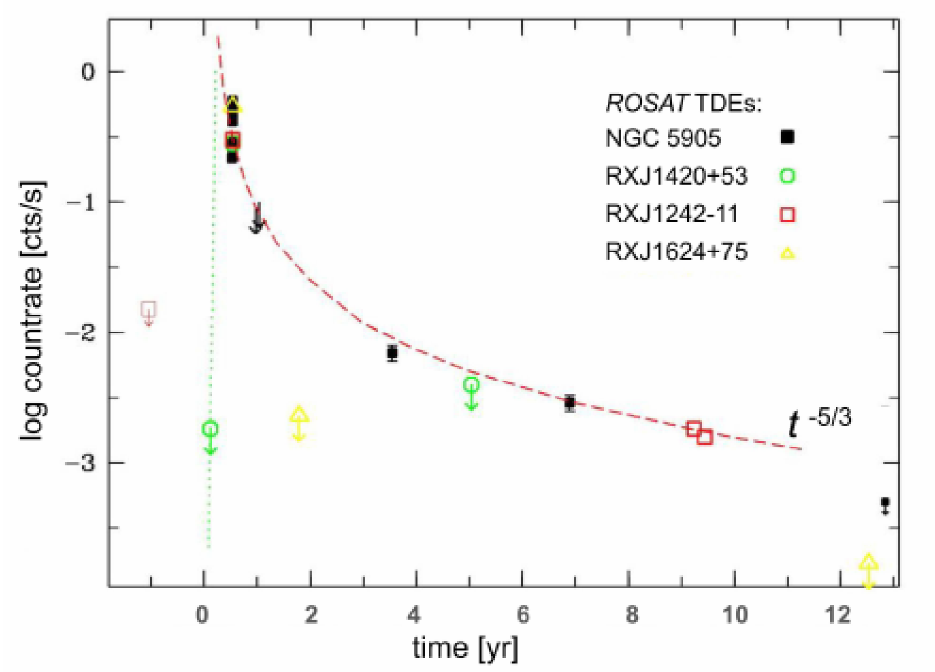


Figure 26: Candidate TDE in the x-rays

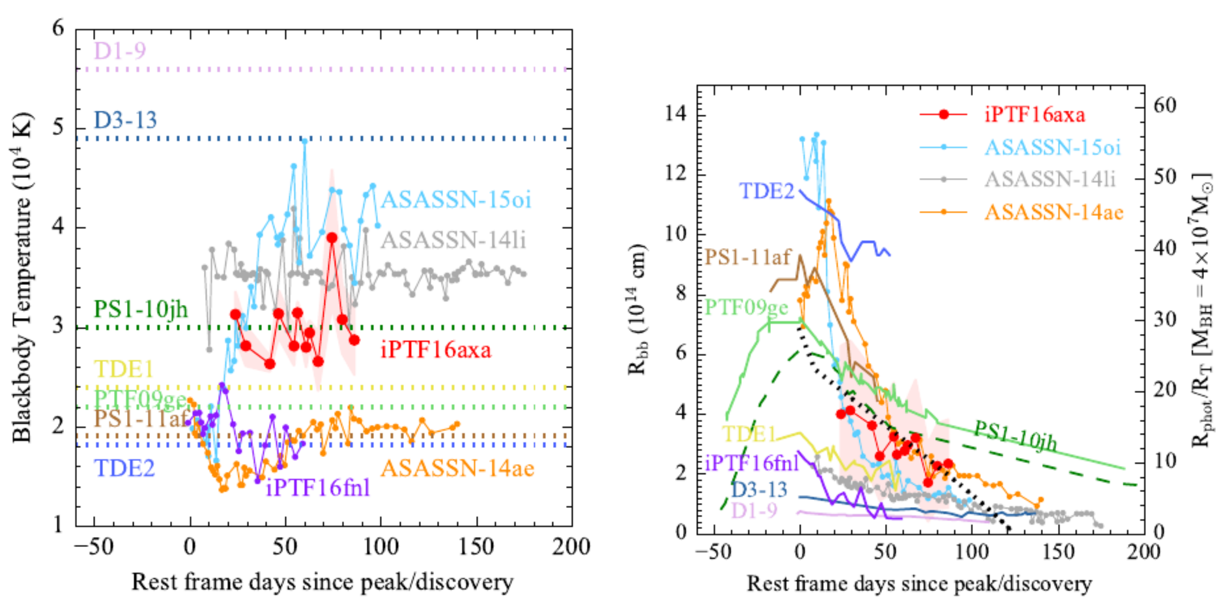


Figure 27: TDE found in the optical

Lecture XXIV Pulsars

Neutron stars were predicted in 1934 (neutrons discovered in 1932) and thought to be associated with SNe, but not seen for $> 30\,\text{yr}$.

Jocelyn Bell was looking at radio emission over time. Studying scintillation, or flickering, which is irregular changes. Saw that there were little blips that appeared regularly, every 1.337 s. This came from the same part of the sky, which rose at a different time every night (*not* from Earth). These are *pulsars*.

Nature of pulsars was deduced by Tommy Gold. Based on facts:

- Periods of ~ 1 s common, although the Crab pulsar has $P = 33\,\text{ms}$
- Periods are very stable, change by $\dot{P} \sim 10^{-15}\,\text{s/s}$

Models:

Binary Star Would need P from orbital period. Can use Kepler's third law to show that for $M = 1\,M_\odot$, $a = 1.6 \times 10^8$ cm. Compare to $R = 7 \times 10^{10}$ cm for the Sun, or 5×10^8 cm for Sirius B.

Could be orbiting NSs. However, GR says that orbit should gradually grow closer, which period is observed to grow longer.

Pulsating Star WDs oscillate with $P = 100 - 1000$ s. Can show that $P \sim 1/\sqrt{G\rho}$. If were a pulsating NS, then would have period of $\sim 10^{-4}$ times that of WD (ρ is 10^8 times higher), so this is OK, but maybe too short for slow pulsars.

Rotating Star As we saw, can rotate at $\sim 1\,\text{ms}$ and be OK for a NS. If were WD, would not periods of ~ 10 s at minimum, so can't work.

Result is that pulsars are rotating NSs.

XXIV.1.1 Crab Pulsar

Center of Crab SNR, from 1054 AD. See $P = 33\,\text{ms}$, with $\dot{P} = 4.2 \times 10^{-13}\,\text{s/s}$. What does this imply about change in rotational energy? $I = \frac{2}{5}MR^2$ for uniform sphere. For NS,

$$I \approx 10^{45}\,\text{g}\,\text{cm}^2 \tag{XXIV.360}$$

See $\dot{P}$. $\omega = 2\pi/P$, so $\dot{\omega} = -2\pi\dot{P}/P^2 = -2.4 \times 10^{-9}\,\text{s}^{-2}$ (increase by a ms every 90 yr). Use $E_\text{rot} = \frac{1}{2}I\omega^2$, so identify:

$$\frac{dE_\text{rot}}{dt} = I\omega\frac{d\omega}{dt} = -4\pi^2 I\frac{\dot{P}}{P^3} \tag{XXIV.361}$$

If we take $I = 10^{45} I_{45}\,\mathrm{g\,cm}^2$, have 4.6×10^{38} erg/s. But we can also measure the amount of energy radiated away by the Crab Nebula and find 5×10^{38} erg/s. These balance, and the energy of the Crab Nebula is supplied by the slowing rotation.

Overall, pulsars were found to be rotating neutron stars. We see blips when the "lighthouse" beam crosses the Earth. The majority of the energy from the spin-down is invisible: the radio blips are a tiny fraction of the energy.

It is the strong magnetic field that makes this happen.

XXIV.1.2 Magnetic Dipole Model

Light cylinder: where v to go around is c. We take the magnetic field to be a dipole, $B(r) = B_0(r/R)^{-3}$. A changing magnet releases electromagnetic power per unit area S (*Poynting flux*) $\sim cB^2$. We can roughly relate the spin-down energy loss $I\omega\dot{\omega}$ to the Poynting flux through the light cylinder:

$$4\pi R_{\mathrm{LC}}^2 S_{\mathrm{LC}} \approx I\omega\dot{\omega} \tag{XXIV.362}$$

with $\omega = 2\pi/P$, $\dot{\omega} = -2\pi\dot{P}/P^2$. $R_{\mathrm{LC}} = cP/2\pi$, so $S_{\mathrm{LC}} = cB_{\mathrm{LC}}^2 = cB_0^2R^6/R_{\mathrm{LC}}^2$. So we have:

$$4\pi R_{\mathrm{LC}}^2 cB_0^2 \frac{R^6}{R_{\mathrm{LC}}^6} = 4\pi R^6 cB_0^2 \left(\frac{cP}{2\pi}\right)^{-4} \sim R^6 \frac{B_0^2}{c^3} P^{-4} \sim I\frac{\dot{P}}{P^3} \tag{XXIV.363}$$

This gives:

$$B_0^2 \sim \frac{c^3 I}{R^6} P\dot{P} \tag{XXIV.364}$$

So from the spin period and the rate at which it is slowing down, we can determine what the magnetic field is!

We can then use this (assuming B =constant) to get $P(t)$. The equation above is a simple ODE. We assume that the spin-down is constant, and have $P(0) = P_0$. Solve:

$$\frac{dP}{dt} = \frac{A}{P(t)} \tag{XXIV.365}$$

With the solution

$$P(t) = \sqrt{2At + P_0^2} \tag{XXIV.366}$$

with

$$A = \frac{R^6 B_0^2}{c^3 I} = \dot{P}_{\mathrm{now}} P_{\mathrm{now}} \tag{XXIV.367}$$

If we assume that $P_{\mathrm{now}} \gg P_0$, then $P(t) \approx \sqrt{2At}$. Alternatively,

$$t = \frac{1}{2A}\left(P(t)^2 - P_0^2\right) \tag{XXIV.368}$$

We find that the age is $\tau \approx P/2/\dot{P}$, so we also get the age of the system from P and $\dot{P}$. Do this for the Crab pulsar get 1250 years, which is very close to the true age of about 950 years (since people saw the supernova).

P-$\dot{P}$ diagram: HR diagram for pulsars. **draw**. Move through the diagram from upper left to lower right until you die from low voltage (don't actually die, just shut off). This takes 10^{7-8} yrs to get to $P = 10$ s from a typical starting point of 10 ms. Usually born with 10^{12} G, but there is a range.

Millisecond pulsars: $P = 1.56\,\mathrm{ms}$, $\dot{P} = 1.1 \times 10^{-19}\,\mathrm{s/s}$. This gives $B = 9 \times 10^{8}\,\mathrm{G}$, so much smaller than normal pulsar. And age of 2.3×10^{8} yr. Cannot get there via normal evolution. Note that many of these are in binaries. Scenario is that MSPs live/die in in binary as normal PSRs. The transfer mass, angular momentum. Reborn (recycled) as MSPs.

XXIV.1.3 Braking Index

Spin-down could come from magnetic dipole or other sources. Parameterize:

$$\dot{\Omega} \propto -\Omega^n \tag{XXIV.369}$$

with n the braking index. For magnetic dipole model get $n = 3$. Can show that for gravitational waves get $n = 5$, but otherwise a similar spin-down law. Measure it through:

$$n = \frac{\Omega\ddot{\Omega}}{\dot{\Omega}^2} \tag{XXIV.370}$$

but can only measure this for young pulsars when higher derivatives are measureable. We find a generic age:

$$\tau = \frac{P}{(n-1)\dot{P}}\left(1 - \frac{P_0}{P}\right)^{n-1} \tag{XXIV.371}$$

So the actual age depends on n (which may not necessarily be constant) and on the ratio between the initial and current periods.

XXIV.1.4 Dispersion

Radio waves propagating through an ionized medium. Plasma frequency is finite:

$$\omega^2 = \omega_p^2 + k^2c^2 \tag{XXIV.372}$$

with $\omega_p^2 = 4\pi n_e e^2/m_e$. So we get a group velocity:

$$v_g \approx c\left(1 - \frac{\omega_p^2}{2\omega^2}\right) \tag{XXIV.373}$$

As ω decreases v_g decreases. So lower frequencies arrive later. Can parameterize in terms of total delay:

$$t(\omega) = \int_0^L \frac{dl}{v_g} = \frac{L}{c} + \frac{2\pi e^2}{m_e c\omega^2} \times DM \tag{XXIV.374}$$

with DM the dispersion measure, the integral of the electron density:

$$DM \equiv \int_0^L n_e\,dl = \langle n_e\rangle L \tag{XXIV.375}$$

Can use this to estimate how far away pulsars are based on DM, since it is proportional to L.

XXIV.1.5 Glitches

Spin-down is mostly steady, based on change in Ω with I constant. But this doesn't have to be the case. The crust+charged particles rotate at one frequency, while the superfluid n's rotate at another. What keeps them together? There isn't a lot of friction between them. So the spin-down (which torques B, and hence the charged particles) will mean that after some time the crust will be slower than the superfluid interior. Some strain will build up, and all of a sudden (vortex unpinning) it will come undone. The two components will come back into equilibrium. This results in a deposit of angular momentum into the crust (and B), so the pulsar appears to spin-up a little bit in a very short amount of time.

We see $\Delta\Omega/\Omega \sim 10^{-6}$ (or lower), as well as a change in $\dot{\Omega}$. These come from recoupling between the two pieces, and then there is a relaxation time after that as the system comes into equilibrium.

Assume I_s is moment of superfluid, and I_n is moment of the rest. After a glitch, assume that ν_n changes discontinuously. Assume coupling between components has timescale τ_c. So:

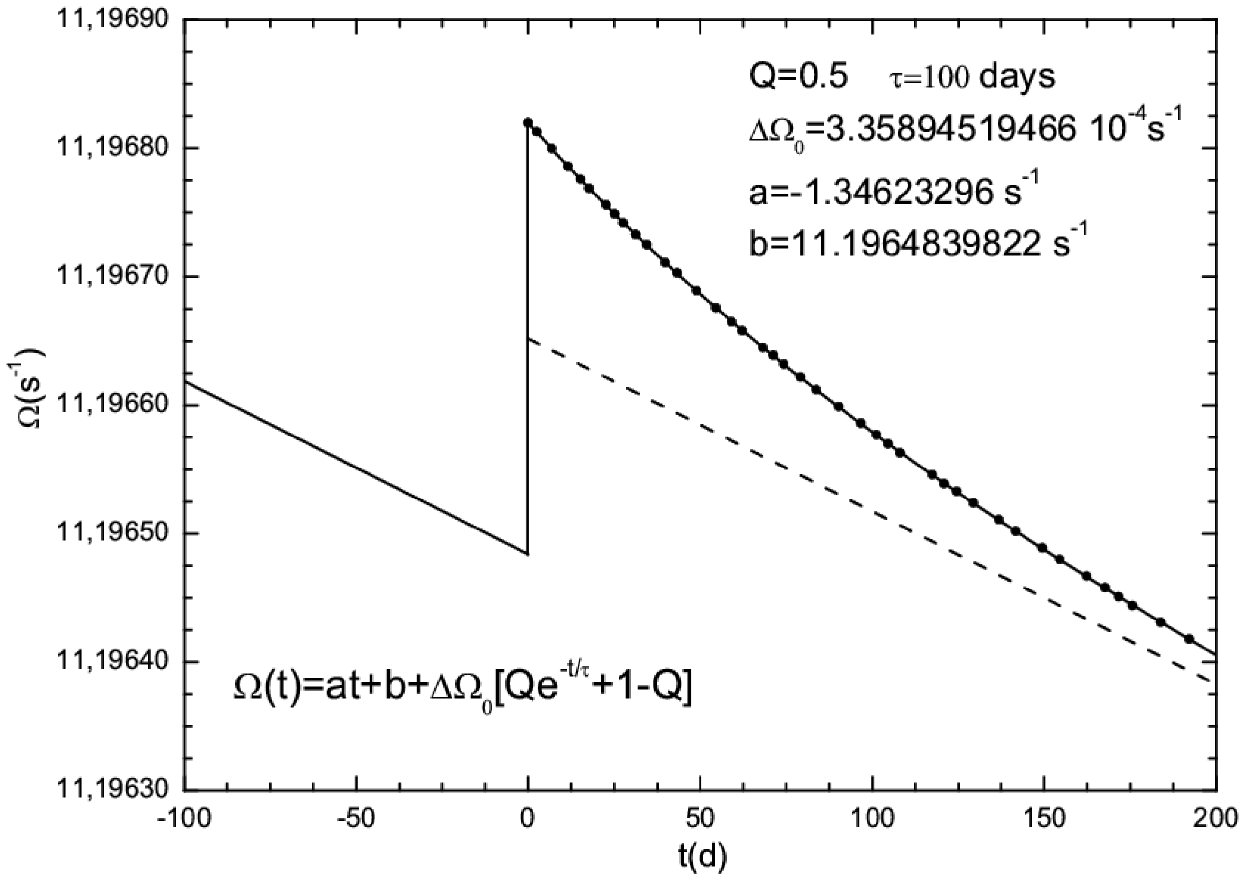
$$I_n\dot{\Omega} = -\alpha - \frac{I_n(\Omega - \Omega_s)}{\tau_c} \tag{XXIV.376}$$

$$I_s\dot{\Omega}_s = \frac{I_n(\Omega - \Omega_s)}{\tau_c} \tag{XXIV.377}$$

α is for external torques (spin-down). This gives:

$$\Omega(t) = \Omega_0(t) + \Delta\Omega_0(Qe^{-t/\tau_c} + 1 - Q) \tag{XXIV.378}$$

where Q describes the healing — how close to the original frequency does it come back. τ_c is weeks to months, so a lot of the interior should be superfluid.



XXIV.1.6 Magnetosphere

A vacuum dipole is not a good approximation. If B is strong, will be induced $E = v \times B$ which can be very high. But this will be cancelled by plasma in the magnetosphere.

Consider a conducting sphere rotating at Ω. Charges would orient on surface to cancel induced E such that:

$$E + (v \times B) = E + ((\Omega \times r) \times B) = 0 \tag{XXIV.379}$$

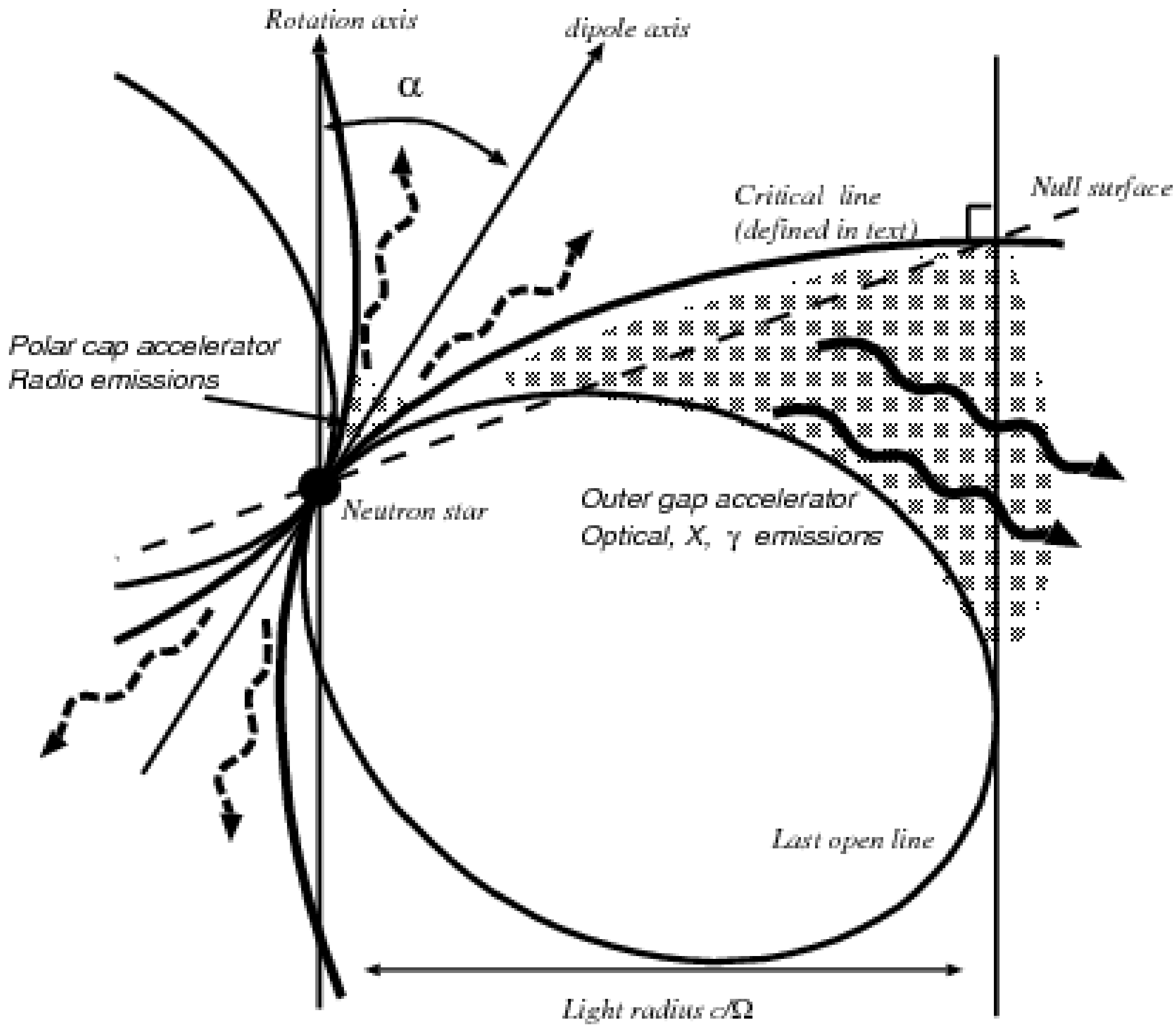
These charges make E through Maxwell's equations. At the surface, we have $\nabla^2 E = 0$, which has an electrostatic potential:

$$\phi = -\frac{B_0 \Omega R^5}{6r^3}(3\cos^2\theta - 1) \tag{XXIV.380}$$

If we then look at $\partial\phi/\partial r$, there is a strong E_r which will pull charges off the surface. $E_r \sim \Omega R B \sim 6 \times 10^{12}\,\mathrm{V/m}$ for a period of 1 s. Compare Lorentz force to gravity:

$$\frac{e(v \times B)}{GMm_e/R^2} \sim \frac{e\Omega R^3 B}{GMm_e} \approx 10^{12} \tag{XXIV.381}$$

The magnetosphere will be filled by charges pulled from the surface.



There are various regions in the magnetosphere with "gaps", where charges can be accelerated, are potential locations for the high-energy emission.

Lecture XXV Neutron Star Cooling

They start out very hot, $> 10^{11}$ K. How does that heat get transported away?

Mostly by neutrinos, especially at the start. T goes to 10^{9-10} K after a day. Neutrinos come from the interior, so they can be very efficient, but eventually they slow down and photons from the surface take over. This happens at 10^8 K, with $T_{\text{Eff}} = 10^6$ K (about what we observe for most NSs). Neutrinos will dominate for the first 10^{3-5} yr.

XXV.2 Neutrino Cooling

For $T < 10^9$ K, cools via neutrino emission from interior. They escape freely. Basic reactions are Urca:

$$n \to p + e^- + \bar{\nu}_e \tag{XXV.382}$$

$$p + e^- \to n + \nu_e \tag{XXV.383}$$

in equilibrium. Most things stay the same, but neutrinos carry away energy. This happens during core-collapse too, but degeneracy of the particles slows things down.

In equilibrium

$$\mu_n = \mu_p + \mu_e \tag{XXV.384}$$

And if degenerate,

$$E_F(n) = E_F(p) + E_F(e) \tag{XXV.385}$$

We can further use:

$$E_F(n) = m_n c^2 + \frac{p_F^2(n)}{2m_n} \tag{XXV.386}$$

and the same for p, and

$$E_F(e) = p_F(e)c \tag{XXV.387}$$

Since $p_F(e)$ depends only on n_e, and $p_F(p)$ depends on n_p, $p_F(e) = p_F(p)$ since $n_e = n_p$. So we have:

$$\frac{p_F(n)^2}{2m_n} = p_F(e)c\left(1 + \frac{p_F(p)}{2m_p c}\right) - Q \tag{XXV.388}$$

with $Q = 1.3$ MeV. But Q will be $\ll$ the other terms, so:

$$E_F'(n) = \frac{p_F^2(n)}{2m_n} \approx p_F(e)c = E_F(e) \tag{XXV.389}$$

where E_F' is the Fermi energy minus the rest mass (the kinetic part). So then:

$$p_F(e) = p_F(p) \ll p_F(n) \tag{XXV.390}$$

$$E_F(p)' \ll E_F'(n) \tag{XXV.391}$$

During the decay, the only neutrons that can do it are at $E_F'(n) \pm k_B T$. And to make electrons and protons, we also need them to have energy $E_F' \pm k_B T$, with the neutrino carrying off $\sim k_B T$. But we have said that $p_F(p) \ll p_F(n)$. We cannot match the energy (which needs to be near E_F) and the momentum (which needs to be near $\sqrt{2mE_F}$) at the same time. So this cannot happen!

Need to modify this with a bystander particle:

$$n + N \to N + p + e^- + \bar{\nu}_e \tag{XXV.392}$$

$$p + e^- + N \to N + n + \nu_e \tag{XXV.393}$$

The N particles make sure that both momentum and energy are conserved.

Put all of the physics together, find $L_\nu \propto MT_9^8$. Very strong power of T, and the luminosity is $\propto M$. This is because all parts of the inside contribute: we are not limited by scattering and the need to get photons out from the surface.

Connect inside temperature and outside. The outside of the star will have a "normal" atmosphere with scale-height ($\sim k_B T/m_p g \approx 1$ cm). The inside will be isothermal at T_I. HSE and radiative diffusion give:

$$\frac{dP}{dr} = -\frac{GM\rho(r)}{r^2} \tag{XXV.394}$$

and

$$\frac{dT}{dr} = -\frac{3\rho(r)\kappa(r)}{4acT(r)^3}\frac{L_\gamma}{4\pi r^2} \tag{XXV.395}$$

Combine:

$$\frac{dP}{dT} = \left(\frac{16\pi acG}{3}\frac{M}{L_\gamma}\right)\frac{T^3}{\kappa} \tag{XXV.396}$$

We assume $\kappa = \kappa_0 \rho T^{-3.5}$ (Kramer's law, which is only true for material with bound states). So we get:

$$\frac{dP}{dT} = C\frac{T^{7.5}}{P} \tag{XXV.397}$$

with

$$C = \frac{16\pi acGk_B}{3\kappa_0 \bar{m}}\frac{M}{L_\gamma} \tag{XXV.398}$$

Integrate this through the envelope with $P = 0$ at $T = 0$:

$$\frac{P^2}{2} = C\frac{T^{8.5}}{8.5} \tag{XXV.399}$$

Both P and T increase going inside. Stop the integration when ideal gas pressure is equal to degeneracy pressure. If you put in the numbers, you get:

$$T_{\text{Eff}} \approx 10^{-2} T_I \left(\frac{T_I}{10^9\,\text{K}}\right)^{-1/8} \tag{XXV.400}$$

so roughly $10^{-2}T$. And:

$$L_\gamma = 4\pi R^2 \sigma T_{\text{Eff}}^4 \approx 7 \times 10^{36}\text{erg/s}\left(\frac{T_{\text{Eff}}}{10^7\,\text{K}}\right)^4 \tag{XXV.401}$$

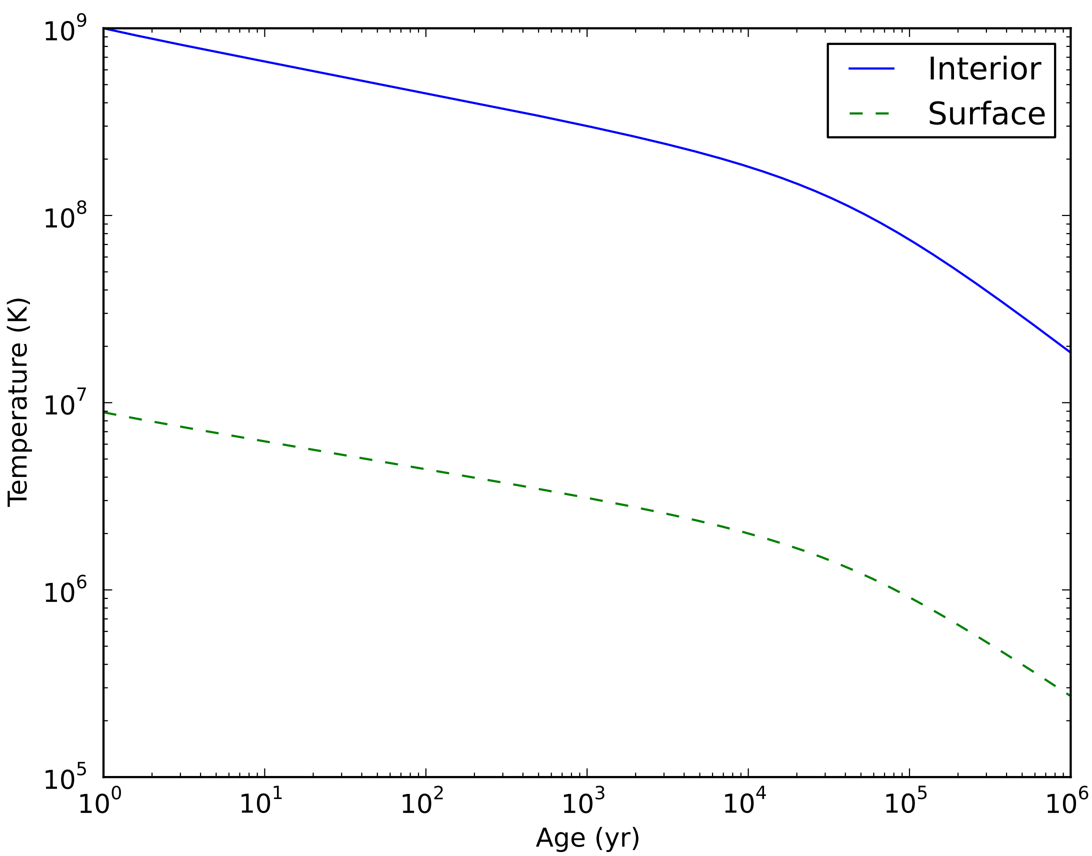
and

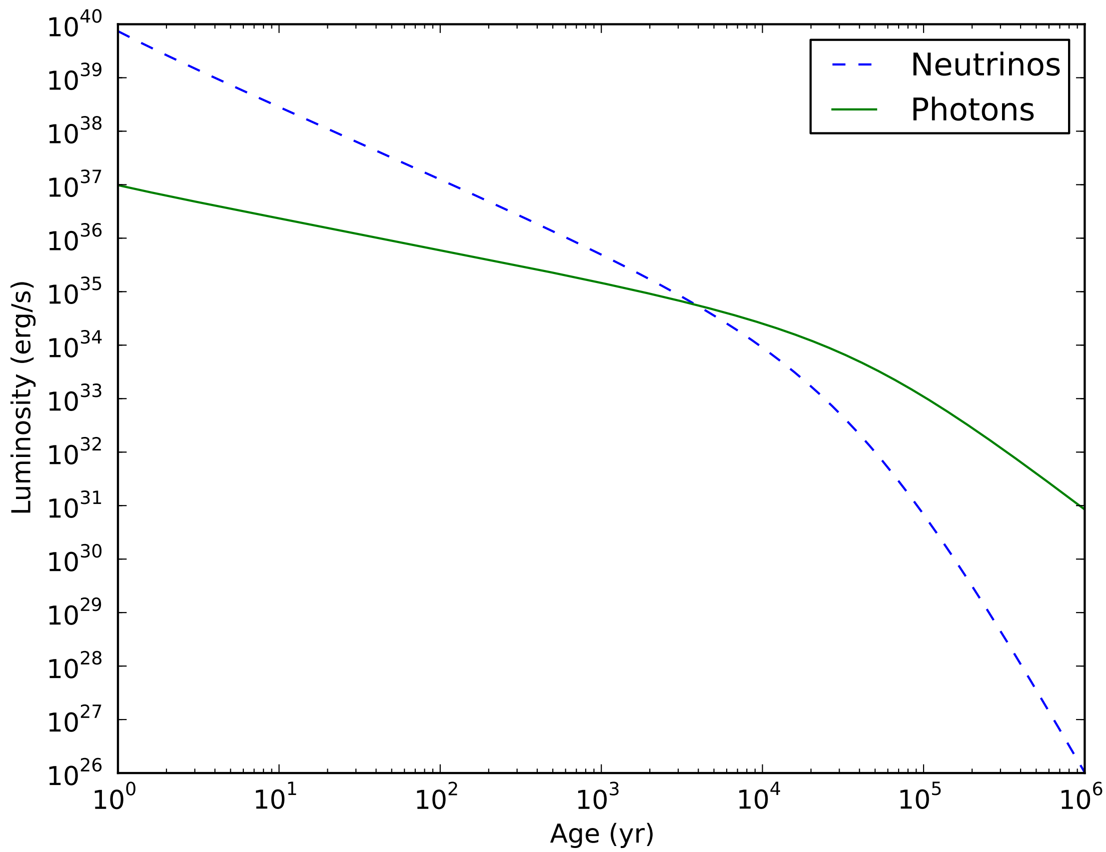
$$L_\nu \approx 5 \times 10^{39} \mathrm{erg/s} \left(\frac{T_I}{10^9\,\mathrm{K}}\right)^8 \tag{XXV.402}$$

So L_ν starts out much higher when T_I is high, but as T_I goes down it will go to lower than L_γ

Total thermal energy: roughly $\propto MT_I^2$. So

$$2T_I M \frac{dT_I}{dt} \propto -L_\gamma - L_\nu \tag{XXV.403}$$





These temperatures on the surface are $\sim 10^6$ K, so X-rays. By measuring the cooling, can get at the physics inside. Probably have measured this for NS in Cas A, with age=300 yr. See change by 80,000 K in 9 years. This is faster than expected, so think that neutrons only recently became superfluid (when this happens, can get rid of a lot of energy).

Lecture XXVI Magnetars!

Most neutron stars: energy is in the form of heat (when they are young) + rotation:

$$K = \frac{1}{2} I\Omega^2 = 8 \times 10^{50}\,\mathrm{erg} = L_\odot \times 10^{10}\,\mathrm{yr} \tag{XXVI.404}$$

(usually are brighter, so do not last as long). Pulsars are magnetized, but the magnet is only important because it gets the rotation energy out. Total luminosity $\dot{E} = I\Omega\dot{\Omega}$, but what we see $L_{\rm radio} \sim 10^{-4}\dot{E}$, $L_{\rm X-ray} \sim 10^{-3}\dot{E}$, $L_\gamma \sim 0.1\dot{E}$, synchrotron nebula $\sim \dot{E}$.

1979: burst of γ-rays from LMC. 10^{45} erg/s ($10^{12} L_\odot$) spike, 3 min decay with 8 s pulsations. Now have seen several "giant flars" plus more numerous (but smaller) bursts (10^{41} erg/s). Call these objects Soft Gamma-ray Repeaters. 1990s: X-ray pulsars with $P \sim 10$ s. Measure $\dot{P}$, find $\dot{E} \sim 10^{34}$ erg/s. But see $L \sim 10^{36}$ erg/s, so $L_X \gg \dot{E}$. How? Call these Anomalous X-ray Pulsars.

Chris Thompson & Rob Duncan: **magnetars** (also Paczynski). NS powered by decay of strong B, $B > 10^{14}$ G. Want total magnetic energy to dominate over $I\Omega^2/2$.

XXVI.1.1 Strong Magnetic Fields

B is high enough to affect structure of atoms on surface (Lai 2001, Rev Mod Phys, 73, 629). Strong B:

- $B_0 = m_e^2 e^3 c/\hbar = 2.4 \times 10^9$ G: cyclotron radius is a_0
- $B_{\rm QED} = m_e^2 c^3/e\hbar = 4.4 \times 10^{13}$ G: $\hbar\omega_{\rm cycl} = m_e c^2$
- $B = \sqrt{6G}MR^2 = 2 \times 10^{18}$ G: magnetic energy is gravitational binding energy

Cyclotron energy:

$$\hbar\omega = \hbar\frac{eB}{m_e c} = 11.577 B_{12}\,\mathrm{keV} \tag{XXVI.405}$$

Leads to orbits transverse to B with radius:

$$r = \left(\frac{\hbar c}{eB}\right)^{1/2} = 2.6 \times 10^{-10} B_{12}^{-1/2}\,\mathrm{cm} \tag{XXVI.406}$$

When $B \gg B_0$, then structure of atoms is very strongly affected. Coulomb forces are $\ll$ magnetic, and electrons end up in ground Landau level. So the electrons can barely move transverse to the protons, and then end up closer to the protons than with $B = 0$. This can have important consequences, e.g., change ionization.

At $B = 0$ ionization energy of H is 13.6 eV, compared to $kT \approx 0.1$ keV on surface. But with $B = 10^{13}$ G, we have ionization energy $\chi_H = 310$ eV. So $X_H/k_BT \approx 5$, and many (but not all) atoms are neutral. They can even form long molecular chains oriented along B fields. You have to worry about quantum states that are no longer spherically symmetric (Landau levels). Doing those atmospheres is hard.

XXVI.1.2 Arguments for Strong B

- $B = 3.2 \times 10^{15}\sqrt{P\dot{P}}$ (like for pulsar) gives $> 10^{14}$ G
- Bursts are $\gg L_{\rm Edd}$. Remember, Eddington limit balances gravity with radiation pressure assuming pressure from σ_T:

  $$L_{\rm Edd} = \frac{4\pi G M m_p c}{\sigma_T} = 10^{38}\,{\rm erg/s}\left(\frac{M}{M_\odot}\right) \tag{XXVI.407}$$

  Does this mean $M \gg M_\odot$? No: we see pulsations, so the object must be small (few $R_\odot$). Instead, change σ. Strong B:

  $$\frac{\sigma}{\sigma_T} \sim \left(\frac{\omega m_e}{eB}\right)^2 = 4\times 10^{-4}\left(\frac{\omega}{10\,{\rm keV}}\right)^2\left(\frac{B}{B_{\rm QED}}\right)^{-2} \tag{XXVI.408}$$

  for a photon with energy $\hbar\omega$ (Canuto et al. 1971, PRD, 3, 2303; Herold 1979, PRD, 19, 2868) with $B_{\rm QED} = 4\times 10^{13}$ G.

  So $10^3 L_{\rm Edd}$ means $B > 30 B_{\rm QED}$.
- Also, fireball to produce giant flares would blow itself apart. Strong B helps keep it tied to the NS surface so that it can rotate around. Total energy of giant flare tail (multiple rotations, so it has to linger) is $E_{\rm tail} \sim 3.6\times 10^{44}$ erg. Confine by magnetic pressure of a loop with outer radius ΔR. Need energy density of field (pressure) to exceed pressure of fireball:

  $$\frac{(B(R_0+\Delta R))^2}{8\pi} > \frac{E_{\rm tail}}{3\Delta R^3} \tag{XXVI.409}$$

  If we integrate a dipole over $R_0 \to \Delta R$, we get:

  $$B > 4\times 10^{14}\,{\rm G}\left(\frac{\Delta R}{10\,{\rm km}}\right)^{-3/2}\left(\frac{1+\Delta R/R_0}{2}\right)^3 \tag{XXVI.410}$$

  (Thompson & Duncan 1995, MNRAS, 275, 255).

XXVI.2 Energetics

Sources are young (found in the Galactic plane, some in SNRs that only stay around for 10^4 yrs). Total energy $\sim L_X\times$ age:

$$E \sim (10^{35}\,{\rm erg/s})\times 10^4\,{\rm yr} \sim 3\times 10^{47}\,{\rm erg} \sim \left(\frac{B^2}{8\pi}\right)\left(\frac{4\pi}{3}R_{\rm NS}^3\right) \tag{XXVI.411}$$

So strong B can power what we see.

XXVI.3 Conclusions

AXP, SGR are magnetars: NS with very strong B. Not powered by rotation but by decay of strong B (e.g., solar flares). Questions:

- How many?
- Why do they form this way?
- How are they different?
- How does the decay happen in detail?

Lecture XXVII Black Holes

XXVII.1 Schwarzschild Geometry

The Schwarzschild metric,

$$ds^2 = -\left(1 - \frac{2M}{r}\right)\, dt^2 + \left(1 - \frac{2M}{r}\right)^{-1}\, dt^2 + r^2 d\Omega^2, \tag{XXVII.412}$$

provides a smooth spacetime in the region

I: $-\infty < t < \infty$ $\qquad 2M < r < \infty$
and another (disconnected) spacetime corresponding to the coordinate values
II: $-\infty < t < \infty$ $\qquad 0 < r < 2M.$
In region I, $t^\alpha = \boldsymbol{\partial}_t$ is a timelike Killing vector, but in region II, $g_{tt} = t^\alpha t_\alpha > 0$, so that t^α, although still a Killing vector, is no longer timelike. Similarly, ∂_r is spacelike in I and timelike in II. In obtaining the form (XXVII.412), we assumed the existence of a timelike Killing vector orthogonal to a family of spacelike (t = constant) hypersurfaces, but in the region $r < 2M$ the field is so strong tht even light is constrained to move inward – no matter can remain at rest (i.e. at constant r), and everything is constrained to collapse to $r = 0$.

[Remaining at rest means moving along t^α, so the fact that t^α is spacelike is exactly the statement that physical particles (which must move along timelike trajectories) cannot remain at rest.] One might hope to join regions I and II by relaxing the condition that there be a scalar field t with ∂_t a Killing vector and t = constant a spacelike surface. Before trying this, it is helpful to learn about the geometry outside $r = 2M$, and one way to do it is to look at the geodesics (another way is to embed the geometry in some $\mathbb{R}^n$ - we'll do this later).

Particle orbits: timelike geodesics
The geodesic equation has the form

$$u^\beta \nabla_\beta u^\alpha = 0, \tag{XXVII.413}$$

for a massive particle with 4-velocity u^α. Associated with any Killing vector ξ^α is a conserved quantity $u_\alpha \xi^\alpha$: That is, from the Killing equation,

$$\mathcal{L}_{\boldsymbol{\xi}} g_{\alpha\beta} = \nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha = 0, \tag{XXVII.414}$$

and the geodesic equation, we have

$$u^\beta \nabla_\beta (u^\alpha \xi_\beta) = (u^\beta \nabla_\beta u^\alpha)\xi_\beta + u^\alpha u^\beta (\nabla_\alpha \xi_\beta) = 0. \tag{XXVII.415}$$

The first term in the middle expression vanishes by the geodesic equation; the second term vanishes because $u^\alpha u^\beta$ is symmetric in α and β, while Killing's equation implies $\nabla_\alpha \xi_\beta$ is antisymmetric.

Associated with the Killing vectors t^α and ϕ^α are the conserved energy and angular momentum per unit mass of a free particle,

$$\mathcal{E} = -u_\alpha t^\alpha = -u_t, \qquad \ell = u_\alpha \phi^\alpha = u_\phi. \tag{XXVII.416}$$

Instead of using the r component of the geodesic equation, it is simpler to use $u_\alpha u^\alpha = -1$, written for equatorial orbits (i.e., for $\theta = \pi/2$) as

$$\begin{aligned}
-1 &= g_{\alpha\beta}u^\alpha u^\beta = g^{\alpha\beta}u_\alpha u_\beta = g^{tt}(u_t)^2 + g^{\phi\phi}(u_\phi)^2 + g_{rr}(u^r)^2 \\
&= -\left(1-\frac{2M}{r}\right)^{-1}\mathcal{E}^2 + \frac{\ell^2}{r^2} + \left(1-\frac{2M}{r}\right)^{-1}\dot r^2 \\
\frac{1}{2}\dot r^2 &= \frac{1}{2}(\mathcal{E}^2-1) - V_{\text{eff}}
\end{aligned} \tag{XXVII.417}$$

where $\dot r = dr/d\tau$ and

$$V_{\text{eff}} = -\frac{M}{r} + \frac{\ell^2}{2r^2} - \frac{M\ell^2}{r^3}. \tag{XXVII.418}$$

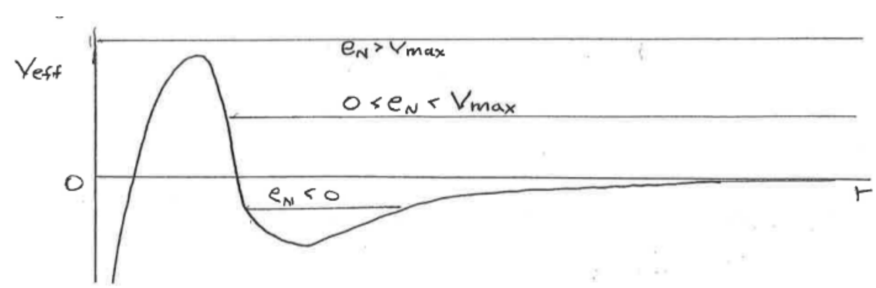
That is, the orbit is described as motion in an effective potential.

For $r >> 2M$, V_{eff} is a Newtonian effective potential, the sum of a $1/r$ attractive potential and an ℓ^2/r^2 centrifugal barrier. (The term $M\ell^2/r^3$ is of order $M/r \times \ell^2/r^2 \sim M/r(v^2)$, smaller than M/r by order $v^2 = v^2/c^2$.) At small r, however, relativistic gravity overcomes the centrifugal barrier. By $r = 2M$, V_{eff} has fallen to zero, and for $r < 2M$ it becomes increasingly negative. As a result, in addition to bound orbits and the analog of hyperbolic unbound orbits, the Schwarzschild geometry has a class of orbits with no Newtonian analog.

Write $e_N := \frac{1}{2}(\mathcal{E}^2-1)$; asymptotically,

$$\mathcal{E} = \frac{dt}{d\tau} = \gamma_\infty\sqrt{\frac{1}{1-v_\infty^2}} \implies e_N = \frac{1}{2}v_\infty^2\,\frac{1}{1-v_\infty^2},$$

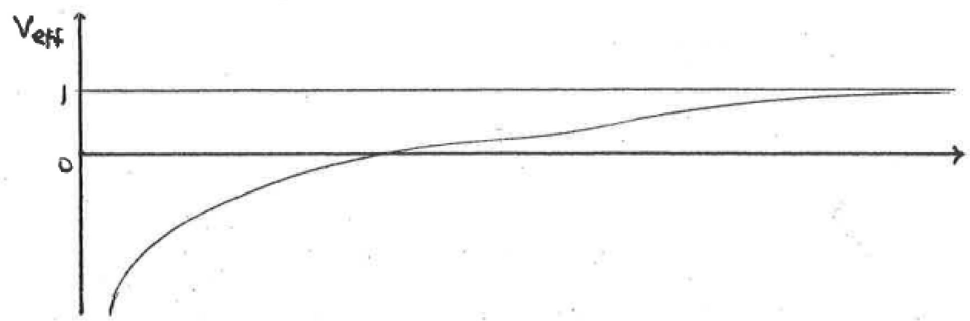
agreeing with the newtonian kinetic energy/mass for small v. A particle moving in the effective potential V_{eff} is allowed to be in the region $e_N > V_{\text{eff}}$, so three types of orbits are possible from the figure below.



For $e_N < 1$, the orbits are bound – the particle moves between an inner and outer radius. For $e_N \geq 1$ but $e_N < V_{\max}$, the particle is unbound and moves from ∞ to a minimum radius (B) and out again. These orbits are similar to the Newtonian hyperbolic orbits. But for $e_N > V_{\max}$, the particle never turns around, and we will see that it reaches $r = 2M$ *in finite proper time*. In the extended geometry, which will include regions I and II, the particle falls to $r = 0$, where it hits an infinite curvature singularity. (Note that by making e_N larger for fixed ℓ, you increase $\dot r$ and thus aim the particle more directly inward.) The time reversed – outgoing – paths with $e_N > V_{\max}$

represent particles emerging from a white hole. (The geometry inside $r = 2M$ is not static because t^α is not timelike, and time reversal will turn out to change a black hole to a white hole).

For $\ell^2 < 12M^2$, V_{eff} has no maxima or minima and all orbits eventually hit $r = 0$.



Circular orbits

For a circular orbit, $\dot{r} = 0$ implies $e_N \equiv \frac{1}{2}(\mathcal{E}^2 - 1) = V_{eff}$, and $\ddot{r} = 0$ implies $V'_{\text{eff}} = 0$. The second condition is

$$\frac{M}{r^2} - \frac{\ell^2}{r^3} - 3\frac{M\ell^2}{r^4} = 0 \Longrightarrow$$
$$\ell^2 = \frac{Mr^2}{r - 3M}. \qquad \text{(XXVII.419)}$$

Then Eq. (XXVII.418) and $e_N = V_{\text{eff}}$ implies

$$\mathcal{E}^2 = \frac{(r-2M)^2}{r(r-3M)}, \qquad \text{(XXVII.420)}$$

and Kepler relation between Ω and r is

$$\Omega = \frac{\dot{\phi}}{\dot{t}} = \frac{g^{\phi\phi}\ell}{-g^{tt}\mathcal{E}} = \frac{r^{-2}\sqrt{Mr^2/(r-3M)}}{(1-2M/r)^{-1}(r-2M)/\sqrt{r(r-3M)}}$$
$$= \sqrt{\frac{M}{r^3}}, \qquad \text{(XXVII.421)}$$

identical, with this choice of radial coordinate, to the Newtonian relation.

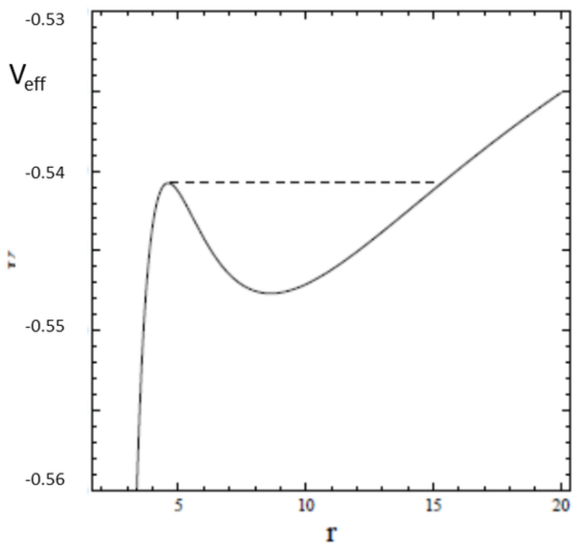
Eqs. (XXVII.419) and (XXVII.420) allow circular orbits for every $r > 3M$. For $r < 6M$, however, the orbits are unstable: $V''_{\text{eff}} < 0$. Using $V_{\text{eff}} = 0$ for a circular orbit, we have

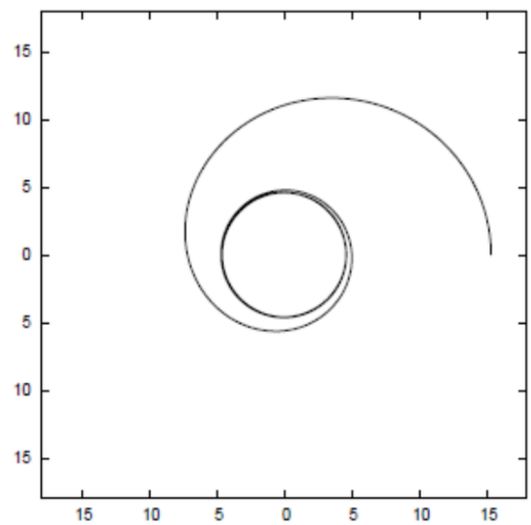
$$r^4 V''_{\text{eff}} = \frac{d}{dr}\left(r^4 \frac{dV\text{eff}}{dr}\right) = \frac{d}{dr}\left(Mr^2 - \ell^2 r + 3M\ell^2\right) = 2Mr - \ell^2 = 2Mr - \frac{Mr^2}{r-3M}$$
$$= \frac{Mr(r-6M)}{r-3M}, \qquad \text{(XXVII.422)}$$

positive for circular orbits only for $r > 6M$, as claimed. The innermost stable circular orbit is denoted by its acronym, ISCO: $r_{\text{ISCO}} = 6M$.

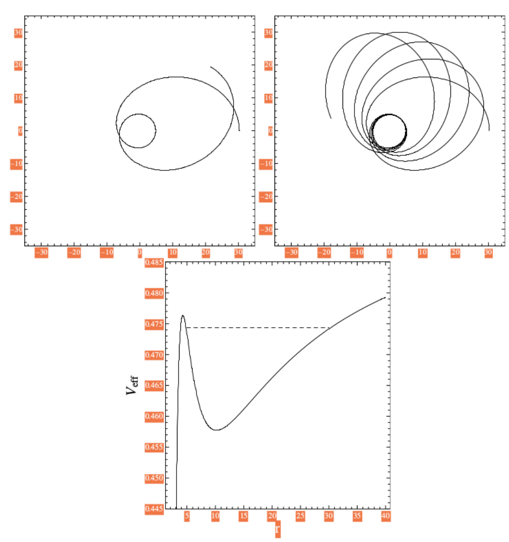
Zoom-whirl orbits

The unstable circular orbits at $r < 6M$ turn out to have an important implication for future observations of LISA and perhaps of LIGO. The figure on the left below shows the effective potential with a value of $\mathcal{E}$ corresponding to an unstable orbit at $r_c \approx 5M$. From the figure you can see that another orbit has this critical energy: A particle with energy E can move from a maximum value of r near $15M$ to a minimum value at r_c. Like any ball with exactly the energy needed to reach the top of a hill. the particle takes an infinite amount of time to reach r_c. But $\Omega = \frac{d\phi}{dt} \to \sqrt{M/r_c^3}$, a constant, finite value. So the particle spirals around the black hole an infinite number of times as it approaches r_c.





For $\mathcal{E}$ slightly below this limiting value, the particle spirals many times and then swings back out before repeating. Here is the resulting *zoom-whirl* orbit for a single whirl between each zoom.



With $e_N < 0$, these are bound zoom-whirl orbits. When e_N and the maximum of the potential are positive (like $V_{\rm eff}$ of the first figure), and e_N is just below the maximum of $V_{\rm eff}$, the particle comes in from infinity, spirals around the black hole and goes back out to infinity.

These are the particle orbits that graze the black hole, that have an impact parameter just large enough to avoid being swallowed. Typical comets in the solar system are the result of chance encounters of ice balls (in the Kuiper belt and Oort cloud) with passing stars that nudge them in a direction that removes enough of their orbital energy that they fall toward the Sun. Because orbital speed at that distance is already small, they start their infall with in nearly parabolic orbits with the small impact parameter associated with their small initial speed. Zoom-whirl orbits are most likely to arise from a similar scenario: Stars initially not too near a galactic black hole that are nudged into a trajectory that grazes the black hole have the best chance of being in the zoom-whirl parameter range.

XXVII.2 Null geodesics and affine parameters

When $c(\lambda)$ is null, $\int [\xi^\alpha \xi_\alpha]^{1/2} d\lambda$ vanishes and $\delta \int [\xi^\alpha \xi_\alpha]^{1/2} d\lambda$ is meaningless, because $[\xi^\alpha \xi_\alpha]^{-1/2}$ appears as a factor when one varies the integral. A null geodesic is in fact not a maximum or minimum of length^2, because nearby spacelike or timelike curves have $\mathrm{length}^2 > 0$. If, however, we define a null geodesic by requiring that its tangent ξ^α be parallel-propagated along the curve,

$$\xi^\beta \nabla_\beta \xi^\alpha = 0 \,, \tag{XXVII.423}$$

then a null curve $c(\lambda)$ is a geodesic if

$$\delta \int_1^2 \xi^\alpha \xi_\alpha d\lambda = 0 \,. \tag{XXVII.424}$$

A null geodesic is an inflection point of the action $\int \xi^\alpha \xi_\alpha d\lambda$. Spacelike or timelike geodesics satisfy (XXVII.423) when parameterized by proper length or time. To include null geodesics, one says a geodesic $c(\lambda)$ is *affinely* parametrized, and λ is an affine parameter, if Eq. (XXVII.423) holds. If μ is any other parameter (with $\lambda = \lambda(\mu)$ smooth and $\lambda' \neq 0$), then $\overline{c}(\mu) = c(\lambda(\mu))$ has tangent

$$\overline{\xi}^\alpha = \lambda'(\mu)\xi^\alpha,$$

and

$$\begin{aligned}
\overline{\xi}^\beta \nabla_\beta \overline{\xi}^\alpha &= \overline{\xi}^\beta \nabla_\beta (\lambda'(\mu)\xi^\alpha) = (\lambda')^2 \underbrace{\xi^\beta \nabla_\beta \xi^\alpha}_{0} + \lambda''(\mu)\, \xi^\alpha \\
&= \frac{\lambda''}{\lambda'} \overline{\xi}^\alpha \,.
\end{aligned} \tag{XXVII.425}$$

Thus only if $\lambda = k\mu + \lambda_0$ is μ still affine.

For arbitrary parameterization, a geodesic can be characterized by the equation

$$\xi^\beta \nabla_\beta \xi^{[\alpha}\, \xi^{\gamma]} = 0. \tag{XXVII.426}$$

Claim: $\bar{c}(\mu)$ has tangent $\bar{\xi}^\alpha$ satisfying (XXVII.426) if and only if $\bar{c}$ is a geodesic (that is, if and only if there is some parameter λ with $\bar{c}(\mu) = c[\lambda(\mu)]$ for which $\xi^\beta\nabla_\beta\xi^\alpha = 0$).

Proof: Clearly, if there is such a λ then by (XXVII.425)

$$\bar{\xi}^\beta\nabla_\beta\bar{\xi}^{[\alpha}\,\bar{\xi}^{\gamma]} = \frac{\lambda''}{\lambda'}\bar{\xi}^{[\alpha}\,\bar{\xi}^{\gamma]} = 0\ .$$

If, on the other hand,

$$\bar{\xi}^{[\alpha}\,\bar{\xi}^{\gamma]} = 0\ ,$$

then $\bar{\xi}^\beta\nabla_\beta\,\bar{\xi}^\alpha$ must lie along $\bar{\xi}^\alpha$. That is, $\bar{\xi}^\beta\nabla_\beta\,\bar{\xi}^\alpha = f(\mu)\bar{\xi}^\alpha$. Defining $\lambda = \int^\mu d\mu'\ \exp[\int^{\mu'} f(\mu'')d\mu'']$, we find, essentially from the computation leading to (XXVII.425) that λ is an affine parameter: That is, the tangent ξ^α to the reparametrized curve $c(\lambda)$ satisfies $\xi^\beta\nabla_\beta\,\xi^\alpha = 0$.
Exercise. Show that this claim is correct, that $\xi^\beta\nabla_\beta\,\xi^\alpha = 0$.

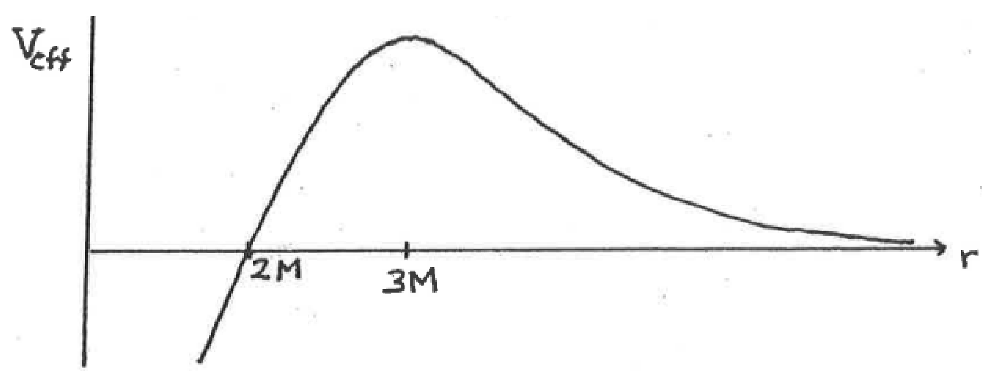
XXVII.3 Null geodesics in Schwarzschild

We denote by k^α the momentum of a photon or other zero rest-mass particle, and choose an affine parameter λ for its orbit, so that $k\cdot\nabla k^\alpha = 0$. Orbits again have two conserved quantities,

$$\mathcal{E} = -k_\alpha t^\alpha = \omega_\infty \text{ and } \ell = k_\alpha\phi^\alpha \tag{XXVII.427}$$

and Eqs. (XXVII.417) and (XXVII.418) are replaced by

$$\frac{1}{2}\dot{r}^2 = \frac{1}{2}\mathcal{E}^2 - V_{\rm eff}, \qquad V_{\rm eff} = \left(1-\frac{2M}{r}\right)\frac{\ell^2}{2r^2} \tag{XXVII.428}$$



Like massive particles, photons moving inward with large $\mathcal{E}$ are trapped by the black hole, even if they have nonzero angular momentum. And there is a surprise: The potential has a maximum at $r = 3M$, implying an unstable circular orbit for photons at this radius.
Check:

$$\frac{d}{dr}V_{\rm eff} = 0 = \frac{d}{dr}[\frac{1}{r^2} - \frac{2M}{r^3}] = -\frac{2}{r^3} + \frac{6M}{r^4} \Longrightarrow r = 3M\ . \tag{XXVII.429}$$

One calls $r = 3M$ the *photon sphere*.

Photons coming in from infinity hit the black hole when $\dot{r} > 0$ at the maximum of V_{eff}, and miss it when $\dot{r} = 0$ before they reach the top of the potential. The top of the potential is at $r = 3M$ with value

$$V_{\text{eff,max}} = \frac{\ell^2}{54M^2}. \tag{XXVII.430}$$

Then photons miss the black hole if and only if

$$V_{\text{eff,max}} > \frac{1}{2}\mathcal{E}^2 \Longleftrightarrow b := \frac{\ell}{M} > 3\sqrt{3}\, M. \tag{XXVII.431}$$

That is, the critical impact parameter for which photons just reach the top of the potential – spiraling an infinite number of times about the $r = 3M$ circular orbit – is black hole is $b = 3\sqrt{3}\, M$.

Radial photons

We now show that radially ingoing light rays reach $r = 2M$ in finite affine parameter length, although the coordinate t becomes infinite along their trajectories. Similarly, radially ingoing particles reach $r = 2M$, in finite proper time. From (XXVII.428) with $\ell = 0$, $dr = \mathcal{E} d\lambda$, so r is itself an affine parameter. Thus photons travel from 8M to 2M in parameter length $\Delta r = 8M$, or $\Delta\lambda = \frac{8M}{\mathcal{E}}$. However, $t \to \infty$:

$$\left(1 - \frac{2M}{r}\right) dt^2 = \left(1 - \frac{2M}{r}\right)^{-1} dr^2$$

$$dt = -\frac{dr}{1 - \dfrac{2M}{r}} \tag{XXVII.432}$$

$$t = v_0 - r - 2M \log\left|\frac{r}{2M} - 1\right|, \tag{XXVII.433}$$

with v_0 the constant of integration. It is useful to introduce a coordinate r^* (called the "tortoise" coordinate), writing

$$t = v_0 - r^*, \tag{XXVII.434}$$

$$r^* := \int^r \frac{dr}{1 - \frac{2M}{r}} = r + 2M \ln\left|\frac{r}{2M} - 1\right|. \tag{XXVII.435}$$

As $r \to 2M$, $r^* \to -\infty$, whence $t \to \infty$.

The same calculation shows that light emitted from $r = 2M(1 + \epsilon)$ reaches a distant observer after a time Δt that blows up like $|\log \epsilon|$ as $\epsilon \to 0$. From (XXVII.434) with a + sign for outgoing photons,

$$\begin{aligned} \Delta t &= r^*(8M) - r^*[2M(1 + \epsilon)] \\ &= (8M - 2M \log 3) - [2M(1 + \epsilon) + 2M \log \epsilon] \\ &\cong 2M \log \epsilon \end{aligned} \tag{XXVII.436}$$

Thus an observer who remains a finite distance outside $r = 2M$ never sees an infalling particle reach $r = 2M$. Moreover, the observed redshift becomes infinite: A photon's frequency as measured by an observer moving along t^α (an observer at fixed r, θ, ϕ) is $-k_\alpha u^\alpha$, with

$$u^\alpha = \frac{t^\alpha}{\sqrt{-g_{tt}}} = \frac{t^\alpha}{\sqrt{1-2M/r}}.$$

$$\omega = \frac{\mathcal{E}}{\sqrt{1-2M/r}} = \frac{\omega_\infty}{\sqrt{1-2M/r}} \qquad \text{(XXVII.437)}$$

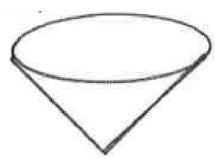
Thus

$$\frac{\omega(r=2M(1+\epsilon))}{\omega(r=R)} = \left[\frac{(1+\epsilon)(1-\frac{2M}{R})}{\epsilon}\right]^{1/2} \rightarrow \infty \text{ as } \epsilon \rightarrow 0. \qquad \text{(XXVII.438)}$$

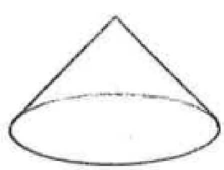
XXVII.4 Eddington-Finkelstein Coordinates

The fact that ingoing photons reach $r = 2M$ in finite value of their affine parameter and that ingoing finite rest mass particles reach $r = 2M$ in finite proper time suggests that one might be able smoothly to join regions I $(r > 2M)$ and II $(r < 2M)$ with coordinates tied to infalling particles. This is easier to do with photons, because they have simpler trajectories. Because the method leads to null coordinates, we'll look first at null coordinates in Minkowski space, to better understand their meaning.

Null coordinates in flat space:
Future null cones are u = constant surfaces, where $u = t - r$



Past null cones are v = constant surfaces, with $v = t + r$



Then

$$t = \frac{u+v}{2}\ , \quad r = \frac{v-u}{2}$$

Outgoing null coodinates: u, r, θ, ϕ

$$\begin{aligned} ds^2 &= -dt^2 + dr^2 + r^2 d\Omega^2 = -(du+dr)^2 + dr^2 + r^2 d\Omega^2 \\ ds^2 &= -du^2 - 2du dr + r^2 d\Omega^2 \end{aligned} \qquad \text{(XXVII.439)}$$

Ingoing null coordinates: v, r, θ, ϕ

$$ds^2 = -dv^2 + 2dvdr + r^2 d\Omega^2. \tag{XXVII.440}$$

Null coordinates: u, v, θ, ϕ

$$\begin{aligned} ds^2 &= -[\frac{1}{2}(du+dv)]^2 + [\frac{1}{2}(dv-du)]^2 + r^2(u,v)d\Omega^2 \\ ds^2 &= -dudv + r^2 d\Omega^2. \end{aligned} \tag{XXVII.441}$$

In this chart, r is a function of u and v: $r^2 = \frac{1}{4}(u-v)^2$.

In the ingoing null coords v, r, θ, ϕ, the path v, θ, ϕ = const is the trajectory of a radially ingoing photon, and r is the affine parameter along the path.

Ingoing and outgoing null coordinates in Schwarzschild
To mimic this in the Schwarzschild geometry, one must take

$$v = t + r_*, \tag{XXVII.442}$$

because by (XXVII.434) the trajectory of an ingoing photon is

$$t + r_* = \text{constant}, \quad \theta, \phi \text{ constant}.$$

Again, r is an affine parameter along the geodesic so we try a (v, r, θ, ϕ) chart:

$$\begin{aligned} ds^2 &= -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2 + r^2 d\Omega^2 \\ &= -\left(1-\frac{2M}{r}\right)(dv - \frac{dr_*}{dr}dr)^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2 + r^2 d\Omega^2 \\ &= -\left(1-\frac{2M}{r}\right)\left[dv - \left(1-\frac{2M}{r}\right)^{-1}dr\right]^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2 + r^2 d\Omega^2 , \end{aligned}$$

using (XXVII.435) to infer $\dfrac{dr_*}{dr} = \left(1-\dfrac{2M}{r}\right)^{-1}$. The form of the metric is surprisingly simple:

$$\boxed{ds^2 = -\left(1-\frac{2M}{r}\right)dv^2 + 2dvdr + r^2 d\Omega^2} \tag{XXVII.443}$$

The metric components are well behaved for $0 < r < \infty$. For $r \gg 2M$ (XXVII.443) looks like the flat space form (XXVII.440); for $r = 2M$ it is

$$ds^2 = 2dvdr + r^2 d\Omega^2, \tag{XXVII.444}$$

similar to the form (XXVII.441); and for $r < 2M$, the coefficient of dv^2 is positive.

The sign change in g_{vv} shows that that the "timelike" Killing vector, $t^\alpha = \boldsymbol{\partial}_v$, becomes null at $r = 2M$ and spacelike for $r < 2M$. [If (XXVII.444) looks strange to you, write the metric in terms of

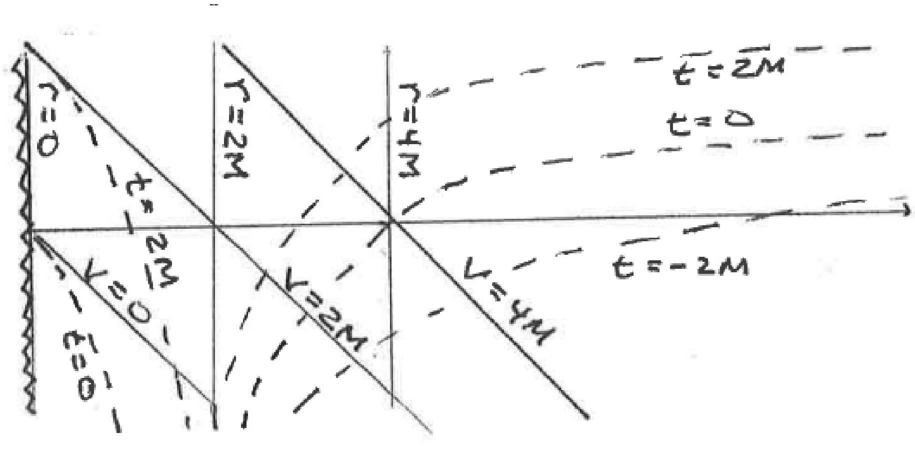
$$T, R, \theta, \phi \text{ with } T = \frac{1}{\sqrt{2}}(v - r), \ \ R = \frac{1}{\sqrt{2}}(v + r);$$

near $r = 2M$ it has the form

$$ds^2 = -dT^2 + dR^2 + (2M)^2 d\Omega^2 + \mathrm{O}(r - 2M)]$$

Note that at $r = 2M$ the completely null form $(g_{vv} = g_{rr} = 0)$ reflects the fact that both $\boldsymbol{\partial}_v$ and $\boldsymbol{\partial}_r$ are null there.

How do regions I and II in the old (t, r, θ, ϕ) charts fit into the new extended geometry in the (v, r, θ, ϕ) chart? It's easier to visualize the v, r, θ, ϕ description with v drawn so that the v = constant $(\theta = \frac{\pi}{2}, \phi = 0)$ null geodesics are 45° lines (representing ingoing photons).



The images of some t = constant lines are shown in the v, r, θ, ϕ diagram as dotted lines.

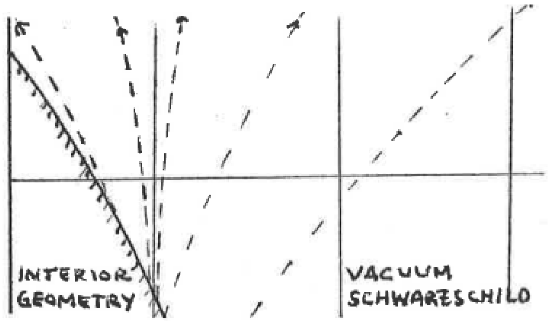
Note that the $r = 2M$ line in t, r, θ, ϕ coordinates has been pushed to $v = \rightarrow \infty$.

Light cones:

Ingoing radial photons move along v = constant lines. Outgoing photons move along $u = t - r^*$ =constant lines. Then $u = v - 2r^*$ = constant, and along the u = constant lines (i.e., along the outward radial null geodesics) $dv = 2dr^* = 2(1 - 2M/r)^{-1}\, dr$. Or, with $T = v - r$, ingoing rays are given by $dT = dr$, outgoing rays by

$$dT = \frac{r + 2M}{r - 2M} dr.$$

Stellar collapse can be described using Eddington-Finkelstein coordinates: inside the star the geometry is regular initially, and outside the geometry is vacuum Schwarzschild. The surface of the star follows a timelike trajectory and so once inside $r = 2M$, it is constrained to go to $r = 0$ in finite proper time.



"Outgoing" photons: - - - - - Outwardly directed photons, emitted after the star's surface passes $r = 2M$, never reach $r = 2M$.

The $r = 2M$ surface is called an event horizon. Observers outside it are unable to receive signals from within. A spacelike slice of the region inside an event horizon is called a black hole.

In a general asymptotically flat spacetime, event horizons and black holes are characterized this way: The set of all future directed null geodesics that make it out to infinity (that eventually leave any compact region) is called the past of future null infinity (in Schwarzschild this in the region outside $r = 2M$). The boundary of this region [”the boundary of the past of future null infinity”] is called the event horizon; and a spacelike section of the spacetime inside the event horizon is called a black hole.

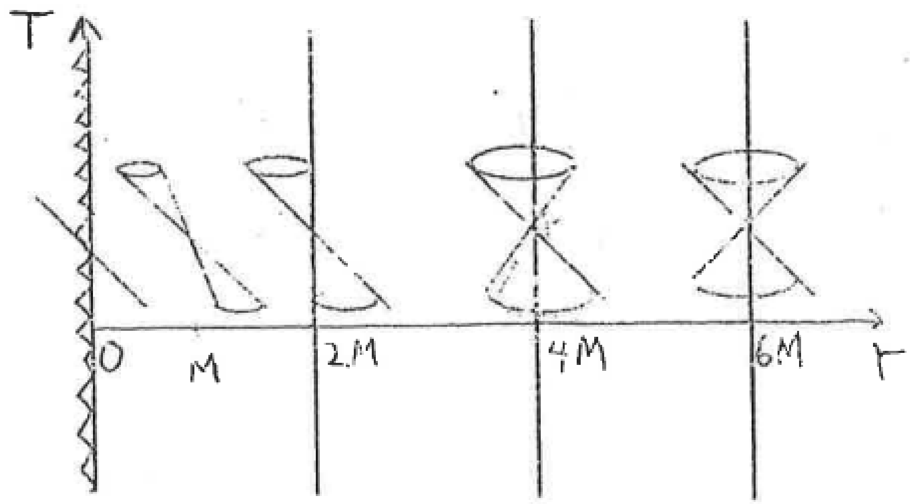
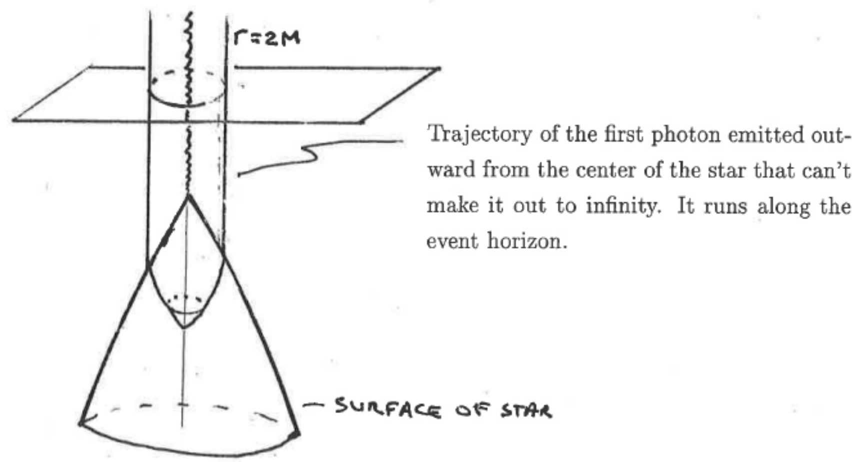
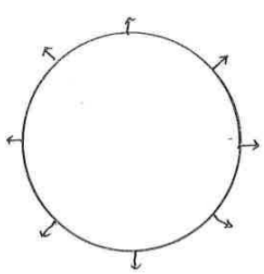


Figure 28: Light cones



Trapped Surfaces:
From the light cone diagram, Fig. 28, it is apparent that any photon emitted ith $r = 2M$ eventually hits $r = 0$. Because the radial coordinate r has the meaning that $4\pi r^2$ is the area of a v, r = constant surface, the following peculiar phenomenon occurs. Consider the set of all outgoing photons emitted from an r = constant sphere.



For large r, the emitted photons comprise a sphere of increasing area – they are diverging. For $r < 2M$ however, even the "outgoing" photons are moving to smaller values of r, so the sphere of outgoing photons is shrinking in area – the photons are converging, trapped.

Note, however, that an outgoing photon, emitted by a flashlight pointing radially outward, moves away from the flashlight in the direction that it is emitted at the speed of light, according to an observer at rest relative to the flashlight. This is consistent, because the flashlight itself is falling inward faster that the emitted photons. In general, a *trapped surface* S is one whose outward directed null geodesics ($\perp$ to S) are all converging. Thus in Schwarzschild, all r, v constant surfaces with $r < 2M$ are trapped.

It used to be thought likely that the singularity at $r = 0$ in Schwarzschild represented the end of spherical collapse only – that nonspherical collapsing stars would never encounter singularities. In 1965, however, Penrose proved that if energy dominates pressure in the sense that

$$(T_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} T) u^\alpha u^\beta \geq 0 \qquad \text{(XXVII.445)}$$

all timelike u^α, and if there is a trapped surface, then the spacetime is singular. For a fluid this is satisfied unless $p < -\frac{1}{3}\epsilon$ (attractive pressure exceeding the relativistic limit); for an electromagnetic field it's always true; for a massive scalar field the condition can fail in regions on the order of

Compton wavelength, $\sim 10^{10}$ cm for pions. Eq. (XXVII.445) is called the *strong energy condition*. The singularity may not be as violent as the $r = 0$ singularity of Schwarzschild where $R_{\alpha\beta\gamma\delta}$ becomes singular ($R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$, for example, blows up). The theorem only says that at least one null geodesic is incomplete - the geodesic leaves the spacetime in finite affine parameter length. The coordinate singularity at $r = 2M$ had that character, but that was the result of picking coordinates that cover only a piece of the spacetime – in effect, by choosing Schwarzschild coordinates, one is cutting a hole out of the spacetime at $r = 2M$. In that case, one can smoothly extend the truncated geometry. Penrose's theorem guarantees that there will be no way to avoid the singularity by, for example, extending the spacetime. As we have already seen, null geodesics hit $r = 0$ in finite parameter length, and this is the singularity implied by Penrose's theorem.

XXVII.5 Embedding Diagrams

If you are handed the components of a metric in some coordinates, it is often difficult to understand what the metric means physically or geometrically. Commonly one draws light cones and constructs the geodesics, as we have done for the Schwarzschild geometry. Another, more powerful, technique, when it can be used, is to construct an embedding diagram. The idea is to regard a section of the spacetime (e.g., a t = constant surface) as a curved surface in flat space (some $\mathbb{R}^n$). For example, a sphere can be characterized intrinsically by the metric with components

$$ds^2 = r^2[d\theta^2 + \sin^2\theta d\phi^2]$$

or pictured as an r = constant subset of $\mathbb{R}^3$. In other words, one looks for a submanifold of $\mathbb{R}^n$ whose metric is that of the surface you want to understand.

Let us first embed a t = constant surface of a spherical star

$$ds^2 = \left(1 - \frac{2m(r)}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \qquad \text{(XXVII.446)}$$

To visualize, we can't look at more than a 2-dimensional surface embedded in $\mathbb{R}^3$, so we'll take a $\theta = \frac{\pi}{2}$ plane:

$$ds^2 = (1 - \frac{2m}{r})^{-1} dr^2 + r^2 d\phi^2. \qquad \text{(XXVII.447)}$$

The object is to find a surface $z = z(r, \phi)$ in $\mathbb{R}^3$ with this metric. The flat metric of $\mathbb{R}^3$ has z, r, ϕ components

$$ds^2 = dz^2 + dr^2 + r^2 d\phi^2 \qquad \text{(XXVII.448)}$$

The metric of a $z = z(r, \phi)$ surface is thus

$$ds^2 = \left[\frac{\partial z}{\partial r} dr + \frac{\partial z}{\partial \phi} d\phi\right]^2 + dr^2 + r^2 d\phi^2. \qquad \text{(XXVII.449)}$$

Axisymmetry $\Rightarrow z = z(r)$,

$$ds^2 = (z'^2 + 1) dr^2 + r^2 d\phi^2.$$

To agree with (XXVII.447) we need

$$z'^2 + 1 = \left(1 - \frac{2m}{r}\right)^{-1}$$

$$z = \int_0^r \frac{dr}{\left[\dfrac{r}{2m(r)} - 1\right]^{1/2}},$$ when there is no black hole – when $2m(r)$ is everywhere greater than r.

(XXVII.450)

Near the center of the star

$$m = \frac{4}{3}\pi\, \rho_c\, r^3 + \mathrm{O}(r^4)$$

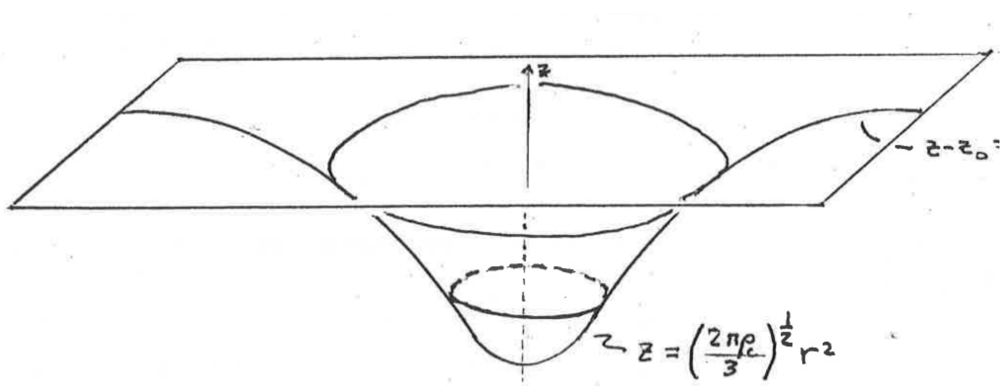
$$\begin{aligned} z &= \int_0^r (\frac{8\pi\rho_c}{3})^{1/2} r\, dr + \mathrm{O}(r^3) \\ &= (\frac{2\pi\rho_c}{3})^{1/2} r^2 + \mathrm{O}(r^3) \end{aligned} \tag{XXVII.451}$$

Outside the star

$$z = z_0 + [8M(r-2M)]^{1/2} \text{ or } (\mathrm{r} - 2\mathrm{M}) = \frac{1}{8\mathrm{M}}(\mathrm{z} - \mathrm{z_0})^2 \tag{XXVII.452}$$

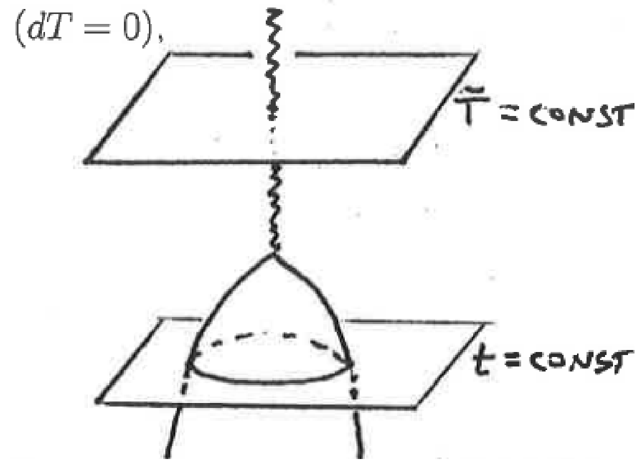
Check:

$$\frac{dz}{dr} = (8M)^{1/2}\frac{1}{2}(r-2M)^{-1/2} = \frac{1}{\left(\dfrac{r}{2M} - 1\right)^{1/2}}$$



Note that the surface is coordinate independent – specified by demanding that its metric inherited from the flat metric on $\mathbb{R}^3$ be the same as the metric of an equatorial plane in the Schwarzschild geometry.

After the star collapses one can embed a T = constant surface:



From (XXVII.443), with $dv = dr \quad (dT = 0)$,

$$ds^2 = \left[-(1 - \frac{2M}{r}) + 2\right] dr^2 + r^2 d\Omega^2 = \left(1 + \frac{2M}{r}\right) dr^2 + r^2 d\Omega^2$$

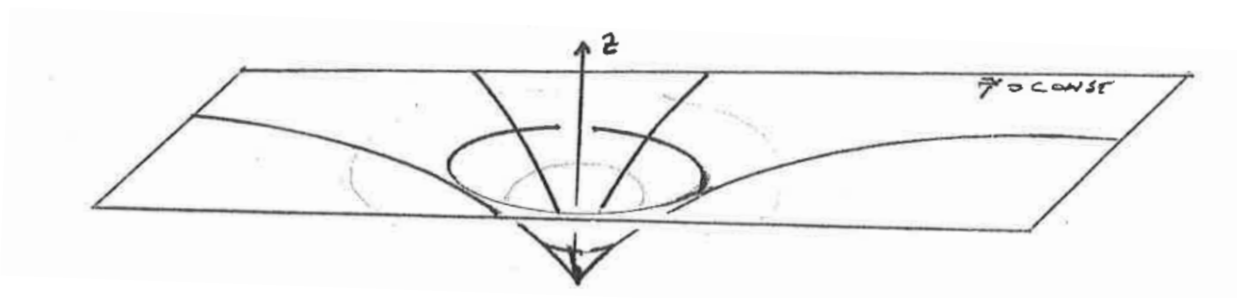
Embed:

$$(z')^2 + 1 = 1 + \frac{2M}{r}$$

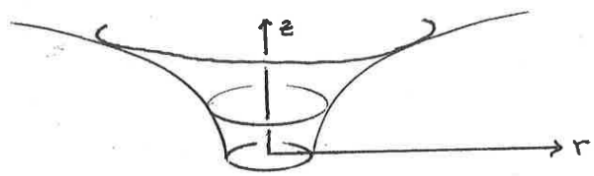
$$z' \sqrt{\frac{2M}{r}} \quad z = (8Mr)^{1/2}$$

$$z = (8Mr)^{1/2} \quad \text{or} \quad r = \frac{1}{8M} z^2 \tag{XXVII.453}$$

After the collapse the surface is no longer smooth at $r = 0$ – there is a cusp, implying infinite curvature. (Old figure has $\tilde{T}$ instead of T.)



Although Eddington-Finkelstein ingoing coordinates are fine for describing collapse, and provide a completion for the future-directed null and timelike geodesics that are cut off in the $\{t, r, \theta, \phi\}$ chart, the $\{v, r, \theta, \phi\}$ coordinates are not good for describing past-directed geodesics that move inward (equivalently, the past of future-directed outward geodesics). These run off the manifold in finite length as before. When the past was a star, the problem disappears, but if one wants to look formally at the vacuum Schwarzschild geometry a further completion is needed. One way to see what to expect is to look at the embedding of a t = constant surface of vacuum Schwarzschild. From (XXVII.452), the embedding looks like $r - 2M = \frac{1}{8M}(z - z_0)^2$, and it ends abruptly at $r = 2M,\ z = 0$ (where, without loss of generality, we have taken $z_0 = 0$).

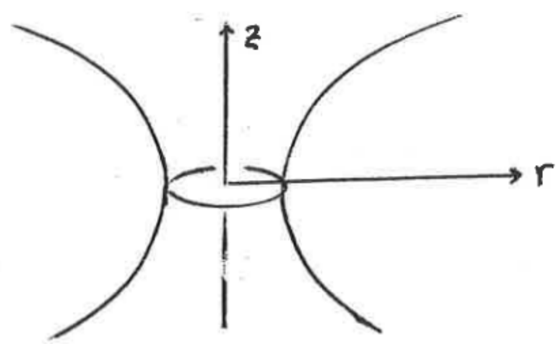


But the bottom half of the paraboloid is identical to the top half and is also described by the metric

$$ds^2 = \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \qquad \text{(XXVII.454)}$$

of a t = constant surface.

Thus the full paraboloid is a smooth extension of the t = constant surface, and it has the spatial Schwarzschild metric (XXVII.454) everywhere.



This suggests that one can extend the spacetime in such a way as to get two identical asymptotically flat regions.

One way is to patch together a bunch of different charts. But it turns out to be possible to find a single chart that covers the complete extension of the geometry.

XXVII.6 Kruskal Coordinates

Part of the problem with Eddington-Finkelstein is that, by choosing ingoing coordinates, one imposes a preferred time direction (future) on a geometry that doesn't know future from past. As in flat space, one could have picked outgoing null coordinates $u = t - r^*, r, \theta, \phi$ and found a metric $ds^2 = -\left(1 - \dfrac{2M}{r}\right) du^2 - 2du dr + r^2 d\Omega^2$ for a spacetime with complete past directed geodesics, but incomplete future geodesics.

So to regain past-future symmetry one could try null coordinates u, v, θ, ϕ. Then, using

$$t = \frac{u+v}{2}, \;\; r^* = \frac{v-u}{2}, \quad \text{and} \;\; \frac{dr}{dr^*} = (1 - \frac{2M}{r}), \tag{XXVII.455}$$

we have

$$\begin{aligned} ds^2 &= -\left(1 - \frac{2M}{r}\right)\frac{1}{4}(du+dv)^2 + \left(1 - \frac{2M}{r}\right)^{-1}\frac{1}{4}(dv-du)^2\left(1 - \frac{2M}{r}\right)^2 + r^2 d\Omega^2 \\ &= -\left(1 - \frac{2M}{r}\right) dudv + r^2 d\Omega^2 \end{aligned} \tag{XXVII.456}$$

Unfortunately, at $r = 2M$ the metric is degenerate (not invertible): $ds^2 = r^2 d\Omega^2$, implying $\sqrt{-g} = 0$. This, however, is the only remaining obstacle, and it is easily overcome by a change of scale:

$$\tilde{u} = -e^{-u/4M} \qquad \tilde{v} = e^{v/4M} \tag{XXVII.457}$$

$$d\tilde{u} = \frac{1}{4M}e^{-u/4M}du \qquad d\tilde{v} = \frac{1}{4M}e^{v/4M}dv$$

$$\begin{aligned} du\,dv &= 16M^2 e^{(-v+u)/4M} d\tilde{u}d\tilde{v} = 16M^2\, e^{-r^*/2M} d\tilde{u}d\tilde{v} \\ &= 16M^2 \exp\left[-\frac{r}{2M} - \log\left|\frac{r}{2M} - 1\right|\right] d\tilde{u}d\tilde{v} \\ &= 16M^2 e^{-r/2M}\frac{1}{\frac{r}{2M} - 1} d\tilde{u}d\tilde{v}, \;\; r > 2M. \end{aligned} \tag{XXVII.458}$$

Finally,

$$\begin{aligned} ds^2 &= -16M^2\, e^{-r/2M}\, \frac{2M}{r} d\tilde{u}d\tilde{v} + r^2(\tilde{u},\tilde{v})d\Omega^2, \\ ds^2 &= -\frac{32M^3}{r}\, e^{-r/2M}\, d\tilde{u}d\tilde{v} + r^2 d\Omega^2. \end{aligned} \tag{XXVII.459}$$

This looks good for $r > 0$, but we still need to find out when $r(\tilde{u},\tilde{v})$ is well behaved. From (XXVII.435), (XXVII.455) and (XXVII.457), r is given implicitly for $r > 2M$ by

$$\left(\frac{r}{2M} - 1\right) e^{r/2M} = -\tilde{u}\tilde{v} \equiv f(r), \tag{XXVII.460}$$

and we *define* $r(\tilde{u},\tilde{v})$ by this relation for all $r > 0$. $f(r)$ is invertible to $r > 0$ when $-\tilde{u}\tilde{v}$ is in the range of f:

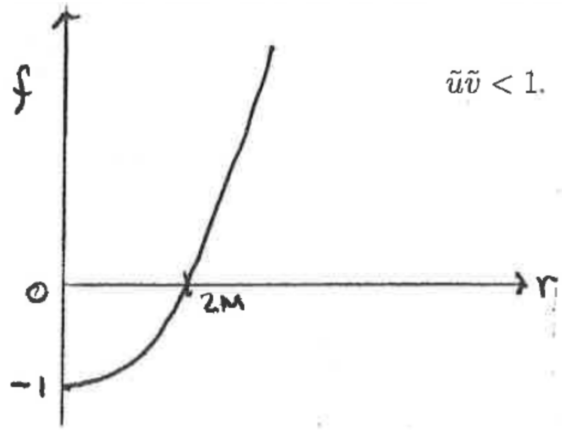
$$\tilde{u}\tilde{v} < 1. \tag{XXVII.461}$$

One commonly introduces the spacelike and timelike coordinates

$$X = \frac{1}{2}(\tilde{v} - \tilde{u}) \qquad T = \frac{1}{2}(\tilde{v} + \tilde{u}). \tag{XXVII.462}$$

Then

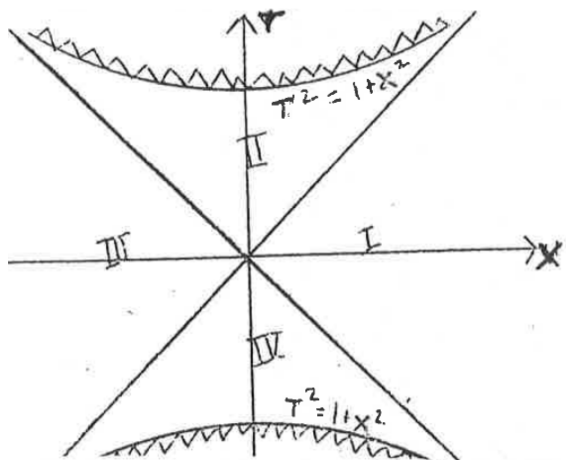
$$f(r) = X^2 - T^2 \tag{XXVII.463}$$



and the metric has components

$$ds^2 = +\frac{32M^3}{r}\, e^{\frac{r}{2M}}(-dT^2 + dX^2) + r^2 d\Omega^2.$$

Eq. (XXVII.463) is invertible to a unique $r > 0$ when $-X^2 + T^2 < 1, or T^2 < 1 + X^2$, a region between two hyperbolae; it has four parts, I, II, III, IV.



In I, one obtains a t, r patch with $r > 2M$ by setting

$$\text{I }(r > 2M): \qquad X = (\frac{r}{2M} - 1)^{1/2}\, e^{r/4M} \cosh\frac{t}{4M}$$
$$T = (\frac{r}{2M} - 1)^{1/2}\, e^{r/4M} \sinh\frac{t}{4M}$$

Region II provides a t, r patch with $r < 2M$

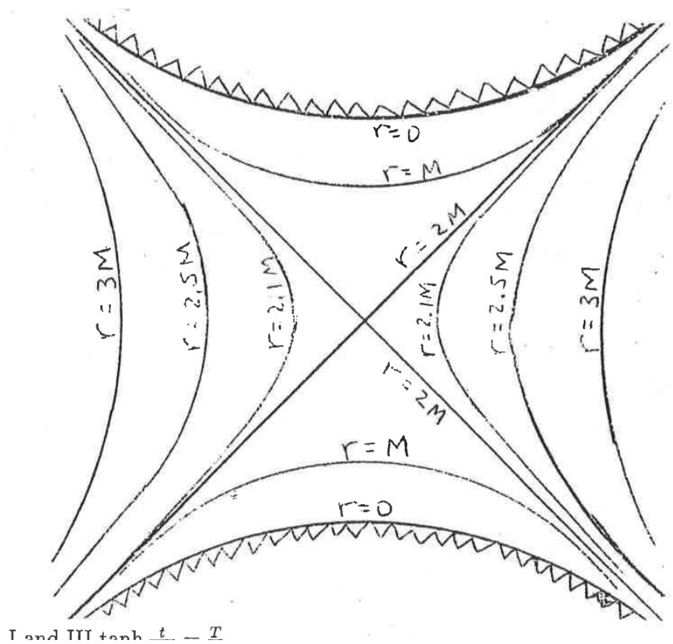
$$\text{I }(r < 2M): \qquad X = (1 - \frac{r}{2M})^{1/2}\ e^{r/4M}\ \sinh\frac{t}{4M}$$
$$T = (1 - \frac{r}{2M})^{1/2}\ e^{r/4M}\ \cosh\frac{t}{4M}$$

Similarly regions III and IV give t, r patches isometric to I and II, respectively, with

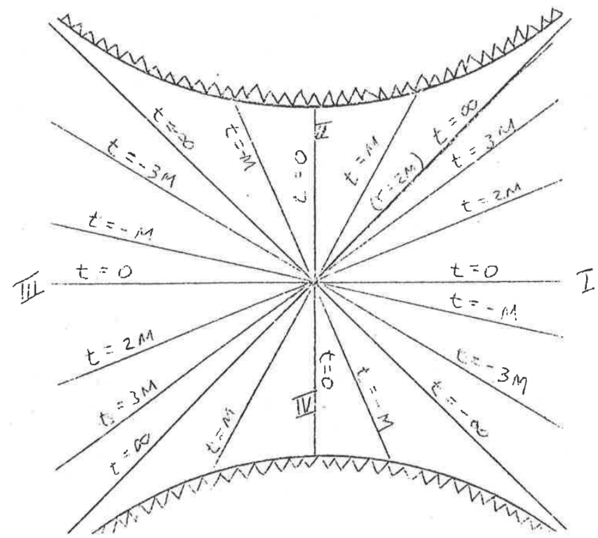
$$\text{III } (r > 2M): \qquad \begin{aligned} X &= -(\frac{r}{2M} - 1)^{1/2}\, e^{r/4M} \cosh\frac{t}{4M} \\ T &= -(\frac{r}{2M} - 1)^{1/2}\, e^{r/4M} \sinh\frac{t}{4M} \end{aligned}$$

$$\text{IV } (r < 2M): \qquad \begin{aligned} X &= -(\frac{r}{2M} - 1)^{1/2}\, e^{r/4M} \sinh\frac{t}{4M} \\ T &= -(\frac{r}{2M} - 1)^{1/2}\, e^{r/4M} \cosh\frac{t}{4M} \end{aligned}$$

As we assumed to begin with in (XXVII.460) and (XXVII.463), r is given everywhere by $X^2 - T^2 = (\frac{r}{2M} - 1)e^{r/2M}$ so that the $r =$ constant curves are hyperbolae:

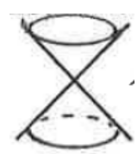


In regions I and III $\tanh\frac{t}{4M} = \frac{T}{X}$
In regions II and IV $\tanh\frac{t}{4M} = \frac{X}{T}$



Features of the complete Schwarzschild geometry in Kruskal coordinates: All values of t for $r = 2M$ are mapped to a single point, the center point of the figure above. With the angular dimensions included, this is a sphere of radius $r = 2M$ in spacetime. This problem with the coordinate t is analogous to the coordinate singularity in polar coordinates where all values of ϕ at $\theta = 0$ are mapped to a single point (a line, the z-axis, when one includes r).

In the Kruskal diagram, the radial null geodesics are 45° lines: The light cones are 45^o cones.



There are two asymptotically flat regions. Spacelike surfaces, for example the t = constant surfaces of regions I and III, extend from one asymptotic region to another, via a wormhole (initially called an Einstein-Rosen bridge). It is immediately clear from the Kruskal diagram that no particle can go through through the wormhole and reach the other asymptotically flat region: Only spacelike lines do that – timelike ($> 45^\circ$ angle with horizontal) lines all hit the $r = 0$ singularity and don't make make it through from I to III or III to I.

It is also apparent from the diagram that no particle in II, in the black-hole interior, can escape from the black hole – to regions I or III.

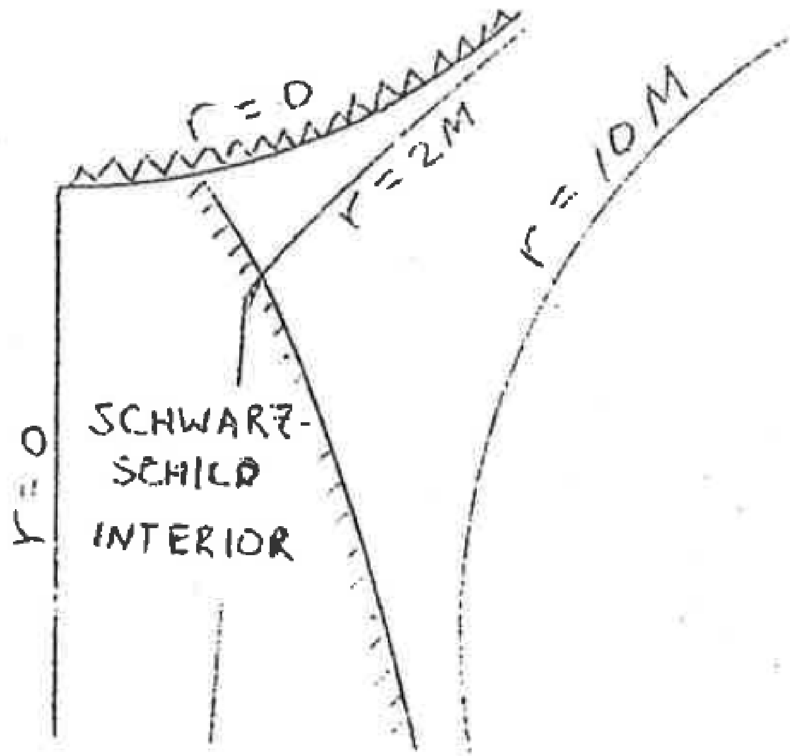
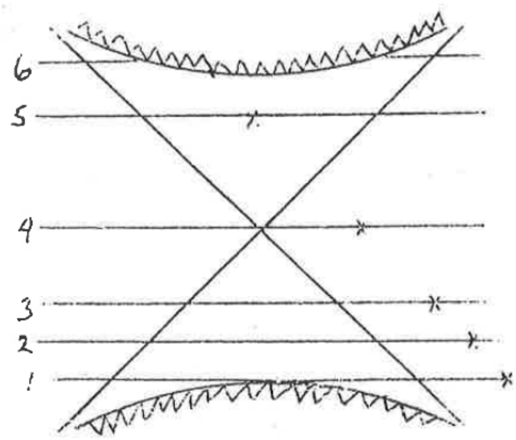


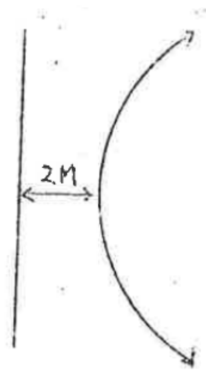
Figure 29: A collapsing star in a Kruskal diagram looks like the figure. The left half of the Kruskal spacetime doesn't exist here. Because the Eddington-Finkelstein ingoing chart is mapped to regions II and III, the Eddington-Finkelstein chart covers the full collapse spacetime, if one continues the coordinates v, r to the star's interior.

The Schwarzschild geometry is time dependent: the Killing vector that is timelike outside $r = 2M$ becomes spacelike inside. Here are two histories given à la MTW as sequences of embedding diagrams.

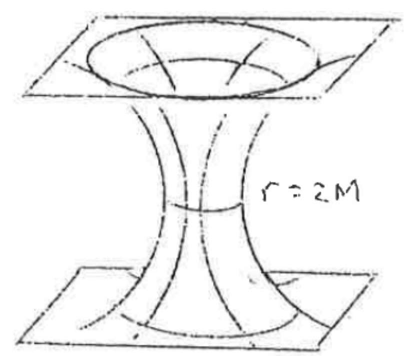


In figuring out what the embedding should like qualitatively, the key point is that distances on the embedded surface are proper distances in the spacetime. In particular, a radial coordinate r always corresponds to a distance r from the z-axis in the embedding diagram. This is because the area of a sphere is $4\pi r^2$ in the Schwarzschild geometry and must therefore be $4\pi r^2$ in the embedding as well (with the dimension reduced, the circumference of a $0 \leq \phi \leq 2\pi$ circle is $2\pi r$ in the embedding as in the geometry). And distance along a radial curve in an embedding diagram is proper spacetime distance.

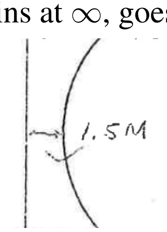
Slice 4: r begins on the right at ∞, then goes to $r = 2M$ (the central point in the diagram), and then out to ∞ again:

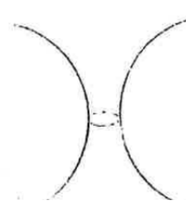


rotate: each point of Kruskal is a sphere; here each point goes to a circle

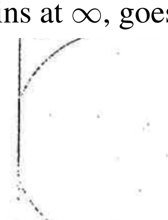


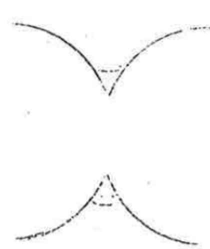
Slice 5: r begins at ∞, goes to $r = 1.5$, then out to ∞:



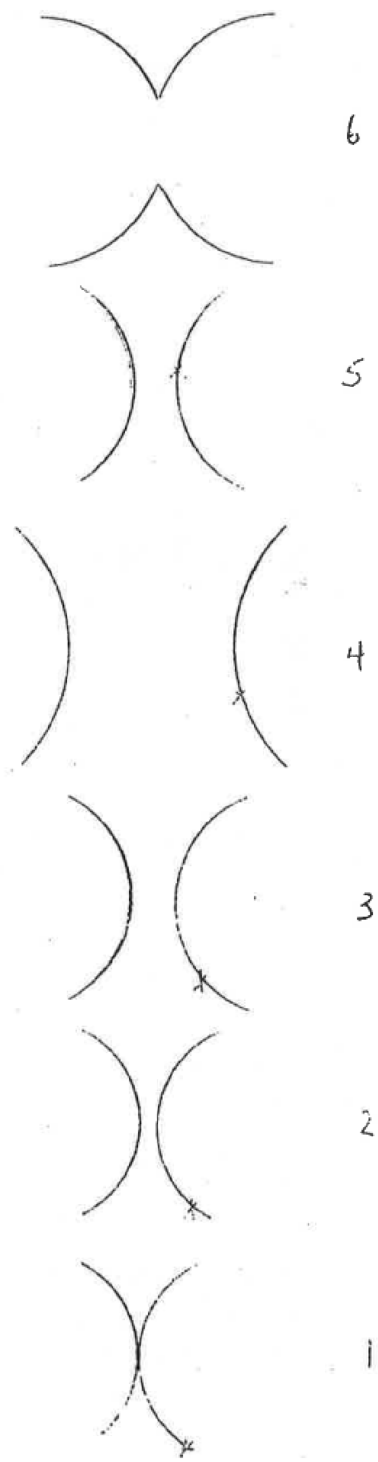


Slice 6: r begins at ∞, goes to $r = 0$; a disconnected piece starts at $r = 0$ and goes to ∞:



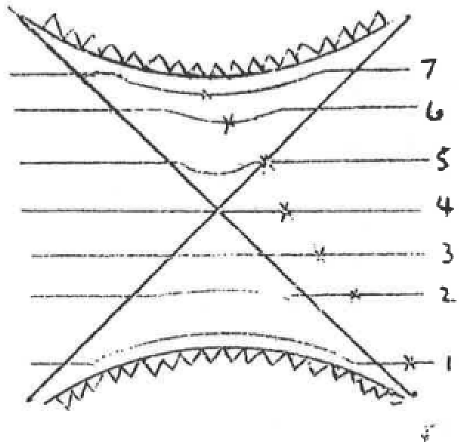


So the slicing 1-6 of the history looks like this.

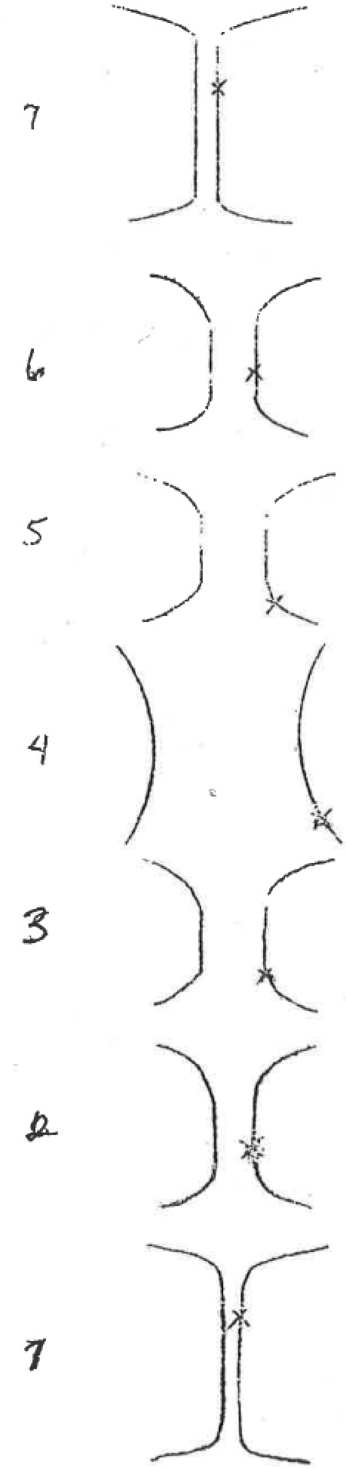


The crosses mark the trajectory of an ingoing particle that fails to make it through the wormhole before the asymptotic regions pinch off.

Another slicing



You can also slice (foliate) the entire spacetime with spacelike slices that never touch the singularities. They then spend a long stretch at $r \approx 0$:



XXVII.7 Kerr Black Holes

Had anyone taken seriously the idea that stars might really collapse to within their Schwarzschild radius, the Kerr solution could have been found in the 1920s by asking what would happen if a rotating star collapsed. Instead, Roy Kerr found it forty years later by looking for exact vacuum solutions whose Riemann tensor had a particularly simple form. Kerr black holes are the unique stationary asymptotically flat vacuum spacetimes with event horizons. For each mass M, there is a 1-parameter family of Kerr black holes with that mass; the parameter is the angular momentum J about the symmetry axis, and at $J = 0$ the family begins with Schwarzschild. There is a maximum angular momentum for each mass M: no star can contract past the point its rotational speed exceeds the speed of light, and this limits the angular momentum of a black hole that can be physically generated. The physical limitation is mirrored in the solution set: For $J > M^2$ (think of $J_{\max} = M v_{\max} R = McM = M^2$), Kerr vacuum metrics exist, but they have no horizons. Instead a naked singularity deprives these spacetimes of physical meaning.

Several features of a rotating black hole could have been predicted at the outset. The spacetime ought to have two Killing vectors ϕ^α and t^α. By looking at the metric of any slowly rotating ball or star, it is not difficult to show that it has the asymptotic form

$$g_{ti} = 2\epsilon_{ijk}\frac{x^j J^k}{r^3}\ , \tag{XXVII.464}$$

where $J_z = J \neq 0$, $\qquad J_x = J_y = 0$. Then

$$g_{t\phi} = x g_{ty} - y g_{tx} = -x\frac{2xJ}{r^3} - y\frac{2yJ}{r^3} = -2\frac{\sin^2\theta}{r}J.$$

Note that the t-ϕ part of the metric just involves the Killing vectors:

$$g_{t\phi} = g_{\alpha\beta}t^\alpha\phi^\beta = t^\alpha\phi_\alpha, \qquad g_{tt} = t^\alpha t_\alpha, \qquad g_{\phi\phi} = \phi^\alpha\phi_\alpha.$$

Furthermore, the geometry should be symmetric under a simultaneous reversal of time and the direction of rotation, under

$$t \to\ -t, \qquad \phi \to\ -\phi\ . \tag{XXVII.465}$$

The Kerr metric, in a chart that becomes the (t, r, θ, ϕ) chart of Schwarzschild when $J = 0$, is

$$ds^2 = -\frac{\Delta - a^2\sin^2\theta}{\rho^2}\,dt^2 - 2\,\frac{2Mar\sin^2\theta}{\rho^2}dt\,d\phi + \frac{(r^2+a^2)^2 - a^2\Delta\sin^2\theta}{\rho^2}\,\sin^2\theta d\phi^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2\ , \tag{XXVII.466}$$

where

$$a = J/M,\ \ \rho^2 = r^2 + a^2\cos^2\theta\ = \Sigma_{S\&T},\ \ \Delta = r^2 - 2Mr + a^2\ . \tag{XXVII.467}$$

Notation: The notes follow MTW and Chandrasekhar's *Mathematical Theory of Black Holes* in using ρ^2, while Shapiro-Teukolsky and Bardeen-Press-Teukolsky use Σ.

(t, r, θ, ϕ) are called Boyer-Lindquist coordinates, after their discoverers. The metric (XXVII.466) can also be written in the factored form

$$ds^2 = -\frac{\Delta}{\rho^2}\left(dt - a\sin^2\theta d\phi\right)^2 + \frac{\sin^2\theta}{\rho^2}[-adt + (r^2+a^2)d\phi]^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 \quad \text{(XXVII.468)}$$

Its determinant is

$$\sqrt{-g} = \rho^2\sin\theta, \quad \text{(XXVII.469)}$$

and the determinant of the t-ϕ part of the metric is

$${}^2g = \boldsymbol{t}\cdot\boldsymbol{t}\,\boldsymbol{\phi}\cdot\boldsymbol{\phi} - (\boldsymbol{t}\cdot\boldsymbol{\phi})^2 = g_{tt}g_{\phi\phi} - (g_{t\phi})^2 = \Delta\sin^2\theta. \quad \text{(XXVII.470)}$$

The inverse metric is then

$$(g^{\mu\nu}) = \begin{pmatrix} -\dfrac{(r^2+a^2)^2 - a^2\Delta\sin^2\theta}{\rho^2\Delta} & 0 & 0 & -2\dfrac{aMr}{\rho^2\Delta}\sin^2\theta \\ 0 & \dfrac{\rho^2}{\Delta} & 0 & 0 \\ 0 & 0 & \rho^2 & 0 \\ -2\dfrac{aMr}{\rho^2\Delta}\sin^2\theta & 0 & & \dfrac{\Delta - a^2\sin^2\theta}{\rho^2\Delta\sin^2\theta} \end{pmatrix} \quad \text{(XXVII.471)}$$

When $a = 0$ and $M \neq 0$, $\rho^2 = r^2,\ \Delta = r^2(1 - \frac{2M}{r})$, and the metric is

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + r^2\sin^2\theta d\phi^2 + \left(1 - \frac{2M}{r}\right)^{-1}dr^2 + r^2 d\theta^2,$$

the Schwarzschild metric in Schwarzschild coordinates.

The most important difference between the geometry of a rotating star or black hole and the geometry of a stationary object is the *dragging of inertial frames* (or just *frame dragging*), a nonzero t-ϕ metric component, meaning that time-translation is not orthogonal to rotation, $t^\alpha\phi_\alpha \neq 0$. The angular velocity

$$\omega \equiv -\frac{t^\alpha\phi_\alpha}{\phi^\beta\phi_\beta}, \quad \text{(XXVII.472)}$$

measures the frame dragging in the sense that particles with zero angular momentum move along trajectories whose angular velocity relative to infinity is $d\phi/dt = \omega$. That is, the angular momentum of a particle is $L = p_\alpha\phi^\alpha = p_\phi = mu_\phi$. For a freely falling particle (and hence for an inertial observer), L is conserved. But the angular velocity of an observer, measured from infinity is $\Omega = d\phi/dt = u^\phi/u^t$. Now

$$u_\phi = g_{\phi t}u^t + g_{\phi\phi}u^\phi + = u^t\boldsymbol{\phi}\cdot\boldsymbol{t} + u^\phi\boldsymbol{\phi}\cdot\boldsymbol{\phi}, \quad \text{(XXVII.473)}$$

so a particle with zero angular momentum has angular velocity

$$\Omega \equiv \frac{u^\phi}{u^t} = -\frac{\boldsymbol{t}\cdot\boldsymbol{\phi}}{\boldsymbol{\phi}\cdot\boldsymbol{\phi}} = \omega. \quad \text{(XXVII.474)}$$

A zero-angular momentum observer (ZAMO) at constant radius r has 4-velocity u^α normal to a $t =$ constant surface: That is, the vectors $\boldsymbol{\partial}_\phi = \boldsymbol{\phi}$, $\boldsymbol{\partial}_r$ and $\boldsymbol{\partial}_\theta$ span the vectors tangent to a constant t surfacace; u^α is proportional to $t^\alpha + \omega\phi^\alpha$, which is obviously orthogonal to the vectors in the r and θ directions, and it is orthogonal to ϕ^α because

$$(t^\alpha + \omega\phi^\alpha)\phi_\alpha = t^\alpha\phi_\alpha - t^\alpha\phi_\alpha = 0. \tag{XXVII.475}$$

For Kerr, from its definition and the form (XXVII.466) of the metric ω is

$$\omega = \frac{2Mar}{(r^2+a^2)^2 - a^2\Delta\sin^2\theta}. \tag{XXVII.476}$$

More on frame dragging when we get to particle orbits.

The horizon

Because the spacetime is stationary and axisymmetric, rotations and time-translations map the horizon to itself. This implies that t^α and ϕ^α are tangent to the horizon H. (This is because ϕ^α is tangent to the circular trajectories of rotated points, and t^α is tangent to the trajectories of time-translated points.) We now show that the horizon of the Kerr geometry is the set of points where $\Delta = 0$. We begin with an outline of the argument and then fill in the argument for each step. Here's the outline:

1. The event horizon is a null surface; that is, it is a hypersurface whose tangent space includes one null vector and no timelike vectors.
2. The null vector field of the Kerr horizon (and of any stationary axisymmetric horizon) is a linear combination of the two Killing vectors, $t^\alpha + f\phi^\alpha$.
3. The metric on a null surface has vanishing determinant.
4. Because the space of vectors spanned by t^α and ϕ^α is a plane and that plane is null on the horizon, the determinant ${}^2g = \Delta\sin^2\theta = 0$ on the horizon.

Here are the details:
1. The fact that the horizon H is a null surface follows from its definition as boundary of the set of points from which timelike curves can reach null infinity. From H and from points inside H, no null or timelike curves reach null infinity. From each point outside H, outwardly directed null rays do reach null infinity. Outwardly directed null rays beginning at points of the horizon, however, remain on the horizon. Thus a null geodesic passes through each point of H. Because one can regard H as formed by these rays, the horizon is said to be *generated* by its null geodesics or by the null vector field ℓ^α tangent to them. A horizon then H is a null hypersurface, a hypersurface whose tangent space includes one null direction and no timelike vectors.

Why can H have no timelike vectors? If u^α is a timelike vector on H and v^α a spacelike vector pointing out from H, then for small enough λ, $u^\alpha + \lambda v^\alpha$ is still timelike, and it points out of H. Then a timelike curve tangent to $u^\alpha + \lambda v^\alpha$ reaches a point outside H. Because all points outside H are joined by timelike curves to null infinity, H itself would be joined by a timelike curve to null infinity, contradicting its definition. (The argument is equally valid for black hole with an axisymmetric accretion disk outside the horizon, and H is again composed of t-ϕ surfaces for which $\boldsymbol{t}\cdot\boldsymbol{t}\ \boldsymbol{\phi}\cdot\boldsymbol{\phi} - (\boldsymbol{t}\cdot\boldsymbol{\phi})^2 = 0$.)

2. We next show that the generators ℓ^α of the horizon are linear combinations of t^α and ϕ^α, that they have no components along θ or r. Because $t^\alpha + k\phi^\alpha$ is a Killing vector for every constant k, it

must be tangent to the horizon. (Combinations of rotations and time translations map the horizon to itself.) Suppose that the null generator had a part along θ or r: $\ell^\alpha = (t^\alpha + f\phi^\alpha) + v^\alpha$, with v^α orthogonal to the t-ϕ plane. Because the r and θ directions are spacelike, $v_\alpha v^\alpha \geq 0$, vanishing only if $v^\alpha = 0$. But

$$0 = \ell_\alpha \ell^\alpha = (t_\alpha + f\phi_\alpha)(t^\alpha + f\phi^\alpha) + v_\alpha v^\alpha.$$

Because each term on the right is non-negative, $v^\alpha = 0$ and $\ell^\alpha = t^\alpha + f\phi^\alpha$ as claimed.

3. Consider an element of area in a null plane with one segment along the null direction, and one in a perpendicular direction (call them the 1 and 2 directions). Because the null segment has proper length $d\ell_1 = 0$, the area is $dA = d\ell_1 d\ell_2 = 0$. Thus a null plane has ${}^2g = 0$.

4. At each point of the horizon of Kerr, the t-ϕ plane is null, whence Eq. (XXVII.470) implies $\Delta = 0$. $\Box$

For a Schwarzschild black hole, the asymptotically timelike Killing vector t^α becomes null on the horizon and is therefore its null generator. For Kerr, the vector field $t^\alpha + \omega\phi^\alpha$ is null on the horizon, because

$$\begin{aligned}
(t^\alpha + \omega\phi^\alpha)(t_\alpha + \omega\phi_\alpha) &= t^\alpha t_\alpha + 2\omega t^\alpha \phi_\alpha + \omega^2 \phi^\alpha \phi_\alpha \\
&= (\phi^\beta\phi_\beta)^1[(t^\alpha t_\alpha)(\phi_\beta\phi_\beta) - 2(t^\alpha\phi_\alpha)^2 + (t^\alpha\phi_\alpha)^2] \\
&= (\phi^\beta\phi_\beta)^{-1}[(t^\alpha t_\alpha)(\phi^\beta\phi_\beta) - (t^\alpha\phi_\alpha)^2] \\
&= 0 \ \text{at} \ \Delta = 0.
\end{aligned}$$

We have shown that the horizon is at $r = r_+$ and that it is generated by the null vector field $t^\alpha + \omega_H \phi^\alpha$, where ω_H is the value of the frame-dragging angular velocity ω on H. Using $\Delta = 0$, $r = r_+ = M + (M^2 - a^2)^{1/2}$, we have

$$\omega_H = -\left.\frac{g_{t\phi}}{g_{\phi\phi}}\right|_{r_+} = \frac{a}{2Mr_+}. \tag{XXVII.477}$$

Because the generators of the horizon rotate relative to infinity with angular velocity ω_H, one regards the horizon itself as an object rotating with angular velocity ω_H.

Within the horizon (for $r < r_+$), there are no nonspacelike constant r paths, and $g_{rr} < 0 \Rightarrow \boldsymbol{\partial}_r$ is a timelike vector; future directed timelike curves are then constrained to have $u^r < 0$. (One can analytically extend Kerr across $\Delta = 0$, as in the Kruskal or Eddington-Finkelstein extensions of Schwarzschild.)

Asymptotically, the metric (XXVII.466) has the form

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 - \frac{4Ma}{r}\sin^2\theta dt d\phi + \left(1 + \frac{2M}{r}\right)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\ ,$$

This is also the asymptotic form of the metric of a rotating star. In the case of a rotating star, the star's angular momentum is $J = Ma$, and that is defined to be the angular momentum of a black hole as well.[12] Note however, that the rest of the geometry of Kerr – the quadrupole and

[12] One can also define a symmetry group of rotations and translations at spatial infinity and use the symmetries to define the associated asymptotic conserved quantities.

higher multipole moments of the metric – does not agree with the geometry of any rotating star. And rotating stars with different equations of state have different shapes and different gravitational fields (in both GR and in the Newtonian approximation).

The asymptotically timelike Killing vector becomes null on a surface that lies outside the horizon (except at $\theta = 0, \pi$ where it touches the horizon). t^α is spacelike ($t^\alpha t_\alpha > 0$) inside the surface $t^\alpha t_\alpha = 0$:

$$\begin{aligned} 0 = t^\alpha t_\alpha &= g_{tt} = \frac{\Delta - a^2\sin^2\theta}{\rho^2} \implies r^2 - 2Mr + a^2 - a^2\sin^2\theta = 0 \\ r^2 &- 2Mr + a^2\cos^2\theta = 0. \end{aligned} \tag{XXVII.478}$$

This region, where t^α is spacelike, is called the *ergosphere* and is discussed in some detail below.

The rotational Killing vector is spacelike everywhere (at least for $r > 0$):

$$\phi^\alpha\phi_\alpha = g_{\phi\phi} = \frac{(r^2+a^2)^2 - a^2\Delta\sin^2\theta}{\rho^2}\sin^2\theta.$$

Then

$$\rho^2 > 0,\ \ r^2 + a^2 > a^2,\ \ r^2 + a^2 > r^2 - 2Mr + a^2 = \Delta \ \text{ and }\ 1 > \sin^2\theta$$

$\Rightarrow$

$$\frac{(r^2+a^2)^2}{\rho^2} > \frac{a^2\Delta\sin^2\theta}{\rho^2} \Rightarrow \phi^\alpha\phi_\alpha > 0\ .$$

[If $\phi^\alpha\phi_\alpha < 0$ as happens if we allow $r < 0$, there are closed timelike curves – namely the curves of constant t, r, θ, the circular trajectories along the vector field ϕ^α.]

When $a > M, (M^2 - a^2)^{1/2}$ is imaginary and thus Δ is nowhere zero. The components of the Kerr metric (XXVII.466) are singular at $r = 0$, and this singularity cannot be avoided by extending Kerr via a coordinate change: The scalar $R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}$, for example, blows up as $r \to 0, (\theta = \pi/2)$. In this case, unlike Schwarzschild, or Kerr with $a < M$, null geodesics from all points outside $r = 0$ escape to infinity, so the singularity is not shrouded by a horizon and is said to be naked. However, as mentioned above, it appears that naked singularities cannot, in fact, form from nonsingular initial data of physical fields, and, in particular, attempts to find processes that take a black hole with $a < M$ and add enough angular momentum to make $e > M$ have failed (see e.g. Wald, Annals of Physics, **82**, 548 (1974)). One reason they fail is this: A typical attempt is to drop a gyroscope with $a_{\rm gyro} \equiv \frac{J_{\rm gyro}}{M_{\rm gyro}} \gg M_{\rm gyro}$ into the $a < M$ black hole. But there is a spin-spin repulsion and in order to push the gyroscope into the black hole, you have to do so much work that $(W + M_{\rm gyro} + M)^2 > J_{\rm gyro} + J$, so the new mass of the hole, $W + M + M_{\rm gyro}$ exceeds its new angular momentum per unit mass. The statement that naked singularities cannot evolve from nonsingular initial data of physical fields is called the cosmic censorship hypothesis and its proof is a fundamental unsolved problem of classical relativity. We henceforth assume $a < m$.

Particle trajectories: General features

The appearance of accreting black holes depends in part on the trajectories of photons and orbits of massive particles. We'll restrict ourselves here to orbits in the equatorial plane. If, at ∞, $u^\phi = 0$, then $\ell = u_\phi = 0$ as well, because the metric is asymptotically Minkowski; but, as we have seen, zero angular momentum particles have nonzero angular velocity

$$\Omega = \omega. \tag{XXVII.479}$$

Because $\omega \sim r^{-3}$ as $r \to \infty$ this dragging is hard to measure unless one observes objects near a dense, rapidly rotating object – in this case, a Kerr black hole.

Consider an arbitrary timelike or null trajectory:

$$u^\alpha = u^t(t^\alpha + \Omega\phi^\alpha + v^\alpha)$$

where $v^\alpha \perp t^\alpha,\ \phi^\alpha$. In Minkowski space, a particle can have a timelike or null trajectory only if $r\sin\theta\Omega < 1$, implying $-1/r < \Omega < 1/r$. Here the angular velocity Ω is limited to a range of values that is not symmetric as seen by an observer at infinity. (The range is symmetric as seen locally by a observer with zero angular momentum). From $u_\alpha u_\alpha = -1 < 0$, we have

$$t^\alpha t_\alpha + 2\Omega t^\alpha\phi_\alpha + \Omega^2\phi^\alpha\phi_\alpha \le 0.$$

Then $g_{rr} > 0,\ g_{\theta\theta} > 0 \Longrightarrow v^\alpha v_\alpha > 0, \Rightarrow \Omega_- \le \Omega \le \Omega_+$, where

$$\Omega_\pm = -\frac{\boldsymbol{t}\cdot\boldsymbol{\phi}}{\boldsymbol{\phi}\cdot\boldsymbol{\phi}} \pm [(\frac{\boldsymbol{t}\cdot\boldsymbol{\phi}}{\boldsymbol{\phi}\cdot\boldsymbol{\phi}})^2 - \frac{\boldsymbol{t}\cdot\boldsymbol{t}}{\boldsymbol{\phi}\cdot\boldsymbol{\phi}}]^{1/2}$$

or

$$\Omega_\pm = \omega \pm (\omega^2 - \frac{\boldsymbol{t}\cdot\boldsymbol{t}}{\boldsymbol{\phi}\cdot\boldsymbol{\phi}})^{1/2}. \tag{XXVII.480}$$

These extrema are reached when $v^\alpha = 0$, that is, for motion in the equatorial plane at constant r.

For large r,

$$\omega \sim \frac{2J}{r^3} \qquad t^\alpha t_\alpha = g_{tt} \sim -1 \qquad \phi^\alpha\phi_\alpha = g_{\phi\phi} \sim r^2,$$

and $\Omega_\pm \approx \pm\frac{1}{r}$, as in Minkowski space. As long as $t^\alpha t_\alpha < 0, \Omega_- < 0$ and $\Omega_+ > 0$ and particles can rotate with or opposite to the black hole rotation. But when $t^\alpha t_\alpha > 0$ both Ω_+ and Ω_- are greater than zero: *All timelike or null trajectories rotate with the geometry* – no physical particle can remain at rest as seen by an observer at infinity. In other words, when t^α is spacelike, no physical particle can move along this Killing vector. In contrast to the Schwarzschild case, however, particles within the region where t^α is spacelike can escape to infinity because for $\Omega_- < \Omega < \Omega_+,\ \ u^r$ can be positive.

The region in which t^α is spacelike is called the ergosphere (The term is similar to *atmosphere*; the ergosphere of a dense, rapidly rotating stellar model is a solid torus, not a topological sphere.) The boundary, where $t^\alpha t_\alpha = 0$ is called the static limit by MTW (stationary limit by most other people). Observers inside the static limit can remain at constant distance from the black hole (constant r)

as long as they rotate in the positive direction with angular velocity Ω in the range $\Omega_- < \Omega < \Omega_+$. But as $\Delta \to 0$, $\Omega_- \to \Omega_+$:

$$\begin{aligned}\Omega_+ - \Omega_- &= [\omega^2 - \frac{t^\alpha t_\alpha}{\phi^\beta \phi_\beta}]^{1/2} \\ &= (\phi^\alpha\phi_\alpha)^{-1}[(t^\beta\phi_\beta)^2 - (t^\beta t_\beta)(\phi^\gamma\phi_\gamma)]^{1/2}\end{aligned} \tag{XXVII.481}$$

From eq. (XXVII.470), namely $(t^\beta\phi_\beta)^2 - (t^\beta\phi_\beta)(\phi^\gamma\phi_\gamma) = \Delta\sin^2\theta$,

$$\Omega_+ - \Omega_- \to 0 \text{ as } \Delta \to 0 .$$

So the allowed range of rotation shrinks to zero as $\Delta \to 0$, and for $\Delta = 0$ the only nonspacelike constant r path is the null trajectory along the generator of the horizon, with $\Omega = \Omega_+ = \Omega_- = \omega_H$.

Photon orbits

We will look at orbits in the equatorial plane. Massive particles have orbits for each value of the particle's speed. Photon orbits have only one speed, and so only one orbit for a given spatial direction. Where massive particles can adjust their speed to have circular orbits for all radii greater than some minimum value, photons have only a single circular orbit outside the horizon.

As in Schwarzschild, we write $k^\alpha k_\alpha = 0$ in terms of $\dot r$ and the conserved energy/$\hbar$ and angular momentum/$\hbar$,

$$\mathcal{E} = -k_\alpha t^\alpha = -k_t, \qquad \ell = k_\alpha\phi^\alpha = k_\phi. \tag{XXVII.482}$$

At $r = \infty$, a stationary observer has 4-velocity t^α, so $E = -\hbar k_t$ and $L = \hbar k_\phi$ ($E = \hbar\mathcal{E}$ and $L = \hbar\ell$) are the energy and angular momentum of a photon, if the affine parameter is chosen to equal t at infinity. The orbit of a photon moving in the equatorial plane is determined by its impact parameter[13]

$$b := \frac{\ell}{\mathcal{E}}.$$

The equatorial plane is the $\theta = \pi/2$ surface, with

$$\sin\theta = 1, \quad \rho^2 = r^2.$$

We have

$$\begin{aligned} 0 &= g^{tt}\mathcal{E}^2 - 2g^{t\phi}\mathcal{E}\ell + g^{\phi\phi}\ell^2 + g_{rr}\dot r^2 \\ &= -\frac{(r^2+a^2)^2 - a^2\Delta}{r^2\Delta}\mathcal{E}^2 + 4\frac{aM}{\Delta r}\mathcal{E}\ell + \frac{\Delta - a^2}{r^2\Delta}\ell^2 + \frac{r^2\dot r^2}{\Delta} \\ r^2\dot r^2 &= \left(r^2 + a^2 + \frac{2M}{r}a^2\right)\mathcal{E}^2 - 4\frac{aM}{r}\mathcal{E}\ell - \left(1 - \frac{2M}{r}\right)\ell^2. \end{aligned} \tag{XXVII.483}$$

[13]The term is accurate because L/E is the impact parameter in flat space, the spacetime is asymptotically flat, and $\hbar\ell$ and $\hbar\mathcal{E}$ are the energy and angular momentum measured by a stationary observer at infinity. So it is the impact parameter in our curved spacetime. Note that changing $\mathcal{E}$ and ℓ to $k\mathcal{E}$ and $k\ell$ in (XXVII.483) changes $\dot r = dr/d\lambda$ to $k\dot r$: This corresponds to a change $\lambda \to \lambda/k$ in the affine parameter, and it leaves $dr/dt = \dot r/\dot t$ unchanged. Adopting our convention that $\dot t = 1$ at infinity keeps $\dot r$ fixed for all values of k.

Dividing by r^2 and grouping terms with the same powers of r gives

$$\dot{r}^2 = \mathcal{E}^2 + \frac{2M}{r^3}(\ell - a\mathcal{E})^2 - \frac{\ell^2 - a^2\mathcal{E}^2}{r^2}, \tag{XXVII.484}$$

the equation of a particle moving in an effective potential $U = -\frac{2M}{r^3}(\ell - a\mathcal{E})^2 + \frac{\ell^2 - a^2\mathcal{E}^2}{r^2}$.

For a circular orbit, $\dot{r} = 0$, implying $\mathcal{E}^2 = U$, and $dU/dr = 0$:

$$\mathcal{E}^2 + \frac{2M}{r^3}(\ell - a\mathcal{E})^2 - \frac{\ell^2 - a^2\mathcal{E}^2}{r^2} = 0, \tag{XXVII.485}$$

$$-\frac{6M}{r^4}(\ell - a\mathcal{E})^2 + 2\frac{\ell^2 - a^2\mathcal{E}^2}{r^3} = 0. \tag{XXVII.486}$$

These are two equations relating $r, \mathcal{E}$ and ℓ; notice, however, that they are homogeneous in $(\mathcal{E}, \ell)$, that every term is proportional to $\mathcal{E}^2$, ℓ^2 or $\mathcal{E}\ell$. Dividing each equation by ℓ^2 then gives two equations for the two variables r and $b = \ell/\mathcal{E}$ and so determines the values of each, giving r and the angular velocity of the orbit. The second equation immediately gives

$$r = 3M\frac{\ell - a\mathcal{E}}{\ell + a\mathcal{E}} = 3M\frac{b - a}{b + a}. \tag{XXVII.487}$$

Substituting this value of r in the first equation (XXVII.485) yields for b the cubic equation

$$1 = U/\mathcal{E}^2 = -\frac{1}{9M^2}\frac{(b+a)^3}{b-a} + \frac{2}{27M^2}\frac{(b+a)^2}{b-a} \tag{XXVII.488}$$

To solve, we simplify as usual by appropriately picking dimensionless variables

$$y := \frac{b+a}{3M}, \quad \chi = \frac{a}{M}. \tag{XXVII.489}$$

Then $(b - a) = 3M(y - \chi/3)$, and the equation is surprisingly simple:

$$y^3 - 3y + 2\chi = 0, \tag{XXVII.490}$$

with a real solution[14] that depends on whether the orbit is corotating or counter-rotating with the black hole. Assuming $\chi = a/M > 0$, the corotating orbit is the solution given by

$$y = (-\chi + i\sqrt{1 - \chi^2})^{1/3} - (\chi - i\sqrt{1 - \chi^2})^{-1/3} \tag{XXVII.491}$$

Because $|\chi + i\sqrt{1 - \chi^2}| = 1$, we can write $-\chi + i\sqrt{1 - \chi^2} = e^{i3\psi}$, with

$$\cos(3\psi) = -\chi, \quad \sin(3\psi) = \sqrt{1 - \chi^2}. \tag{XXVII.492}$$

[14] If you have never solved a cubic, here is how to do it: First change variables to get rid of any y^2 term – in our case it's already gone. Then write $y = p + q$, $y^3 = p^3 + q^3 + 3pq(p + q)$. Our cubic then becomes $p^3 + q^3 + 3pqy - 3y + 2\chi$ =0, satisfied when $pq = 1$, $p^3 + q^3 + 2\chi = 0$, or, replacing q by $1/p$, $(p^3)^2 + 2\chi p^3 + 1 = 0$, a simple quadratic equation for p^3.

Then

$$y = 2\cos\psi = 2\cos\left[\frac{1}{3}\cos^{-1}(-\chi)\right],$$

and, from (XXVII.489) and (XXVII.487), respectively,

$$b = 6M\cos\psi - a, \quad r = 3M(1 - 2\chi/y) = M\left(3 - \frac{\chi}{\cos\psi}\right). \tag{XXVII.493}$$

As χ increases from 0 to 1 (as a increases from 0 to M), ψ increases from from $\pi/6$ to $\pi/3$ and r decreases from

$$r = 3M \text{ at } \chi = 0, \quad \text{to} \quad r = M \text{ at } \chi = 1. \tag{XXVII.494}$$

(When $\chi = 1$, the horizon itself has shrunk to $r = M$.)

Counter-rotating orbits have Ω and a with opposite signs. We can just switch the sign of a (of χ) to write

$$r = M\left(3 + \frac{|\chi|}{\cos\psi}\right), \quad \cos 3\psi = |\chi| \quad \text{(counter-rotating)} \tag{XXVII.495}$$

with r increasing from $2M$ to $4M$ as $|\chi|$ increases from 0 to 1.

To find the angular velocity $\Omega = k^\phi/k^t$, we need $\dot\phi = k^\phi$ and $\dot t = k^t$:

$$\begin{aligned}\dot\phi = k^\phi = g^{\phi\phi}k_\phi + g^{\phi t}k_t &= \frac{\Delta - a^2\sin^2\theta}{\rho^2\Delta\sin^2\theta}\ell + 2\frac{aMr}{\rho^2\Delta}\sin^2\theta\mathcal{E} \\ &= \frac{1}{\Delta}\left[\left(1 - \frac{2M}{r}\right)\ell + a\frac{2M}{r}\mathcal{E}\right], \quad \text{at } \theta = \pi/2, \end{aligned} \tag{XXVII.496}$$

$$\dot t = k^t = \frac{1}{\Delta}\left[\left(r^2 + a^2 + 2\frac{a^2M}{r}\right)\mathcal{E} - a\frac{2M}{r}\ell\right], \tag{XXVII.497}$$

giving Ω in terms of b and r,

$$\Omega = \frac{(1 - 2M/r)\,b + 2aM/r}{(r^2 + a^2 + 2a^2M/r) - (2aM/r)b}. \tag{XXVII.498}$$

Particle orbits

Finding the circular orbits in the equatorial plane is done in essentially the same way as for photons, and the basic equations are easy to write down, just changing $k^\alpha k_\alpha = 0$ to $u^\alpha u_\alpha = -1$: Again multiplying by Δ/r^2 gives Eq. (XXVII.484) with the additional term $-\Delta/r^2$ on the right side:

$$\dot r^2 = \mathcal{E}^2 - U, \quad U = -\frac{2M}{r^3}(\ell - a\mathcal{E})^2 + \frac{\ell^2 - a^2\mathcal{E}^2}{r^2} + \frac{\Delta}{r^2}. \tag{XXVII.499}$$

The conditions for a circular orbit are again $\dot r = 0$, $\ddot r = 0$, or $\mathcal{E}^2 - U = 0$, $dU/dr = 0$. Again grouping powers of r, the first relation is

$$\mathcal{E}^2 + \frac{2M}{r^3}(\ell - a\mathcal{E})^2 - \frac{\ell^2 - a^2\mathcal{E}^2}{r^2} - \frac{\Delta}{r^2} = 0. \tag{XXVII.500}$$

Because $\mathcal{E}^2 - U = 0$, we can write the second relation in the form $\frac{1}{2r}[r^2(\mathcal{E}^2 - U)]' = 0$:

$$\mathcal{E}^2 - \frac{M}{r^3}(\ell - a\mathcal{E})^2 - \frac{M}{r} - 1 = 0. \tag{XXVII.501}$$

The goal is to use the two equations to find the generalized Kepler relation $\Omega(r)$ and to find the innermost stable circular orbit. This took me about 4 pages. Because of the length of the calculation, S&T suppress the details and just give the results. We won't do much more than that, but if you want the calculation, I'll supply it on demand.

We have two equations for the three variables r, $\mathcal{E}$ and ℓ. The strategy is to solve them for $\mathcal{E}$ and ℓ in terms of r, to write Ω in terms of $\mathcal{E}$ and ℓ and so obtain $\Omega(r)$. First the suprisingly simple result, then a brief outline of the calculation. The Newtonian Keplerian angular velocity is $\Omega_N = \sqrt{\frac{M}{R^3}}$, and we have seen the form is identical for Schwarzschild. For Kerr it is modified by the frame dragging, the nonzero value of $g_{t\phi}$:

$$\begin{aligned}\Omega &= \frac{\Omega_N}{1 + a\Omega_N}, \quad \text{corotating} \\ \Omega &= -\frac{\Omega_N}{1 - a\Omega_N}, \quad \text{counter-rotating}\end{aligned} \tag{XXVII.502}$$

Outline of calculation:
This time use the variable[15] $x = \ell - a\mathcal{E}$ and write the equations in terms of $x, \mathcal{E}$ and r, using $\ell^2 - a^2\mathcal{E}^2 = x^2 + 2a\mathcal{E}x$ in the first equation. The equations are now

$$\begin{aligned}\mathcal{E}^2 &= -\frac{2M}{r^3}x^2 + \frac{x^2 + 2a\mathcal{E}x}{r^2} + \frac{\Delta}{r^2} \\ \mathcal{E}^2 &= \frac{Mx^2}{r^3} - \frac{M}{r} + 1.\end{aligned} \tag{XXVII.503}$$

We now eliminate $\mathcal{E}$ to get an equation for x and r alone and then solve for x in terms of r. First get rid of $\mathcal{E}^2$ by subtracting the second equation from the first. This lets us solve for $\mathcal{E}$ in terms of x and r without having to solve a quadratic equation:

$$2ax\mathcal{E} = -\left(1 - \frac{3M}{r}\right)x^2 + Mr - a^2 \tag{XXVII.504}$$

We now have independent expressions (XXVII.504) for $\mathcal{E}$ and (XXVII.503) for $\mathcal{E}^2$. So we get an equation for x and r alone by
[right side of (XXVII.504)]^2 = [right side of (XXVII.503)]$\times(2ax)^2$: Notice that, although it's quartic in x, it's quadratic in x^2! So we're almost done. The equation has the form

$$\alpha x^4 - 2\beta x^2 + \gamma = 0, \tag{XXVII.505}$$

[15] If you're going to try the calculation, you might want to set $M = 1$ or divide variables by M to make everything dimensionless. On the other hand, keeping M has the advantage that you can spot errors by checking that every term has the right dimension as a power of $L = [M]$.

with

$$\alpha = \left(1 - \frac{3M}{r}\right)^2 - 4a^2\frac{M}{r^3},$$
$$\beta = \left(1 - \frac{3M}{r}\right)\left(Mr - a^2\right) + 2a^2\left(1 - \frac{M}{r}\right),$$
$$\gamma = (Mr - a^2)^2, \tag{XXVII.506}$$

and the next happy fact is that, although α and β are somewhat long, the discriminant is short

$$\beta^2 - \alpha\gamma = 4a^2\Delta^2 M/r^3 = (2a\Delta\Omega_N)^2 \tag{XXVII.507}$$

(This is one line in Mathematica, using `Factor[]`. By hand it took a page.) The solution is then (after a bit more effort)

$$x = \frac{r^2\Omega_N - a}{\sqrt{1 - 3M/r - 2a\Omega_N}}. \tag{XXVII.508}$$

Everything has this same denominator, and it just gets carried along, so we'll call it Q:

$$Q = \sqrt{1 - 3M/r - 2a\Omega_N} \tag{XXVII.509}$$

Eq. (XXVII.503) gives

$$\mathcal{E} = \frac{1 - 2M/r + a\Omega_N}{Q}, \tag{XXVII.510}$$

and we have

$$\ell = x + a\mathcal{E} = \frac{(r^2 + a^2)\Omega_N - 2aM/r}{Q}. \tag{XXVII.511}$$

Finally, we find $\Omega(r)$ from Eq. (XXVII.496) and (XXVII.497) for $\dot\phi$ and $\dot t$. This is quick:

$$\dot\phi = \frac{\Omega_N}{Q}$$
$$\dot t = \frac{1 + a\Omega_N}{Q}, \tag{XXVII.512}$$

whence

$$\Omega = \frac{\Omega_N}{1 + a\Omega_N}, \tag{XXVII.513}$$

as claimed. This is the solution for a corotating orbit, with $a > 0$. For a counter-rotating (retrograde) orbit change a to $-a$ with the convention that $\Omega > 0$, or replace a by $-a$ and change the sign of Ω for the standard convention that $a > 0$ and $\Omega < 0$ for counter-rotating orbits.

The condition for stability of the circular orbit, $U'' > 0$ can be written as $[r^3(\mathcal{E}^2 - U)]'' = 0$, because $\mathcal{E}^2 - U = 0$ and $(\mathcal{E}^2 - U)' = -U' = 0$. From Eq. (XXVII.501), we have at the ISCO (innermost stable circular orbit),

$$\mathcal{E}^2 = 1 - \frac{2}{3}\frac{M}{r}. \tag{XXVII.514}$$

For the result in terms of r, see the three-equation expression (12.7.24) of S&T. The radius of the ISCO decreases from $3M$ to M as a increases from 0 to M for corotating orbits, and it increases from $3M$ to $9M$ for counter-rotating orbits.

XXVII.8 Ergospheres: The Penrose Process

Ergospheres are regions in which an asymptotically timelike Killing vector t^α becomes spacelike. They are present not only in rotating black holes, but also in models of dense, rapidly rotating stars (perfect fluids) which have been constructed numerically by Ipser & Butterworth (Ap. J. **200**, L103 (1975)) (earlier, for dust disks, by Bardeen & Wagoner, and later by a few other groups). Thus even when no horizon is present, the rotation of a star can be sufficient to force all physical particles within a region to rotate with respect to a distant observer.

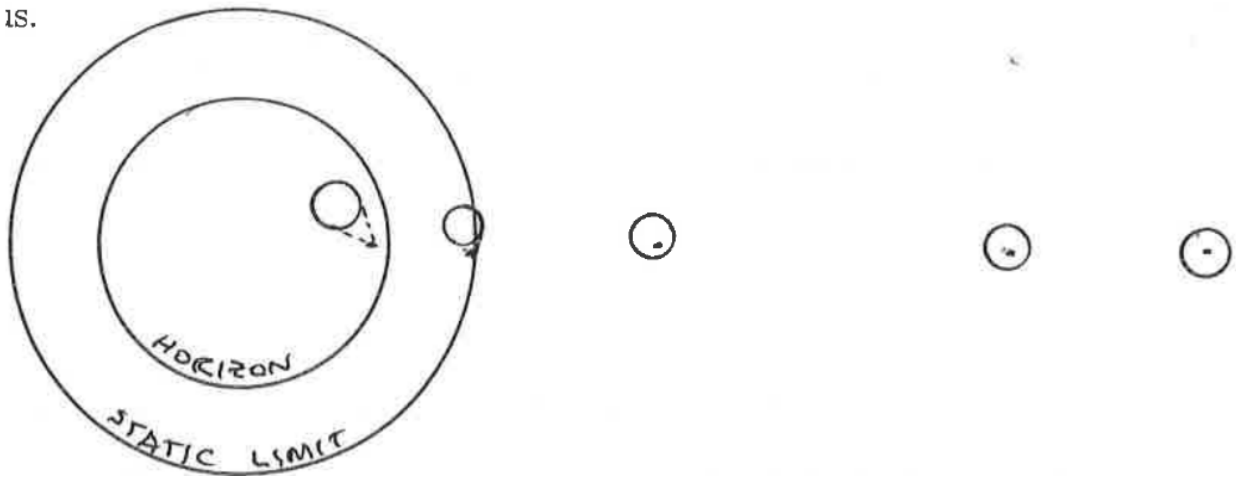


Figure 30: A t = const, $\theta = \frac{\pi}{2}$ plane showing the horizon, static limit, and ergosphere. Light cones are represented by dots for their vertices and circles for the cone - a circle represents the position of flash of light emitted a short time after it was emitted from the dot. Picture the light cone by imagining time out of the paper, so the circles sit just above the paper.

It is possible to extract rotational energy from a geometry with an ergosphere in the following way. A particle (1) sent from ∞ to the ergosphere can decay into (2) and (3), leaving (3) in the ergosphere and sending (2) back out to ∞ with greater energy than (1) had to begin with. Let particle (1) begin with momentum ${p_1}^a$. When, along its trajectory, $-{p_1}^\alpha t_\alpha = E_1$ is constant; E_1 is its energy measured by an observer at ∞ who sends the particle in (at ∞, t^α is a unit timelike vector and can be chosen as the velocity of our observer).

Inside the ergosphere, t^α is spacelike, so $p_1^\alpha t_\alpha$ is not the energy measured by any observer. In particular, consider a ZAMO, an observer with velocity $u^\alpha = u^t(t^\alpha + \omega\phi^\alpha)$. Let (1) decay into (2) + (3). The observer u^α must see $u^\alpha p_{2\alpha} < 0$ and $u^\alpha p_{3\alpha} < 0$ because the locally measured energy is positive. But $t^\alpha p_{3\alpha}$ can be positive:
$0 > u^\alpha p_{3\alpha} = u^t(t^\alpha p_{3\alpha} + \omega\phi^\alpha p_{3\alpha}) = u^t(t^\alpha p_{3\alpha} + \omega L)$,
L the angular momentum of (3).

Thus, if we choose $L < 0$, the momentum $t^\alpha p_{3\alpha}$ need only satisfy $t^\alpha p_{3\alpha} < |\omega L|$; because all decays can occur that satisfy conservation of 4-momentum, we can choose a decay for which

$t^\alpha p_{3\alpha} > 0$.
Momentum conservation in the decay is

$$p_1{}^\alpha = p_2{}^\alpha + p_3{}^\alpha.$$

Its component along t^α,

$$p_1{}^\alpha t_\alpha = p_2{}^\alpha t_\alpha + p_3{}^\alpha t_\alpha,$$

implies

$$E_2 = -p_2{}^\alpha t_\alpha = -p_1{}^\alpha t_\alpha + p_3{}^\alpha t_\alpha > E_1,$$

and particle (2) reaches infinity with energy greater than that particle (1) had. Note that particle (3) cannot escape to infinity, or even to a region where t^α is timelike because then the physical energy of (3) measured by an observer with velocity along t^α would be negative. In the case of an ergosphere about a Kerr black hole, particle (3) would spiral into the black hole. When a geometry has an ergosphere an no horizon (as in the case of uniform-density stellar models of rapidly rotating stars), a particle with $L < 0$ (or a field in the ergosphere with negative "energy") will radiate and its "energy" $p^\alpha t_\alpha$ will grow increasingly negative until ultimately the enough angular momentum is radiated to ∞ that no ergosphere remains.