Physics 718: White dwarfs, neutron stars, black holes and related topics in relativistic astrophysics

Lecture I Preliminaries

Course Description:

Texts:
Black Holes, White Dwarfs, and Neutron Stars: The Physics of Compact Objects,
by Shapiro and Teukolsky [S-T]
We will also use a set of class notes and
The Physics of Stars, by A. C. Phillips.

Additional books that might be useful:

Accretion Power in Astrophysics, by Frank, King, & Raine

High Energy Astrophysics, by Malcolm S. Longair

An Introduction to Modern Astrophysics by Carroll & Ostlie

Rotating Relativistic Stars by Friedman & Stergioulas

Gravitation (Misner, Thorne & Wheeler), *General Relativity* (Wald), *A First Course in General Relativity* (Schutz), or *Gravitation & Cosmology* (Weinberg)

In addition, much material will be related to the modern astrophysical literature. You will have to read papers. The normal place to start is ADS:
`http://adsabs.harvard.edu/abstract_service.html`.
You can search by author, title, keyword, object, etc.

After you find a paper you can click on the links to read it. Note that some papers are published in journals that you cannot read from home. Most modern papers, though, are also listed at
`http://arxiv.org`
where you can read them for free. [Note that most papers will use cgs units. Be careful!]

Evaluation will be:

- Biweekly problem sets (70%), with the lowest grade dropped
- Final Project (30%). The final project will entail a 1-page project proposal that is due on March 11 that must include expected sources, an in-class oral presentation (20 minutes), and a final written report.

I.1 Precision

We often do not know things very precisely. So we use $\sim$ and $\approx$ and related symbols. $\sim$ is for when we know something to *an order of magnitude*. So we if we know that $x \sim 5$, we know that x is between $5/3$ and $5 * 3$, where 3 is roughly $\sqrt{10}$. This means that the possible range for x is in total a factor of 10. We will also sometimes use $\sim$ to mean *scales as*. For example, if you were to estimate the height of a person as a function of their weight (for a wide range of people), you might expect that as you double the weight, the height changes by $2^{1/3}$. We could write height$\sim$ weight$^{1/3}$. There will be a lot of variation, but this is roughly correct.

$\approx$ means more precision. It doesn't necessarily have an exact definition. But generally, if we say $x \approx 5$, that means that 4 is probably OK but 2 is probably not.

Finally, we have $\propto$, which means *proportional to*. This is more precise that the *scales as* use of $\sim$. So while for a person height$\sim$ weight$^{1/3}$ is OK, for a sphere (where we know that volume is $4\pi/3r^3$) we could write volume$\propto r^3$: we take this as correct, but leave off the constants ($4\pi/3$ in this case).

I.2 Small Angles

For small angles θ, $\sin\theta \approx \tan\theta \approx \theta$ and $\cos\theta \approx 1$. We need θ to be in radians. But we also often deal with fractions of a circle. A circle has $360°$. We break each degree into 60 minute (or *arcminutes*): $1° = 60'$. And each arcminute into 60 seconds (or *arcseconds*): $1' = 60''$, so $1° = 3600''$. But we also know that 2π radians is $360°$, so we can convert between radians and arcsec. This will come up frequently: $1'' = 360 \times 3600/2\pi \approx 1/206265$ radians.

I.3 Units and Celestial Constants

Astronomy emphasizes *natural* units ($\odot$ is for the Sun, $\oplus$ is for the Earth). We also use cgs units.

- $M_\odot = 2 \times 10^{33}$ g (solar mass)
- $R_\odot = 7 \times 10^{10}$ cm (solar radius)
- $M_\oplus = 6 \times 10^{27}\,\mathrm{g} \approx 3 \times 10^{-6}\,M_\odot$(earth mass)
- $M_\mathrm{J} = 2 \times 10^{30}\,\mathrm{g} \approx 10^{-3}\,M_\odot$(Jupiter)
- $L_\odot = 4 \times 10^{33}$ erg/s (solar luminosity or power)
- light year $= 10^{18}$ cm: the *distance* light travels in one year (moving at $c = 3 \times 10^{10}\,\mathrm{cm\,s^{-1}}$)
- Astronomical Unit = AU = 1.5×10^{13} cm (distance between earth and sun)
- parsec = parallax second (we will understand this later) = pc = $3 \times 10^{18}\,\mathrm{cm} = 206,265\,\mathrm{AU}$

- energies: eV=electron volt=1.6×10^{-12} erg (typical chemical reaction is eV; typical nuclear reaction is MeV)
- temperatures: often express as $k_B T$, where $k_B = 1.4 \times 10^{-16}$ erg/K is Boltzmann's constant. $k_B T$ is an energy, can express in eV; 10^6 K is 86 eV
- Masses often expressed as energies (also in eV) via $E = mc^2$, so:
  - $m_e = 511\,\mathrm{keV}$ (electron)
  - $m_n \approx m_p \approx 1\,\mathrm{GeV}$ (neutron or proton)
  - $m_\gamma = 0$ (photon — rest mass)

And then we use usual metric-style prefixes to get things like kpc, Mpc, etc.

Google/Wolfram Alpha/astropy can be very helpful when checking unit conversions.

Lecture II Basic Concepts

Phillips Chapter 1. These are (in many cases) things we will go back over later.

II.2 Big Bang Nucleosynthesis

What ingredients do we have to make a star? Universe is mostly H, then He. Rest is details. How did it get that way?

It started out very hot (we know this since the Universe is expanding and we see the left-over radiation at 3 K now). Was a soup of interacting sub-atomic particles (electrons, positrons, neutrinos, quarks). Eventually (after 10^{-4} s) free quarks got bound up into neutrons, protons, ... Neutrons and protons in particular were in equilibrium:

$$\nu_e + n \to e^- + p$$

and

$$\bar{\nu}_e + p \to e^+ + n$$

at the same time. However, n is slightly more massive than p. At a temperature T, the ratio of these is given by the mass (energy) difference:

$$\frac{N_n}{N_p} = e^{-\Delta mc^2/k_BT}$$

This is a **Boltzmann factor** (will come back). Δmc^2 is energy difference, 1.3 MeV (remember that $m_pc^2 \approx 1$ GeV, so difference is 0.1%).

As T goes down and universe expands, the reactions go more slowly and we get more protons wrt neutrons. Finally it effectively stopped, and the ratio was frozen. This happened at $T \sim 10^{10}$ K, with $N_n/N_p \approx 1/5$. Then went down a little more (to 1/7) through natural decay of n.

At 10^9 K, could make deuteron:

$$n + p \to d + \gamma$$

From these could make ^{3}He, then ^{4}He. ^{4}He is very stable, so a lot of things got stuck there, except for a little ^{7}Li. But there were still a lot of protons left over. How much He?

$N_n/N_p \approx 1/7$. So take 2 neutrons, 14 protons (16 particles total, or a mass of ≈ 16 amu). Make a single ^{4}He nucleus, then 12 protons left. So out of 16 amu, 4 amu are in ^{4}He, or mass of He is $\approx 25\%$ total mass. This is pretty close to what we see.

II.3 Gravitational Contraction

Stars are one big fight against gravity. Temporary relief from thermonuclear fusion. But what are they fighting against?

Spherical system with M, R. Only have pressure, gravity. density is $\rho(r)$, pressure is $P(r)$.

Start at the center. How much mass out to r?

$$m(r) = \int_0^r dr' \rho(r') 4\pi r'^2$$

($dr'\rho(r')4\pi r'^2$ is the mass of a shell at r'). Gravity only cares about enclosed mass, so:

$$g(r) = \frac{Gm(r)}{r^2}$$

What about pressure? Pressure on a parcel between r and $r + \Delta r$ (area=ΔA, volume=$\Delta r \Delta A$). If pressure at the top is the same as the bottom, no net force. But what if it is not the same?

$$P(r + \Delta r) \approx P(r) + \frac{dP}{dr}\Delta r$$

so difference (top $-$ bottom, or inward) in force (pressure times area) is:

$$\left[P(r) + \frac{dP}{dr}\Delta r - P(r)\right] \Delta A = \frac{dP}{dr}\Delta r \Delta A$$

But acceleration is force / mass, and mass is volume times density ($\Delta M = \rho(r)\Delta r \Delta A$). So acceleration from pressure is:

$$\frac{dP}{dr}\frac{1}{\rho(r)}$$

The total acceleration is then:

$$\frac{d^2 r}{dt^2} = -g(r) - \frac{1}{\rho(r)}\frac{dP}{dr}$$

So if the star isn't moving, then P must increase toward the center ($dP/dr < 0$).

II.3.1 Free Fall

What if $P = 0$? Deal with energies, not acceleration. Convert potential energy to kinetic. Start at r_0, mass enclosed m_0. Initial $K = 0$, $U = -Gm_0^2/r_0$. $K + U$ is always the same, and $K = m_0 v^2/2 = m_0(dr/dt)^2/2$. So:

$$\frac{1}{2}\left(\frac{dr}{dt}\right)^2 - \frac{Gm_0}{r} = \frac{-Gm_0}{r_0}$$

Can get the time to go all the way to the center ($r = 0$):

$$t_{\rm FF} = \int_{r_0}^0 dr \frac{dt}{dr} = -\int_{r_0}^0 dr \left[\frac{2Gm_0}{r} - \frac{2Gm_0}{r_0}\right]^{-1/2}$$

The integral is a little messy, but you can show that the free-fall time $t_{\rm FF}$is just:

$$\frac{\pi}{2}\left(\frac{r_0^3}{2Gm_0}\right)^{1/2}$$

Only depends on m_0/r_0^3. What has these units? $\rho = m_0/(4\pi r_0^3/3)$! So

$$t_{\rm FF} = \sqrt{\frac{3\pi}{32G\rho}}$$

For the Sun, $1/2$ hour. But for most things, eventually Pressure will stop collapse.

II.3.2 Hydrostatic Equilibrium

Assume 0 acceleration. Then:

$$\frac{dP}{dr} = -\frac{Gm(r)\rho(r)}{r^2} = -\rho(r)g(r)$$

This is a very important result: the equation of **hydrostatic equilibrium** (HSE). Applies to any stable system (atmospheres, stars, etc).

If the whole thing is in equilibrium at all r, then this will be true everywhere. Can then look at total potential energy:

$$\int_0^R dr\, 4\pi r^3 \frac{dP}{dr} = -\int_0^R dr\, \frac{Gm(r)\rho(r)4\pi r^3}{r^2}$$

where we multiplied both sides by $4\pi r^3$ and integrated. RHS is:

$$U_{\rm G} = -\int_0^M dm\, \frac{Gm(r)}{r}$$

where $dm = 4\pi r^2 \rho(r) dr$. Integrate LHS by parts:

$$P(r)4\pi r^3|_0^R - 3\int_0^R dr\, 4\pi r^2 P(r)$$

The first term is 0 ($P(R) = 0$). Second is average P times V: $\langle P \rangle V$. So:

$$\langle P \rangle = -\frac{U_G}{3V}$$

This is a very important result — one way of expressing the **virial theorem**. Can work for lots of things. What about particles in a box?

II.3.3 Kinetic Origin of Pressure

Box with side L has N particles. Particle hits top/bottom at a rate $v_z/2L$ (collisions/s) and imparts $2p_z$ (redirects with equal velocity). So momentum per time per area is $2p_z v_z/2L/L^2 = p_z v_z/L^3$. But momentum per time is force, and force per area is pressure. Total of N particles:

$$P = \frac{N}{L^3}\langle p_z v_z \rangle$$

Assume all directions are the same, so:

$$\langle p_x v_x \rangle = \langle p_y v_y \rangle = \langle p_z v_z \rangle = \langle \vec{p} \cdot \vec{v} \rangle/3 = \frac{n}{3}\langle \vec{p} \cdot \vec{v} \rangle/3$$

n is N/V or number density.

Can then do this for different types of particles. Total energy of a particle $\epsilon^2 = p^2c^2 + m^2c^4$ (kinetic + rest-mass). NR: $p \ll mc$, so $\epsilon = mc^2 + p^2/2m$. UR: $p \gg mc$ so $\epsilon = pc$. Can show:

$$P_{NR} = \frac{2}{3}\frac{K}{V}$$

$$P_{UR} = \frac{1}{3}\frac{K}{V}$$

So if NR:

$$\langle P \rangle = \frac{2}{3}\frac{K}{V} = -\frac{U_G}{3V}$$

or $U_G = -2K$. Other ways to write this are ($E = U + K$): $E = -K$, $E = U/2$. Overall $E < 0$ so the system is bound.

Strange consequence: add a little energy slowly. Add 1% of total energy, U goes down by 2%, K goes up by 1%. So for a contracting cloud converting gravitational energy to radiation (early model for the Sun) will get hotter (K goes up) as it contracts.

What if energy comes from nuclear reactions at the center? E goes up, so K goes down. Therefore it cools down! Adding energy makes it cooler?

II.4 Ideal Gas Law

You may know from chemistry:

$$PV = n_{\mathrm{mol}}RT$$

with n_{mol} the # of moles, and R the ideal gas constant. But we are physicists. So the total number of molecules is $N_{\mathrm{molec}} = N_A n_{\mathrm{mol}}$, and we can write:

$$PV = N_{\mathrm{molec}}\frac{R}{N_A}T$$

Divide both sides by V:

$$P = \frac{N_{\mathrm{molec}}}{V}\frac{R}{N_A}T$$

Have $k_B = R/N_A$ is Boltzmann's constant. And:

$$n \equiv \frac{N_{\mathrm{molec}}}{V}$$

is the **number density**: the number of particles per volume (units are cm^{-3}, since number doesn't have a unit).

$$P = nk_BT$$

Can also write in terms of **mass density** ρ ($\mathrm{g\,cm}^{-3}$):

$$\rho \equiv m_{\mathrm{molec}}n$$

so

$$P = \frac{\rho}{m_{\mathrm{molec}}}k_BT$$

II.5 Star Formation

Cloud collapses under influence of gravity. Details complicated. But basic conditions must be satisfied. Gravity must be stronger than pressure (kinetic energy).

$$U = -f\frac{GM^2}{R}$$

(f depends on density distribution, $f \sim 1$).

$$K = \frac{3}{2}Nk_BT$$

Need $|U| > K$ for collapse. Can write this as:

$$M > M_J = \frac{3k_BT}{2G\bar{m}}R$$

where $\bar{m}$ is average mass of particle ($M = N\bar{m}$). Or

$$\rho > \rho_J = \frac{3}{4\pi M^2}\left(\frac{3k_BT}{2G\bar{m}}\right)^3$$

These are the **Jeans mass and density**.

So want a big cloud to collapse. But does a big cloud make a big star? Generally it breaks up along the way (*fragmentation*).

$T = 20\,\mathrm{K}$, $M = 10^3 M_\odot$, needs $\rho = 10^{-25}\,\mathrm{g/cm^3}$ ($n = 0.1\,cm^{-3}$) to collapse (not too bad). But for $1M_\odot$ density needs to be 10^6 times higher.

II.6 The Sun

$M = 1M_\odot$, $R = 1R_\odot$. So average density is $1.4 \times 10^0\,\mathrm{g\,cm^{-3}}$. $t_{\mathrm{FF}} = 30\,\mathrm{min}$, which isn't happening, so there must be pressure.

$$\langle P\rangle = \frac{-U}{3V} \approx \frac{1}{3}\frac{GM_\odot^2}{R_\odot}\frac{3}{4\pi R_\odot^3} = \frac{GM_\odot^2}{4\pi R_\odot^4} \approx 10^{14}\,\mathrm{Pa}$$

Can also say $\langle P\rangle = \langle\rho\rangle k_BT/\bar{m}$ (ideal gas law). $\bar{m} \approx 0.5\,\mathrm{amu}$ (ionized H). So

$$k_BT \approx \frac{GM_\odot\bar{m}}{3R_\odot} \approx 0.5\,\mathrm{keV}$$

or $T \approx 6 \times 10^6\,\mathrm{K}$. Hotter (and denser etc.) toward center.

II.6.1 What Powers the Sun and How Long Will It Last?

We take the Solar luminosity to be 4×10^{33} erg/s, and try to find a way to get that amount of energy out over a long time.

The first estimate was due to Lord Kelvin (1862, in Macmillan's Magazine). This estimate (known now at the Kelvin-Helmholtz time, $t_{\rm KH}$) was shown to be $< 100\,$Myr. But Darwin said (at the time) that fossils were at least 300 Myr old. So something weird was going on. Kelvin's estimate may have been wrong by a bit, but it couldn't be that bad. So there had to be some unknown energy source.

The lifespan of the Sun could be due to:

1. Chemical energy
2. Gravitational energy
3. Thermal energy (could it have just been a lot hotter in the past?)
4. Fission?

The answers for all of these are no. Kelvin's estimate concerned specifically gravitational. Chemical energy isn't enough, since we know about how much chemical energy a given reaction can release for a given amount of stuff. Same with fission.

II.6.2 Gravito-Thermal Collapse, or the Kelvin-Helmholtz Timescale

This ascribes the luminosity to the change in total energy: L is change in $E = K + U$.

If you do this you get a timescale of $t_{\rm KH} \sim 10^7$ yr, which is $\gg t_{\rm ff}$:

$$t_{\rm KH} \sim \frac{E}{L}$$

But $E \sim GM_\odot^2/R_\odot \sim 10^{48}$ erg.

That is because as collapse occurs, $|U|$ increases so K increases too. That heats up the star, which slows down the collapse.

We can use the Virial theorem to get the central temperature T_c of the Sun. We assume that the center (the hottest/densest bit) dominates K:

$$K \sim \frac{3}{2} k_B T_c \frac{M}{\bar{m}}$$

with $\bar{m} \approx m_p$ the average particle mass. And $K = -U/2$, with $U \sim -GM_\odot^2/R_\odot$. So we find $T_c \sim GM_\odot m_H/k_B R_\odot \sim 10^7\,$K. This is pretty good (the real number is about $1.6 \times 10^7\,$K).

II.6.3 Stellar Radiation

Stars are not quite blackbodies. But we describe them by their effective (surface) temperature $T_{\rm eff}$ and defined by

$$L = 4\pi R^2 \sigma T_{\rm eff}^2. \tag{II.1}$$

where R is the radius and L the luminosity. We also infer the luminosity from the flux F with

$$L = 4\pi d^2 F \tag{II.2}$$

(with d =distance). Note, though, that instead of fluxes and luminosities we often use magnitudes, with

$$m = -2.5 \log_{10} \frac{F}{F_0} \tag{II.3}$$

, comparing the observed flux against some reference flux F_0. Apparent magnitudes relate to fluxes, and absolute magnitudes to lumonisities (defined by apparent magnitude at a distance of 10 pc).

We can contrast "bolometric" quantities (integrated over all wavelengths) and actual observed quantities, which are defined over finite bandpasses. Then instead of measuring temperature directly we use "color" as a proxy:

$$B - V = m_B - m_V = -2.5 \log_{10} F_B/F_V \tag{II.4}$$

is the magnitude difference (flux ratio) between the B(lue)=4000 Å and V(isual)=5500 Å bands. So a hotter star would be bluer and would have $B - V < 0$. We put these together on a "color-magnitude" or Hertzprung-Russell (HR) diagram. Eventually people recognized different parts of the diagram: main sequence (where stars spend most of their lives burning H→He) with $L \sim M^{3.5}$, followed by giant branc(es) and white dwarfs.

For the Sun, $T_{\rm Eff} \approx 6000$ K. This means the blackbody peaks in the visible portion of the spectrum. And this is much cooler than the interior.

How does it get from very hot interior to cool exterior?

Center of the Sun: nuclear reaction releases energy in the form of neutrinos (which escape) and photons (gamma-rays). How long to get out? A naive answer is $\sim R_\odot/c = 2$ s. But not for photons.

It actually takes $\sim 10^7$ yrs. Why? Because a star is a very crowded place, and photons (even though they move fast) cannot move very far before they wack into something else and end up going in another direction. They easily bounce (scatter) off of ions, electrons, and atoms, and even other photons.

Each bounce tends to make the photon lose energy, but more photons are then produced, conserving energy. In the center the photons start out as X-ray photons, but by the time they get to the surface of the star they are optical photons. They get there via a *random walk*.

Assume that a photon will move (on average) a distance $l_{\rm mfp}$ before it hits something and changes direction. That distance is the *mean free path*. It travels a distance d after N collisions. We can determine what $d(N)$ is. Assume each one moves $\vec{l}_i$ for $i = 1 \dots N$, with $|\vec{l}_i| = l_{\text{mfp}}$. So the total distance is the vector sum:

$$\vec{d} = \sum^{N} \vec{l}_i$$

We want the magnitude of this, $|\vec{d}| = \sqrt{\vec{d} \cdot \vec{d}}$. But

$$\vec{d} \cdot \vec{d} = \sum_{i}^{N} \vec{l}_i \cdot \vec{l}_i + \sum_{i \neq j} \vec{l}_i \vec{l}_j$$

The second term there will go to 0 on average, since the directions are different. So $|\vec{d}|^2 = N|\vec{l}| = Nl_{\text{mfp}}$, or $d = \sqrt{N} l_{\text{mfp}}$. This is in fact a general result with applicability to a wide range of areas.

From this we can determine how long does it take for a photon to diffuse out of the star. To go a distance d, it takes:

$$\begin{array}{cl} \frac{d}{c} & l_{\text{mfp}} > d \\ N\frac{l_{\text{mfp}}}{c} = \frac{d^2}{l_{\text{mfp}} c} & l_{\text{mfp}} < d \end{array}$$

This is also often referred to as a "drunkard's walk". So to go $R_\odot$ it takes:

$$\frac{R_\odot^2}{lc}$$

which is a factor of $R_\odot / l$ longer than basic escape. So luminosity (energy per time) also changes by that factor. Naive luminosity for central temperature is:

$$L = 4\pi R_\odot^2 \sigma T_I^4$$

but in reality it is $T_{\text{Eff}} = 6000\,\text{K} \ll T_I = 6{,}000{,}000\,\text{K}$. So:

$$T_{\text{Eff}} \approx T_I \left(\frac{l}{R_\odot} \right)^{1/4}$$

which would give $l \sim 1\,\text{mm}$ (very small!). Which would give about 50,000 yr to diffuse (too small, but not horrible).

II.7 Stellar Life Cycles

Big Bang: mostly H and He. Stars make the rest. T at the center of a star is pretty close to constant, set by fusion (hotter $\rightarrow$ faster $\rightarrow$ bigger $\rightarrow$ cooler). So $M/R \sim$ constant.

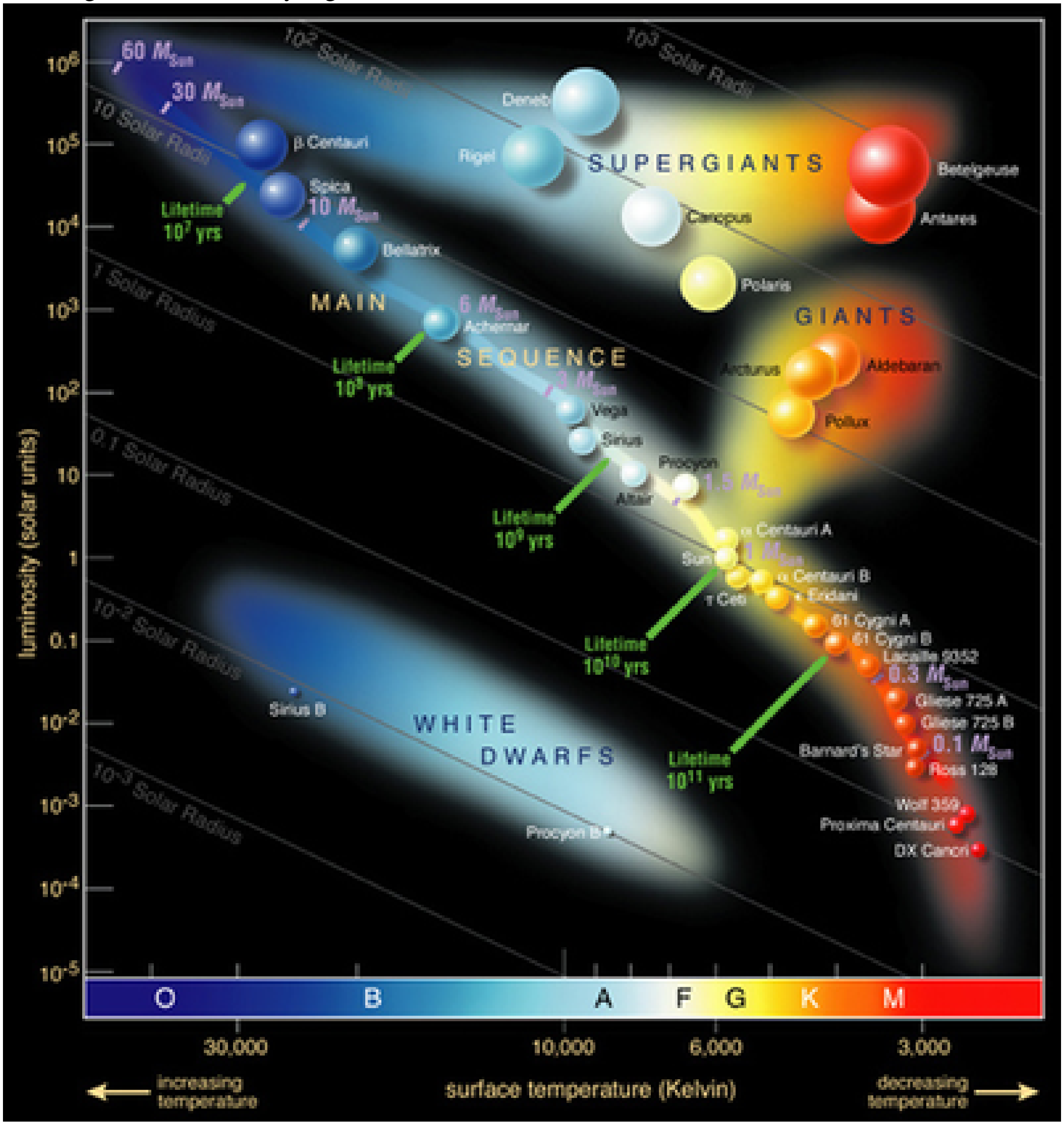
Energy escape determines luminosity. Since $L \sim R^2 T_I^4 (l/R) \sim R^2 (M/R)^4 (l/R) \sim \rho M^3$.

Since $L \sim M^3$ (roughly), determined by how fast energy can escape. So lifetime is $\sim M/L \sim M^{-2}$: bigger stars use up their fuel much faster. About 10^{10} yr for the Sun.

II.8 Color-Magnitude Diagram

Plot $T_{\rm Eff}$ increasing to the left, L increasing up. Hotter is the same as bluer, so often plot color (blue to the left) on the x-axis. We can directly observe color. And instead of L plot magnitude, where $m = m_\odot - 2.5\log_{10}(L/L_\odot)$. So it decreases going up, but that still means brighter.

Most of the stars define the **Main Sequence**. This turns out to be where normal H fusion is occuring. Can also identify regions for **Red Giants** and **White Dwarfs**.



Lecture III Fluids

(See, e.g., *The Feynman Lectures in Physics*, vol. II, Chap. 40, http://www.feynmanlectures.caltech.edu; and the Thorne-Blandford giant *Modern Classical Physics*, Chapter 13, http://www.pmaweb.caltech.edu/Courses/ph136/yr2012/)

In physics, the word *fluid* refers both to liquids and gases. More generally, what distinguishes a fluid from a solid is that a fluid cannot maintain a shear stress.[1] The stress is normal to any surface that is at rest relative to the fluid, and its magnitude is independent of the orientation of the surface. Suppose that inside a fluid one makes a small cut, say the vertical cut on shown in the figure below. The cut separates the matter on one side of a small plane from the matter on the other side. What force $\boldsymbol{F}$ is needed to keep the matter on the left side of that plane in the state it would have been in had there been no cut?

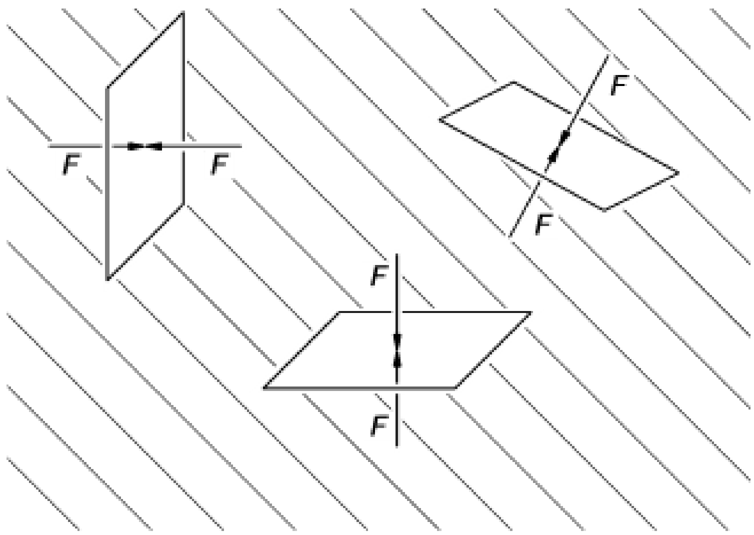


Figure 1: (This is Fig. 40-1 of the *Feynman Lectures*, vol. II)

This $\boldsymbol{F}$ is, of course, the same force that the right side was exerting on the left side just before the cut was made (and is equal and opposite to the force exerted by the left side side on the right side). Let $\boldsymbol{A} = \boldsymbol{n}A$, where A is the area of the cut and $\boldsymbol{n}$ is a unit normal pointing to the right. Then for small cuts, $\boldsymbol{F} \propto \boldsymbol{A}$ and its magnitude is independent of the orientation of the surface: $\boldsymbol{F} = P\boldsymbol{A}$, with P the pressure.

Solids, on the other hand, maintain shear stresses: The force needed to keep the matter on one side of a cut in place is not, in general, normal to the cut, and it depends on the orientation: For small cuts, the force is linear in the area, but the general linear map from an element of area to a force (from a vector to a vector) is a tensor,

$$F^a = T^a{}_b A^b. \tag{III.5}$$

[1] An *imperfect fluid*, a fluid with viscosity, is intermediate between a perfect fluid and a solid. We are assuming here that viscosity is negligible.

The map $T^a{}_b$ from areas to forces is is called the *stress tensor*. For a fluid, $T^a{}_b$ is independent of orientation – invariant under rotations, and the only 2-index tensors invariant under rotations are multiples of the identity:

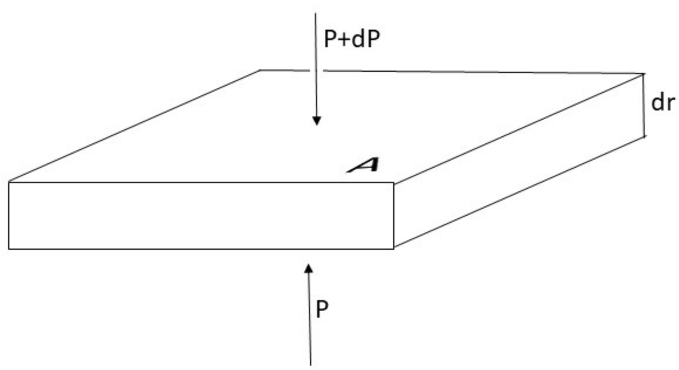
$$T^a{}_b = P\delta^a_b, \tag{III.6}$$

implying $F^a = PA^a$.

III.1 Hydrostatic Equilibrium of a Newtonian Star

A star is pulled together by gravity and supported against collapse by pressure: The pressure below a column of fluid supports the weight of the fluid above it in a spherical (nonrotating) star. This means that the pressure $P(r)$ at radius r is equal to the weight of a column of unit area extending from r to the surface of the star.

The structure of a star is determined by the balance of pressure and gravity on a slab of fluid (*fluid element*) of area A and thickness dr shown in the figure. Let M, R be the total mass and radius of the star, $\rho(r)$ the density of the fluid at radius r and $m(r)$ the mass inside the radius r. The pressure P at the bottom of the fluid element is larger than the pressure $P + dP$ at the top ($dP < 0$), and the difference between the force on the bottom and the force on the top is equal to the weight of the fluid.



The pressure P at radius r exerts a force $F_r = PA$ on the bottom of the fluid element; the smaller pressure $P + dP$ at radius $r + dr$ exerts a force $F_r = -(P + dP)A$ on the top. The buoyant force, the net force on the fluid element due to the pressure difference, is then

$$PA - (P + dP)A = -AdP.$$

The gravitational force on the mass dm of the fluid element cares only about the enclosed mass $m(r)$: $F_r = -g\, dm = -\dfrac{Gm(r)dm}{r^2}$.

The balance between the pressure gradient and gravity is then given by

$$\begin{aligned} -AdP = \frac{Gm(r)dm}{r^2} &= \frac{Gm(r)}{r^2}\rho A dr \\ \frac{dP}{dr} &= -\frac{Gm(r)}{r^2}\rho. \end{aligned} \tag{III.7}$$

This is the equation of hydrostatic equilibrium.

Summary:
Hydrostatic equilibrium of a star is governed by the equations (III.7),

$$m(r) = \int_0^r dr' \rho(r') 4\pi r'^2, \tag{III.8}$$
$$\frac{d\Phi}{dr} = -\frac{1}{\rho}\frac{dP}{dr}, \quad r \leq R \tag{III.9}$$

and

$$\nabla^2\Phi = 4\pi G\rho, \quad \Phi \to 0 \text{ as } r \to \infty. \tag{III.10}$$

The last two equations, (III.9) and (III.10), are redundant inside the star.

If we know the equation of state, the relation $P = P(\rho)$ between pressure and density, we can simultaneously integrate Eqs. (III.7) and (III.8) to find the quantities $P(r)$, $\rho(r)$ and $m(r)$ that describe the structure of the star: For each central density, there will be one star, obtained by integrating the equations outward until reaching the radius at which the pressure vanishes – the surface of the star.

This is how one models white dwarfs and neutron stars (with the relativistic equation of hydrostatic equilibrium replacing the Newtonian approximation for neutron stars). For living stars, the pressure depends on temperature as well as density $P = P(\rho, T)$ and one must adjoin equations determining the temperature. These could include heat flow (radiative transfer), reaction rates, and densities of each nuclear species.

III.2 Hydrodynamics A: $\boldsymbol{F} = m\boldsymbol{a}$ for radial motion

If a spherical star is not static – if it is contracting, expanding or spherically oscillating, the sum of the forces due to pressure and gravity is mass× acceleration $= dm\, a_r = \rho\, Adr\, \dfrac{d^2r}{dt^2}$:

$$-\frac{dP}{dr}Adr - \frac{Gm(r)}{r^2}\rho\, Adr = \rho\, Adr\, \frac{d^2r}{dt^2}.$$

Then the motion is governed by the equations

$$\frac{d^2r}{dt^2} = -\frac{Gm(r)}{r^2} - \frac{1}{\rho(r)}\frac{dP}{dr}, \qquad m(r) = \int_0^r dr' \rho(r') 4\pi r'^2. \tag{III.11}$$

Free Fall

What if $P = 0$? For free fall from rest, the matter initially inside a shell at r_0 stays inside the contracting shell. This is not obvious, but we'll start by assuming it and then show that the resulting solution has that property.[2] For a particle initially at r_0, the mass enclosed is then always its intial

[2] This is enough to verify the assumption, because solutions to the dynamical equations with specified initial conditions are unique: That is, we find a solution, we are guaranteed that it is the unique solution, and it satifies the assumption. The assumption is therefore true.

value, $m(r(t)) = m(r_0) \equiv m_0$. This allows us to write conservation of energy E for a freely falling particle of mass m and initial radius r_0 in the form $E = K + U = \frac{1}{2}\mathrm{m}\dot{r}^2 - \frac{Gm_0\mathrm{m}}{r^2}$. At $t = 0$, we have $\dot{r} = 0$ and $r = r_0$, implying $E_0 = -Gm_0\mathrm{m}/r_0^2$. Then conservation of energy, $E/\mathrm{m} = E_0/\mathrm{m}$, has the form

$$\frac{1}{2}\left(\frac{dr}{dt}\right)^2 - \frac{Gm_0}{r} = -\frac{Gm_0}{r_0} \tag{III.12}$$

with solution

$$t = \int_{r_0}^{r} dr\frac{dt}{dr} = -\int_{r_0}^{r} dr\left[\frac{2Gm_0}{r} - \frac{2Gm_0}{r_0}\right]^{-1/2}. \tag{III.13}$$

This is simpler when written in terms of $x = r/r_0$, the fraction of the initial radius:

$$t = \sqrt{\frac{r_0^3}{2Gm_0}}\int_x^1 dx'\left[\frac{1}{x'} - 1\right]^{-1/2} \equiv f(x). \tag{III.14}$$

The time to reach the center, $r = 0$, is the free-fall time t_{FF}. Using $\displaystyle\int_0^1 dx\left[\frac{1}{x} - 1\right]^{-1/2} = \frac{\pi}{2}$, we have

$$t_{\mathrm{FF}} = \frac{\pi}{2}\sqrt{\frac{r_0^3}{2Gm_0}}, \tag{III.15}$$

a time that depends only on m_0/r_0^3. What is this? The average density inside r_0 at $t = 0$ is $\rho_0 = m_0/(4\pi r_0^3/3)$. So the free-fall time depends only on the initial density: With negligible pressure, a large or a small cloud with the same initial density takes the same time to collapse,

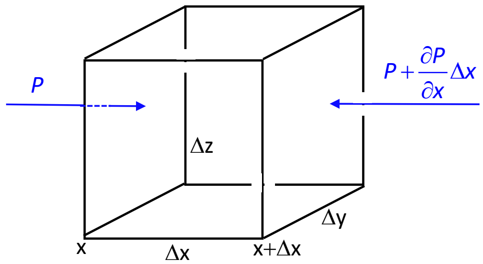
$$t_{\mathrm{FF}} = \sqrt{\frac{3\pi}{32G\rho_0}}. \tag{III.16}$$

Note that the relation $t \propto \rho^{-1/2}$ follows from dimensional analysis: $\frac{1}{\sqrt{G\rho}}$ is the only combination of G, m and r whose dimension is time. Similarly, the period of a particle in circular orbit near the surface of a star or planet, the period of radial oscillations, and the time a sound wave takes to cross a star are each of order $\frac{1}{\sqrt{G\rho}}$. For the Sun, this characteristic time is 1/2 hour.

Because the free-fall time for a shell decreases as the average density inside the shell increases, and the density is higher at smaller radii, our initial assumption is right: Matter inside a collapsing shell stays inside the shell. (Explictly, the integral (III.14) is $t = f(x) = \sqrt{\frac{3}{8\pi G\rho_0}}\left[\cos^{-1}\sqrt{x} + \sqrt{x(1-x)}\right]$ and can be found by the substitution $x = \cos^2\theta$.)

III.2.1 The Euler Equation for general motion

The Euler equation is the equation of motion, $\boldsymbol{F} = m\boldsymbol{a}$ for a fluid element. For radial motion, it is Eq. (III.11), and we'll now derive the 3-dimensional version. Again consider a fluid element, shown here as a small box of fluid with a density ρ and velocity $\mathbf{v}$.



The pressure on the left face is $P(x)$; the pressure on the right face is $P(x+\Delta x) = P(x) + \dfrac{\partial P}{\partial x}\Delta x$. With $A = \Delta y \Delta z$ the area of the left and right faces of the box, the net force in the x-direction is

$$\begin{aligned} F_x &= P(x)A - P(x+\Delta x)A \\ &= -\frac{\partial P}{\partial x} V. \end{aligned}$$

where $V = \Delta x \Delta y \Delta z$ is the volume of the fluid element. Replacing the index x by y and z, we have

$$\boldsymbol{F} = -\boldsymbol{\nabla} P \, V.$$

We want to write $\boldsymbol{F} = m\boldsymbol{a}$ or

$$-\boldsymbol{\nabla} P \, V = \rho V \, \boldsymbol{a},$$

and we need to find $\boldsymbol{a}$ in terms of the velocity field $\mathbf{v}(x,t)$. The vector field $\mathbf{v}(\mathbf{x},t)$ has the meaning that at time t the fluid element at $\mathbf{x}$ has velocity $\mathbf{v}(\mathbf{x},t)$. Thus at time $t+\Delta t$ that same fluid element is at $\mathbf{x} + \mathbf{v}(\mathbf{x},t)\Delta t$ and has velocity $\mathbf{v}(\mathbf{x} + \mathbf{v}(\mathbf{x},t)\Delta t, t+\Delta t)$. The fluid element has changed its velocity by

$$\begin{aligned} \Delta \mathbf{v} &= \mathbf{v}(\mathbf{x} + \mathbf{v}(\mathbf{x},t)\Delta t, t+\Delta t) - \mathbf{v}(\mathbf{x},t) \\ &= \left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) \Delta t \end{aligned}$$

in time Δt, and its acceleration is therefore

$$\boldsymbol{a} = (\partial_t + \mathbf{v} \cdot \nabla)\, \mathbf{v}. \tag{III.17}$$

In this way we obtain Euler's equation of motion ("Principes généraux du mouvement des fluides," Mémoires de l'Académie des Sciences de Berlin, 1757)

$$\rho\, (\partial_t + \mathbf{v} \cdot \boldsymbol{\nabla})\, \mathbf{v} = -\boldsymbol{\nabla} P\,. \tag{III.18}$$

In the presence of a gravitational field, with potential Φ satisfying $\nabla^2 \Phi = 4\pi G \rho$, there is an additional force $-\rho V \, \boldsymbol{\nabla}\Phi$ on each fluid element; and the Euler equation becomes

$$\rho\, (\partial_t + \mathbf{v} \cdot \boldsymbol{\nabla})\, \mathbf{v} = -\boldsymbol{\nabla} P - \rho \boldsymbol{\nabla}\Phi. \tag{III.19}$$

III.2.2 Conservation of mass: The continuity equation

As a fluid element moves its volume changes. Because its mass is conserved (in our present Newtonian approximation) a fractional increase $\Delta V/V$ in its volume is equal to the fractional decrease $\Delta\rho/\rho$ in its density.

$$\rho V = \text{ constant } \Longrightarrow \frac{\Delta\rho}{\rho} = -\frac{\Delta V}{V} \quad \text{or} \quad \frac{d\rho/dt}{\rho} = -\frac{dV/dt}{V}, \tag{III.20}$$

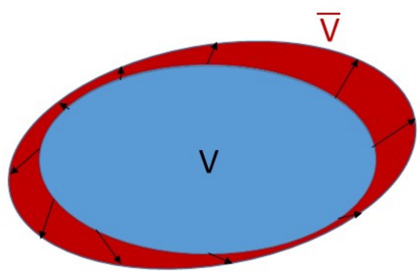
where $\rho = \rho(t, \mathbf{x}(t))$, $V = V(t, \mathbf{x}(t))$.

We'll begin with the change in the volume of the fluid element as it moves. It is helpful first to recall or notice the geometrical meaning of the divergence of a vector field. If each point in a volume V moves by a small amount $\boldsymbol{\xi}(x)$, from an initial position $\mathbf{x}$ to a final position $\bar{\mathbf{x}} = \mathbf{x} + \boldsymbol{\xi}$, the volume of the box changes by $\Delta V = \overline{V} - V = \nabla \cdot \boldsymbol{\xi}\, V$: That is, $\boldsymbol{\nabla} \cdot \xi$ is the fractional change in volume.

This is really Gauss's theorem: As illustrated by the figure below, moving V to $\overline{V}$ moves each point of the surface S of a volume V along $\boldsymbol{\xi}$, changing the the volume of the box by

$$\Delta V = \int_S \boldsymbol{\xi} \cdot d\boldsymbol{S}\,,$$

to lowest order in $\boldsymbol{\xi}$.



Gauss's theorem now implies

$$\Delta V = \int_V \boldsymbol{\nabla} \cdot \boldsymbol{\xi}\, dV. \tag{III.21}$$

For a small volume V the volume then changes by $\Delta V = V \boldsymbol{\nabla} \cdot \boldsymbol{\xi}$, or

$$\frac{\Delta V}{V} = \boldsymbol{\nabla} \cdot \boldsymbol{\xi}, \tag{III.22}$$

to lowest order in V and $\boldsymbol{\xi}$. That is, as claimed, $\nabla \cdot \boldsymbol{\xi}$ is the fractional change in volume.

Go back now to a fluid with a velocity field $\mathbf{v}(x,t)$. In a time Δt the fluid at $\mathbf{x}$ moves to $\mathbf{x} + \boldsymbol{\xi}$, where $\boldsymbol{\xi} = \mathbf{v}\Delta t$. Then, writing $\nabla \cdot \boldsymbol{\xi} = \nabla \cdot (\mathbf{v}\, \Delta t)$, we have

$$\frac{dV/dt}{V} = \boldsymbol{\nabla} \cdot \mathbf{v}. \tag{III.23}$$

The change in density is given by

$$\frac{d}{dt}\rho(t,\mathbf{x}(t)) = \partial_t\rho + \partial_i\rho\frac{dx_i}{dt} = (\partial_t + \mathbf{v}\cdot\boldsymbol{\nabla})\rho. \tag{III.24}$$

Finally, with these expressions for dV/dt and $d\rho/dt$, conservation of mass (III.20) is

$$\frac{1}{\rho}(\partial_t + \mathbf{v}\cdot\boldsymbol{\nabla})\rho = -\boldsymbol{\nabla}\cdot\mathbf{v},$$

or

$$\partial_t\rho + \nabla\cdot(\rho\mathbf{v}) = 0. \tag{III.25}$$

This is commonly called the *continuity equation.*

To summarize: A fluid is characterized by its pressure P, density ρ and 3-velocity $\mathbf{v}$. Its motion is governed by the equations

$$\nabla^2\Phi = 4\pi G\rho, \qquad \lim_{r\to\infty}\Phi = 0, \tag{III.26}$$

$$\partial_t\,\rho + \nabla\cdot(\rho\mathbf{v}) = 0,$$

$$\rho\,(\partial_t + \mathbf{v}\cdot\boldsymbol{\nabla})\,\mathbf{v} = -\boldsymbol{\nabla}P - \rho\boldsymbol{\nabla}\Phi. \tag{III.27}$$

The Euler equation in this form neglects viscosity and magnetic fields, and this is appropriate in computing the structure and oscillations of white dwarfs and neutron stars. Magnetic fields and effective viscosity from neutrino loss are important in collapse and mergers, and viscosity can also be important for stability of rapidly rotating neutron stars. For neutron stars, the Newtonian approximation is off by 10-15%, and we will use the relativistic version of the Euler equation to compute their structure.

Finally, the Newtonian potential Φ satisfies

III.3 Newtonian energy conservation

We'll begin with conservation of energy for a fluid element with no gravitational field. Including the field is simple for a time-independent field but needs a little discussion when the field is time-dependent, and we'll save that for the end.

The energy of a fluid element of mass M is a sum, $E = \frac{1}{2}Mv^2 + U$, of its kinetic energy and internal energy. For a fluid element of volume V, we have seen that conservation of mass written in terms of $\rho = M/V$ has the form (III.25). Denote by ε the internal energy density, $\varepsilon = U/V$ (we'll use u for the energy per unit mass, $u = E/M = \varepsilon/\rho$). Conservation of energy does not have the form $\partial_t\varepsilon + \nabla\cdot(\varepsilon\mathbf{v}) = 0$, because pressure does work as the fluid element moves. The change in internal energy per unit volume, $\partial_t\varepsilon + \nabla\cdot(\varepsilon\mathbf{v})$ is equal to the work done per unit volume per unit time:

$$\partial_t\varepsilon + \nabla\cdot(\varepsilon\mathbf{v}) = -P\boldsymbol{\nabla}\cdot\mathbf{v}, \tag{III.28}$$

where from Eq. (III.23), $\boldsymbol{\nabla}\cdot\mathbf{v}$ is the change in volume per unit volume per unit time. We are assuming that heat flow is negligible, that the entropy of each fluid element is conserved, and this relation is equivalent to the first law of thermodynamics, $dU = TdS - PdV$, when $dS = 0$. That is, dividing by V and writing $\dfrac{d}{dt} = (\partial_t + \mathbf{v}\cdot\boldsymbol{\nabla})$ for the change per unit time along the flow gives the first law in the form

$$\frac{1}{V}(\partial_t + \mathbf{v}\cdot\boldsymbol{\nabla})U = -\frac{P}{V}(\partial_t + \mathbf{v}\cdot\boldsymbol{\nabla})V, \tag{III.29}$$

and Eq. (III.23) then implies (III.28).

The second contribution to the change in the energy of a fluid element is the change in its kinetic energy. Recall that energy conservation for a mass m is obtained by manipulating $(\boldsymbol{F} = m\boldsymbol{a})\cdot\mathbf{v}$, using $m\boldsymbol{a}\cdot\mathbf{v} = \partial_t(\frac{1}{2}mv^2)$. Conservation of energy has the form $\partial_t(\frac{1}{2}mv^2) = \boldsymbol{F}\cdot\mathbf{v}$, equating the rate of change of kinetic energy to the power, the rate at which forces are doing work. So we multiply the Euler equation (III.19) by $\mathbf{v}$:

$$\begin{aligned} 0 &= v^a\rho\,(\partial_t + \mathbf{v}\cdot\boldsymbol{\nabla})\,v_a + v^a\nabla_a P \\ \text{1st term} &= \rho\,(\partial_t + \mathbf{v}\cdot\boldsymbol{\nabla})\frac{1}{2}v^2 = \partial_t(\frac{1}{2}\rho v^2) + \boldsymbol{\nabla}\cdot(\mathbf{v}\frac{1}{2}\rho v^2) \end{aligned} \tag{III.30}$$

where we have used conservation of mass, $\partial_t\rho + \boldsymbol{\nabla}\cdot(\rho\mathbf{v}) = 0$. With no pressure, no internal energy, and no gravitational field, we have energy conservation for dust:

$$\partial_t(\frac{1}{2}\rho v^2) + \boldsymbol{\nabla}\cdot(\mathbf{v}\frac{1}{2}\rho v^2) = 0.$$

In our case, the change in the sum of two contributions to energy satisfies

$$\partial_t\left(\frac{1}{2}\rho v^2 + \varepsilon\right) + \boldsymbol{\nabla}\cdot\left(\mathbf{v}\frac{1}{2}\rho v^2 + \mathbf{v}\varepsilon\right) = -P\boldsymbol{\nabla}\cdot\mathbf{v} - \mathbf{v}\cdot\boldsymbol{\nabla}P$$

or

$$\partial_t\left(\frac{1}{2}\rho v^2 + \varepsilon\right) + \boldsymbol{\nabla}\cdot\left[\rho\mathbf{v}\left(\frac{1}{2}v^2 + \frac{\varepsilon + P}{\rho}\right)\right] = 0.$$

The quantity $\dfrac{\varepsilon + P}{\rho}$ appearing in the energy flux term is the enthalpy per unit mass, written

$$h = \frac{\varepsilon + P}{\rho}, \tag{III.31}$$

and the conservation law is commonly written in the form

$$\partial_t\left[\rho\left(\frac{1}{2}v^2 + u\right)\right] + \boldsymbol{\nabla}\cdot\left[\rho\mathbf{v}\left(\frac{1}{2}v^2 + h\right)\right] = 0, \tag{III.32}$$

where, as mentioned above, $u = \varepsilon/\rho$ is the internal energy per unit mass.

When a time-independent gravitational field is present (e.g. for matter orbiting or falling onto a star or black hole, where one can ignore the field of that matter), the additional energy density of the fluid element is $\rho\Phi$, the additional term in the the Euler equation is $\rho\boldsymbol{\nabla}\Phi$, and our energy conservation equation becomes

$$\partial_t\left[\rho\left(\frac{1}{2}v^2 + u + \Phi\right)\right] + \boldsymbol{\nabla}\cdot\left[\rho\mathbf{v}\left(\frac{1}{2}v^2 + h + \Phi\right)\right] = 0. \tag{III.33}$$

This has the general form of a conserved current,

$$\partial_t \mathcal{E} + \boldsymbol{\nabla} \cdot \mathbf{f} = 0, \tag{III.34}$$

with the energy density $\mathcal{E}$ and the energy flux $\mathbf{f}$ playing the roles of charge density and current density.

When the gravitational field is time dependent, one needs to include the energy flux of the field itself and that exercise will be one of the assigned problems. In GR, however, there is no well-defined local energy density of a time-dependent gravitational field. The reason is that conservation of energy is associated with time-translation invariance of the geometry. When the geometry is time dependent, the invariance is gone. What remains is the total energy of the spacetime, associated with the asymptotic time-translation invariance of an asymptotically flat spacetime. In the Newtonian approximation, gravity is a field on flat space (the metric linearized about flat space), and this allows one to define a local energy.

III.4 Lie derivatives, Gauss's Theorem, and Stokes's Theorem

Lie derivatives

Lie derivatives arise naturally in the context of fluid flow and are a tool that can simplify calculations and aid one's understanding of relativistic fluids.

Begin, for simplicity, in a Newtonian context, with a stationary fluid flow with 3-velocity $\mathbf{v}(\mathbf{r})$. A function f is said to be *dragged along* by the fluid flow, or *Lie-derived* by the vector field $\mathbf{v}$ that generates the flow, if the value of f is constant on a fluid element, that is, constant along a fluid trajectory $\mathbf{r}(t)$:

$$\frac{d}{dt} f[\mathbf{r}(t)] = \mathbf{v} \cdot \nabla f = 0. \tag{III.35}$$

The *Lie derivative* of a function f, defined by

$$\mathcal{L}_{\mathbf{v}} f = \mathbf{v} \cdot \nabla f, \tag{III.36}$$

is the directional derivative of f along $\mathbf{v}$, the rate of change of f measured by a comoving observer.

Consider next a vector that joins two nearby fluid elements, two points $\mathbf{r}(t)$ and $\bar{\mathbf{r}}(t)$ that move with the fluid: Call the connecting vector $\lambda\mathbf{w}$, so that for small λ the fluid elements are nearby: $\lambda\mathbf{w} = \bar{\mathbf{r}}(t) - \mathbf{r}(t)$. Then $\lambda\mathbf{w}$ is said to be *dragged along* by the fluid flow, as shown in Fig. (2). In the figure, the endpoints of $\mathbf{r}(t_i)$ and $\bar{\mathbf{r}}(t_i)$ are labeled $\mathrm{r_i}$ and $\bar{\mathrm{r}}_\mathrm{i}$.

A vector field $\mathbf{w}$ is *Lie-derived* by $\mathbf{v}$ if, for small λ, $\lambda\mathbf{w}$ is dragged along by the fluid flow. To make this precise, we are requiring that the equation

$$\mathbf{r}(t) + \lambda\mathbf{w}(\mathbf{r}(t)) = \bar{\mathbf{r}}(t) \tag{III.37}$$

be satisfied to $O(\lambda)$. Taking the derivative of both sides of the equation with respect to t at $t = 0$, we have

$$\begin{aligned} \mathbf{v}(\mathbf{r}) + \lambda\mathbf{v} \cdot \nabla\mathbf{w}(\mathbf{r}) &= \mathbf{v}(\bar{\mathbf{r}}) = \mathbf{v}[\mathbf{r} + \lambda\mathbf{w}(\mathbf{r})] \\ &= \mathbf{v}(\mathbf{r}) + \lambda\mathbf{w} \cdot \nabla\mathbf{v}(\mathbf{r}) + O(\lambda^2), \end{aligned} \tag{III.38}$$

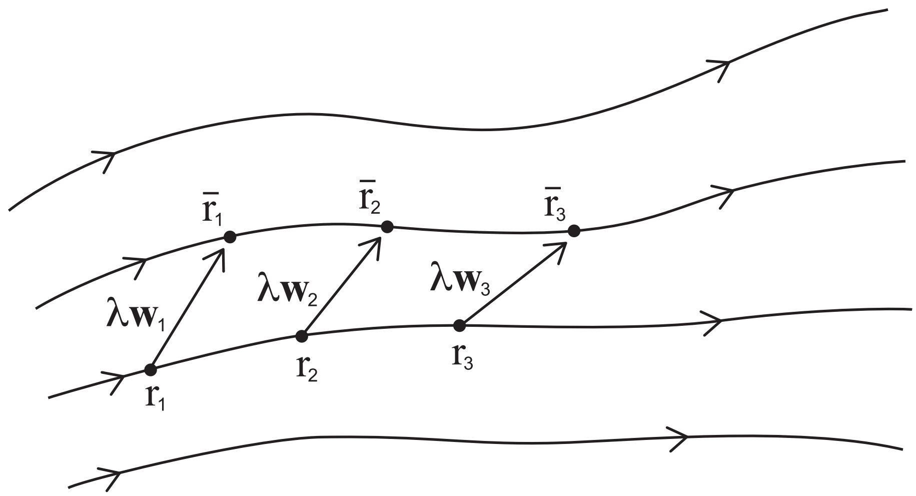


Figure 2: Two nearby fluid elements move along the flow lines, their successive positions labeled $\mathbf{r}_i$ and $\bar{\mathbf{r}}_i$. A vector field $\lambda\mathbf{w}$ is said to be dragged along by the flow when, as shown here, it connects successive positions of two nearby fluid elements.

which holds if and only if

$$[\mathbf{v},\mathbf{w}] \equiv \mathbf{v}\cdot\nabla\mathbf{w} - \mathbf{w}\cdot\nabla\mathbf{v} = 0. \tag{III.39}$$

The commutator $[\mathbf{v},\mathbf{w}]$ is the *Lie derivative* of $\mathbf{w}$ with respect to $\mathbf{v}$, written

$$\mathcal{L}_\mathbf{v}\mathbf{w} = [\mathbf{v},\mathbf{w}]. \tag{III.40}$$

Then $\mathbf{w}$ is Lie-derived by $\mathbf{v}$ when $\mathcal{L}_\mathbf{v}\mathbf{w} = \mathbf{0}$. The Lie derivative $\mathcal{L}_\mathbf{v}\mathbf{w}$ compares the change in the vector field $\mathbf{w}$ in the direction of $\mathbf{v}$ to the change that would occur if $\mathbf{w}$ were dragged along by the flow generated by $\mathbf{v}$.

In a curved spacetime the Lie derivative of a function f is again its directional derivative,

$$\mathcal{L}_\mathbf{u} f = u^\alpha \nabla_\alpha f. \tag{III.41}$$

If u^α is the 4-velocity of a fluid, generating the fluid trajectories in spacetime, $\mathcal{L}_\mathbf{u} f$ is commonly termed the convective derivative of f. The Newtonian limit of u^α is the 4-vector $\boldsymbol{\partial}_t + \mathbf{v}$, and $\mathcal{L}_\mathbf{u} f$ has as its limit the Newtonian convective derivative $(\partial_t + \mathbf{v}\cdot\nabla)f$, again the rate of change of f measured by a comoving observer. (Now the flow is arbitrary, not the stationary flow of our earlier Newtonian discussion.)

A connecting vector is naturally a contravariant vector, the tangent to a curve joining nearby points in a flow; and in a curved spacetime, the Lie derivative of a contravariant vector field is again defined by Eq. (III.40),

$$\mathcal{L}_\mathbf{u} w^\alpha = u^\beta \nabla_\beta w^\alpha - w^\beta \nabla_\beta u^\alpha. \tag{III.42}$$

We have used a fluid flow generated by a 4-velocity u^α to motivate a definition of Lie derivative; the definition, of course, is the same in any dimension and for any vector fields:

$$\mathcal{L}_\mathbf{u} w^a = u^b \nabla_b w^a - w^b \nabla_b u^a. \tag{III.43}$$

Although the covariant derivative operator ∇ appears in the above expression, the Lie derivative is in fact independent of the choice of derivative operator. This is immediate from the symmetry $\Gamma^i_{jk} = \Gamma^i_{(jk)}$, which implies that the components have in any chart the form

$$\mathcal{L}_{\mathbf{u}} w^i = u^j \partial_j w^i - w^j \partial_j u^i. \tag{III.44}$$

We now extend the definition of Lie derivative to arbitrary tensors using the Leibnitz rule, the requirement that, for any vector w^a,

$$\mathcal{L}_{\mathbf{u}}(\sigma_a w^a) = (\mathcal{L}_{\mathbf{u}}\sigma_a) w^a + \sigma_a \mathcal{L}_{\mathbf{u}} w^a. \tag{III.45}$$

Using this and the action (III.41) of the Lie derivative on the scalar $\sigma_a u^a$, we have

$$\mathcal{L}_{\mathbf{u}}\sigma_a = u^b \nabla_b \sigma_a + \sigma_b \nabla_a u^b. \tag{III.46}$$

Because $\mathcal{L}_{\mathbf{u}}(\sigma_a w^a)$ and $\mathcal{L}_{\mathbf{u}} w^a$ in Eq. (III.45) are independent of the choice of derivative operator, definition (III.46) is independent of the choice of derivative operator, and it is easy to check that the components in any chart are given by

$$\mathcal{L}_{\mathbf{u}}\sigma_i = u^j \partial_j \sigma_i + \sigma_j \partial_i u^j. \tag{III.47}$$

Finally, the Lie derivative of an arbitrary tensor $T^{a_1 \cdots a_m}{}_{b_1 \cdots b_n}$ again follows from the Leibnitz rule, using its form for contravariant and covariant vectors to compute $\mathcal{L}_{\mathbf{u}}(T^{c \cdots d}{}_{a \cdots b} v^a \cdots w^b \sigma_c \cdots \tau_d)$ for arbitrary vectors $v^a, \ldots, w^a$ and covectors $\sigma_a, \ldots, \tau_a$:

$$\begin{aligned} \mathcal{L}_{\mathbf{u}} T^{a \cdots n}{}_{c \cdots d} \;&=\; u^e \nabla_e T^{a \cdots b}{}_{c \cdots d} \\ &\quad -T^{e \cdots b}{}_{c \cdots d} \nabla_e u^a - \cdots - T^{a \cdots e}{}_{c \cdots d} \nabla_e u^b \\ &\quad +T^{a \cdots b}{}_{e \cdots d} \nabla_c u^e + \cdots + T^{a \cdots b}{}_{c \cdots e} \nabla_d u^e, \end{aligned} \tag{III.48}$$

independent of the derivative operator, and with components in a chart again given by replacing ∇ by ∂. [3]

Killing vectors

Spacetimes that are symmetric under continuous symmetries like rotations about an axis or time translations have metrics that are invariant under these symmetries. Let's start with rotations of flat space with metric

$$ds^2 = dx^2 + dy^2 + dz^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2. \tag{III.49}$$

The group of rotations about the z-axis maps a point P with coordinates (x, y, z) to the path $\phi \to P(\varphi) = R_\varphi(P)$, with

$$x(\varphi) = x\cos\varphi - y\sin\varphi, \quad y(\varphi) = y\cos\varphi + x\sin\varphi, \quad z(\varphi) = z. \tag{III.50}$$

[3] If, instead of a coordinate basis, an observer uses a frame that is dragged along by $\mathbf{u}$, then a tensor is dragged along by $\mathbf{u}$ (i.e., $\mathcal{L}_{\mathbf{u}}\mathbf{T} = 0$), if its components are constant along a fluid trajectory. It follows that the components of the Lie derivative of any tensor are just $u^m \partial_m T^{i \cdots j}{}_{k \cdots l}$ in a frame dragged along by $\mathbf{u}$.

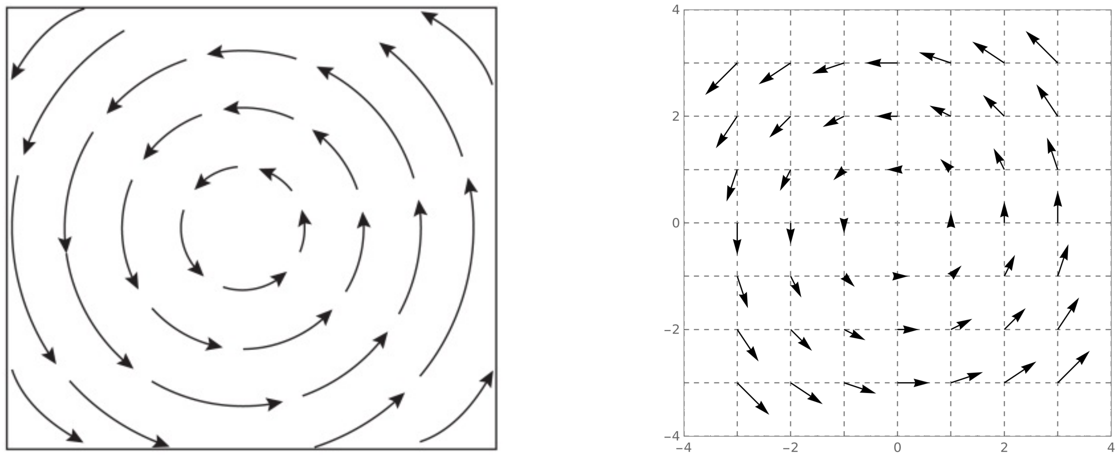


Figure 3: Rotation paths and the vector field $\boldsymbol{\xi}$ tangent to the paths.

The tangent to the path through P is the vector $\boldsymbol{\xi} = \frac{d}{d\phi} R_\phi(P)|_{\varphi=0} = x\boldsymbol{j} - y\boldsymbol{i} \equiv x\boldsymbol{\partial}_y - y\boldsymbol{\partial}_x$, or

$$\xi^x = y, \tag{III.51}$$
$$\xi^y = -x, \tag{III.52}$$
$$\xi^z = 0. \tag{III.53}$$

A fluid rotating with angular velocity Ω has velocity $\mathbf{v} = \Omega\boldsymbol{\xi}$. A vector field $\mathbf{w}$ is rotationally invariant if it is Lie-dragged by this flow, if $\mathcal{L}_{\boldsymbol{\xi}}\mathbf{w} = 0$; for example, the radial vector field $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$, is rotationally invariant.

The metric is rotationally invariant if $\mathcal{L}_{\boldsymbol{\xi}} g_{ab} = 0$. This is equivalent to saying that rotations preserve dot products, that the lengths of vectors and the angles between them are invariant under rotations. In other words, if $\boldsymbol{A}$ and $\boldsymbol{B}$ are Lie-derived by $\boldsymbol{\xi}$, then $\boldsymbol{A} \cdot \boldsymbol{B} = g_{ab}A^aB^b$ is Lie derived by $\boldsymbol{\xi}$.
A check that invariance of dot products of means $\mathcal{L}_{\boldsymbol{\xi}} g_{ab} = 0$:

$$\begin{aligned} 0 &= \mathcal{L}_{\boldsymbol{\xi}}(g_{ab}A^aB^b) = (\mathcal{L}_{\boldsymbol{\xi}} g_{ab})A^aB^b + g_{ab}(\mathcal{L}_{\boldsymbol{\xi}} A^a)B^b + g_{ab}A^a\mathcal{L}_{\boldsymbol{\xi}} B^b = (\mathcal{L}_{\boldsymbol{\xi}} g_{ab})A^aB^b \\ \Rightarrow \mathcal{L}_{\boldsymbol{\xi}} g_{ab} &= 0. \end{aligned} \tag{III.54}$$

Definition. A vector field $\boldsymbol{\xi}$ is a Killing vector of a metric g_{ab} if

$$\mathcal{L}_{\boldsymbol{\xi}} g_{ab} = 0. \tag{III.55}$$

Recalling our definition of Lie derivative, we have

$$\begin{aligned} \pounds_{\boldsymbol{\xi}} g_{\alpha\beta} &= \xi^\gamma \nabla_\gamma g_{\alpha\beta} + g_{\alpha\gamma}\nabla_\beta \xi^\gamma + +g_{\gamma\beta}\nabla_\alpha \xi^\gamma. \\ &= \nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha \end{aligned} \tag{III.56}$$

and, in any coordinate system $\{x^i\}$,

$$\pounds_{\boldsymbol{\xi}} g_{ij} = \xi^k \partial_k g_{ij} + g_{kj}\partial_i \xi^k + g_{ik}\partial_j \xi^k. \tag{III.57}$$

Note that, if we choose as a coordinate the parameter distance ϕ along the rotation paths e.g., choosing coordinates r, θ, ϕ, then the rotation path is $(r, \theta\phi) \to (r, \theta, \phi + \varphi)$, and its tangent $\boldsymbol{\xi}$ has components $\xi^i = \delta^i_\phi$. Then, by Eq. (III.57), invariance of the metric under rotations in the x-y plane is equivalent to requiring

$$\partial_\phi g_{ij} = 0 \text{ when } \phi \text{ is one of the coordinates.} \tag{III.58}$$

More generally, any nonzero vector field $\boldsymbol{\xi}$ is tangent to a family of paths $x^i(\lambda)$: the paths are the solutions to $\dfrac{dx^i}{d\lambda} = \xi^i$. You can think of the paths $x^i(\lambda)$ as the trajectories of fluid elements whose velocity field is $\boldsymbol{\xi}$, and the paths are called the flow of the vector field. Then

$$\mathcal{L}_{\boldsymbol{\xi}} g_{ij} = \partial_\lambda g_{ij} \text{ in a coordinate system with } \lambda \text{ as one of the coordinates.} \tag{III.59}$$

Thus, if there are coordinates t, r, θ, ϕ for which the metric is independent of t, then the vector field t^α with components $t^\mu = \delta^\mu_t$ is a Killing vector.

Gauss's Theorem

Gauss's theorem and Stokes's theorem relate differential to integral conservation laws. One proves them in flat space by dividing a volume into a set of coordinate cubes and proving the theorem on each cube. We can quickly see that the flat-space proofs go through in curved space without change.

Because the divergence of a vector has the form $\nabla_a A^a = \dfrac{1}{\sqrt{|g|}}\partial_i(\sqrt{|g|A^i})$ the integral $\displaystyle\int \nabla_a A^a dV \equiv \int \nabla_a A^a \sqrt{|g|} d^n x$ has the form $\displaystyle\int \partial_i(\sqrt{|g|}A^i) d^n x$, involving only the partial derivative of a vector density, $\mathcal{A}^i := A^i\sqrt{|g|}$.[4] As a result, the flat-space proof of Gauss's Theorem, based on the Fundamental Theorem of calculus, $\int_a^b f'(x)dx = f(b) - f(a)$, holds in curved space as well.

The flat-space proof of Gauss's theorem follows from an integration over a coordinate cube V using the fundamental theorem of calculus for the integral over each coordinate.

$$\int_V \partial_i \mathcal{A}^i dx^1\, dx^2\, dx^3 = \int_S (\mathcal{A}^1 dx^2 dx^3 + \mathcal{A}^2 dx^1 dx^3 + \mathcal{A}^3 dx^1 dx^2);$$

and any volume is approximated by an arbitrarily fine division into coordinate cubes.

The integration over a coordinate cube has the identical form in curved space. If V is an n-

[4] Another way to say this is that the divergence of a vector density $\mathcal{A}^a$, here $A^a\sqrt{|g|}$, is defined without introducing a covariant derivative: In any coordinate system $\nabla_a \mathcal{A}^a := \partial_i A^i$.

dimensional coordinate cube

$$\begin{aligned}
\int_V \partial_i \mathcal{A}^i d^n x &= \int_V \partial_1 \mathcal{A}^1 dx^1 dx^2 \cdots dx^n + \cdots + \int_V \partial_n \mathcal{A}^n dx^n dx^1 \cdots dx^{n-1} \\
&= \int_{\partial_{1+}V} \mathcal{A}^1 dx^2 \cdots dx^n - \int_{\partial_{1-}V} \mathcal{A}^1 dx^2 \cdots dx^n + \cdots \\
&\quad + \int_{\partial_{n+}V} \mathcal{A}^n dx^1 \cdots dx^{n-1} - \int_{\partial_{n-}V} \mathcal{A}^n dx^1 \cdots dx^{n-1} \\
&= \int_{\partial V} \mathcal{A}^i dS_i \quad (\partial V \text{ means the boundary of } V) \qquad \text{(III.60)}\\
&\quad \text{where } dS_i = \pm \epsilon_{ij\cdots k} dx^j \cdots dx^k \frac{1}{(n-1)!}\text{ , with} \\
&\quad dS_1 = +dx^2 \cdots dx^n \ \text{ for } x^1 \text{ increasing outward,} \\
&\quad dS_1 = -dx^2 \cdots dx^n \text{ , for } x^1 \text{ increasing inward.}
\end{aligned}$$

This form is correct for a region in a space with a metric, independent of the signature of the metric. When $\partial\Omega$ has a unit outward normal n_a (along the gradient of a scalar that increases outward), one can write dS_a in the form $dS_a = n_a dS$. In this case,

$$\int \nabla_a A^a d^n V = \int A^a n_a dS. \qquad \text{(III.61)}$$

Example: As we discuss below, the differential form of baryon conservation is $\nabla_\alpha(nu^\alpha) = 0$. The corresponding integral form is

$$\begin{aligned}
0 &= \int_\Omega \nabla_\alpha (nu^\alpha) d^4 V = \int_{\partial\Omega} nu^\alpha dS_\alpha \\
&= \int_{V_2} nu^\alpha dS_\alpha - \left| \int_{V_1} nu^\alpha dS_\alpha \right| .
\end{aligned}$$

Here the fluid is taken to have finite spatial extent, and the spacetime region Ω is bounded by the initial and final spacelike hypersurfaces V_1 and V_2. In a coordinate system for which V_1 and V_2 are surfaces of constant t, with t increasing to the future, we have $dS_\mu = \nabla_\mu t \sqrt{|g|} d^3x = \delta^t_\mu \sqrt{|g|} d^3 x$ on V_2, $dS_\mu = -\delta^t_\mu \sqrt{|g|} d^3x$ on V_1, and

$$\int_\Omega \nabla_\alpha (nu^\alpha) d^4 V = \int_{V_2} nu^t \sqrt{|g|} d^3 x - \int_{V_1} nu^t \sqrt{|g|} d^3 x. \qquad \text{(III.62)}$$

If, on a slicing of spacetime one chooses on each hypersurface V a surface element dS_α along $+\nabla_\alpha t$, the conservation law is then

$$N = \int_V nu^\alpha dS_\alpha = \text{ constant,} \qquad \text{(III.63)}$$

with N the total number of baryons.

Note that the fact that one can write the conserved quantity associated with a current j^α in the form,

$$\int_V j^\alpha dS_\alpha = \int_V j^t \sqrt{|g|}\, d^3x,$$

means that there is no need to introduce n_α and $\sqrt{{}^3g}$ to evaluate the integral. This fact is *essential* if one is evaluating an integral $\int j^\alpha dS_\alpha$ on a null surface, because there is no unit normal. The flux of energy or of baryons across the horizon of a Schwarzschild black hole, for example, can be computed in Eddington-Finkelstein or Kruskal coordinates: In ingoing Eddington-Finkelstein coordinates v, r, θ, ϕ, the horizon is a surface of constant r, and we have

$$\int j^\alpha dS_\alpha = \int j^r \sqrt{|g|}\, dv d\theta d\phi.$$

Stokes's Theorem

The simplest version of Stokes's theorem is its 2-dimensional form, namely Green's theorem:

$$\int_S (\partial_x A_y - \partial_y A_x) dx\, dy = \int_c (A_x dx + A_y dy),$$

where c is a curve bounding the 2-surface S. The theorem involves the integral over a 2-surface of the antisymmetric tensor $\nabla_a A_b - \nabla_b A_a$. In three dimensions, the tensor is dual to the curl of $\mathbf{A}$: $(\nabla\times\mathbf{A})^a = \epsilon^{abc}\nabla_b A_c$; and Stokes's generalization of Green's theorem can be written in either the form

$$\int_S (\boldsymbol{\nabla}\times\boldsymbol{A})\cdot d\mathbf{S} = \int_c \mathbf{A}\cdot\mathbf{dl}$$

or in terms of the antisymmetric tensor $\nabla_a A_b - \nabla_b A_a$

$$\int_S (\nabla_a A_b - \nabla_b A_a) dS^{ab} = \int_c A_a dl^a, \tag{III.64}$$

where, for an antisymmetric tensor F_{ab}, $F_{ab}dS^{ab}$ means $F_{12}dx^1dx^2 + F_{23}dx^2dx^3 + F_{31}dx^3dx^1$. Written in this form, the theorem is already correct in a curved spacetime. The reason is again that the antisymmetric derivative $\nabla_a A_b - \nabla_b A_a$ has in curved space the same form it has in flat space: Its components in any coordinate system are just $\partial_i A_j - \partial_j A_i$. The antisymmetric derivative $\nabla_a A_b - \nabla_b A_a$ is called the exterior derivative, written dA, and it does not need a covariant derivative for its definition.

Here is the three-line proof for a coordinate square S in a surface of constant coordinates t and z:

$$\begin{aligned}\int_S (\partial_x A_y - \partial_y A_x) dx dy &= \int_{y_1}^{y_2} dy \int_{x_1}^{x_2} dx (\partial_x A_y - \partial_y A_x)\\ &= \int_{y_1}^{y_2} dy (A_y|_{x_2} - A_y|_{x_1}) + \int_{x_1}^{x_2} dx (A_x|_{y_2} - A_x|_{y_1})\\ &= \int_c A_i dx^i, \end{aligned}\tag{III.65}$$

with the boundary c of the square traversed counterclockwise as seen from above the square.

In the more mathematical part of the gravitational physics literature and in differential geometry texts, the theorem is written $\int_S dA = \int_{\partial S} A$, where ∂S is the boundary of S.

III.5 Relativistic Euler equation and the TOV equation

In this section units are chosen to make $c = G = 1$. These are called *gravitational units* or *geometrized units*. With $c = 1$, length and time have the same dimension; if time is measured in seconds, length is measured in light-seconds, where 1 light-second = 3×10^{10} cm, the distance light travels in 1 second. MTW (Misner-Thorne-Wheeler) gives distance in cm, and the unit of time is then $1/(3 \times 10^{10})$ s. With $G = 1$, mass, length and time all have the same dimension, which MTW takes to be length with unit 1 cm. Then the unit of mass is (1 cm)$\times c^2/G = 1.3468 \times 10^{28}$ g. Here is the MTW conversion table:

Box 1.8 GEOMETRIZED UNITS

Throughout this book, we use "geometrized units," in which the speed of light c, Newton's gravitational constant G, and Boltzman's constant k are all equal to unity. The following alternative ways to express the number 1.0 are of great value:

$$1.0 = c = 2.997930 \cdots \times 10^{10}\ \text{cm/sec}$$
$$1.0 = G/c^2 = 0.7425 \times 10^{-28}\ \text{cm/g};$$
$$1.0 = G/c^4 = 0.826 \times 10^{-49}\ \text{cm/erg};$$
$$1.0 = Gk/c^4 = 1.140 \times 10^{-65}\ \text{cm/K};$$
$$1.0 = c^2/G^{1/2} = 3.48 \times 10^{24}\ \text{cm/gauss}^{-1}.$$

One can multiply a factor of unity, expressed in any one of these ways, into any term in any equation without affecting the validity of the equation. Thereby one can convert one's units of measure from grams to centimeters to seconds to ergs to .... For example:

$$\begin{aligned}\text{Mass of sun} = M_\odot &= 1.989 \times 10^{33}\ \text{g}\\ &= (1.989 \times 10^{33}\ g) \times (G/c^2)\\ &= 1.477 \times 10^5\ \text{cm}\\ &= (1.989 \times 10^{33}\ g) \times (c^2)\\ &= 1.788 \times 10^{54}\ \text{ergs}.\end{aligned}$$

The standard unit, in terms of which everything is measured in this book, is centimeters. However, occasionally conventional units are used; in such cases a subscript "conv" is sometimes, but not always, appended to the quantity measured:

$$M_{\odot\text{conv}} = 1.989 \times 10^{33}\ \text{g}.$$

Relativistic fluids

First a little more discussion of what is meant by a fluid: A perfect fluid is a model for a large assembly of particles in which a continuous energy density $\epsilon = \rho c^2$, pressure P, and 4-velocity u^α can reasonably describe the macroscopic matter. We will also be assuming that the microscopic particles collide frequently enough that collisions enforce a local thermodynamic equilibrium. In particular, one assumes that a mean velocity field u^α and a mean stress-energy tensor $T^{\alpha\beta}$ (AKA the energy-momentum tensor) can be defined in boxes – fluid elements – small compared to the macroscopic length scale but large compared to the mean free path, and on scales large compared to the size of the fluid elements, the 4-velocity and thermodynamic quantities can be accurately described by continuous fields. Finally, the statement that a fluid cannot maintain a shear stress again means that the stress-energy tensor has no preferred orientation for an observer moving with the average velocity u^α of the fluid:

Denote by

$$q^{\alpha\beta} = g^{\alpha\beta} + u^\alpha u^\beta \tag{III.66}$$

the projection operator orthogonal to u^α. The lack of a preferred orientation implies that the stress-energy tensor has the form

$$T^{\alpha\beta} = \rho u^\alpha u^\beta + P q^{\alpha\beta} = (\rho + P) u^\alpha u^\beta + P g^{\alpha\beta}. \tag{III.67}$$

That is, the only nonzero parts of $T^{\alpha\beta}$ are the rotational scalars [5]

$$\rho \equiv T^{\alpha\beta} u_\alpha u_\beta \tag{III.68}$$

and

$$P \equiv \frac{1}{3} q_{\gamma\delta} T^{\gamma\delta}. \tag{III.69}$$

In an orthonormal frame with $\mathbf{e_0}$ along $\boldsymbol{u}$, $T^{\alpha\beta}$ has components

$$\|T^{\mu\nu}\| = \left\| \begin{matrix} \rho & & & \\ & P & & \\ & & P & \\ & & & P \end{matrix} \right\|. \tag{III.70}$$

Conservation of baryons

Within the standard model of particle physics and to within the accuracy of observations, baryon number is conserved. (The lower limit on the lifetime of the proton against decay to a positron or positive muon is 10^{34} yr.) Let n be the number density of baryons, measured in the rest frame of a fluid. Conservation of baryons has the form

$$\nabla_\alpha (n u^\alpha) = 0, \tag{III.71}$$

and it can be derived heuristically by essentially following our derivation of conservation of mass in the Newtonian approximation.

The proper volume of a fluid element is the volume V of a piece of fluid orthogonal to u^α through the history of the fluid element. With N the number of baryons in the fluid element, conservation

[5] Because the momentum current

$$q^\alpha_\gamma T^{\gamma\beta} u_\beta$$

is a vector in the 3-dimensional subspace orthogonal to u^α, it is invariant under rotations of that subspace only if it vanishes. Similarly, the symmetric tracefree tensor ${}^3T^{\alpha\beta} - \frac{1}{3} q^{\alpha\beta}\, {}^3T \equiv q^\alpha_\gamma q^\beta_\delta T^{\gamma\delta} - \frac{1}{3} q^{\alpha\beta} q_{\gamma\delta} T^{\gamma\delta}$
transforms as a $j = 2$ representation of the rotation group and can be invariant only if it vanishes. More concretely, in an orthonormal frame with $\mathbf{e_0}$ along $\boldsymbol{u}$, T^{0i} and $T^{ij} - \frac{1}{3}\delta^{ij} T^k_k$ must vanish, implying that $T^{\alpha\beta}$ has the form (III.70)

of baryons means $0 = \Delta N = \Delta(nV)$. The fractional change in V in a proper time $\Delta\tau$ is given by the 3-dimensional divergence of the velocity, in the subspace orthogonal to u^α:

$$\frac{\Delta V}{V} = q^{\alpha\beta}\nabla_\alpha u_\beta \Delta\tau. \tag{III.72}$$

Because $u^\beta u_\beta = -1$, we have $u^\beta \nabla_\alpha u_\beta = \frac{1}{2}\nabla_\alpha(u_\beta u^\beta) = 0$, implying

$$q^{\alpha\beta}\nabla_\alpha u_\beta = \nabla_\beta u^\beta. \tag{III.73}$$

Noting that $u^\alpha \nabla_\alpha n = \dfrac{d}{d\tau} n$, we can write the conservation law in the form

$$0 = \frac{\Delta(nV)}{V} = \Delta n + n\frac{\Delta V}{V} = (u^\alpha \nabla_\alpha n + n \nabla_\alpha u^\alpha)\Delta\tau, \tag{III.74}$$

or

$$\nabla_\alpha(n u^\alpha) = 0. \tag{III.75}$$

The conservation law is often written as conservation of rest mass or baryon mass. To do this one has to define a rest mass per baryon: For example, one could imagine dispersing to infinity all the nucleons and electrons in a neutron star, letting the neutrons decay, and taking the rest mass of the star to be the rest mass of the resulting collection of protons, electrons and neutrinos. Because the electron rest mass is about 1/2000 that of a proton, and the neutrino rest mass is about a millionth that of an electron, the result, to within 1 part in 2,000, is to assign the mass of the proton as the rest mass per baryon: $m_p = 1.67 \times 10^{-24}\text{g} = 938$ MeV. The rest mass density is then $\rho_0 = m_p n$, and conservation of baryons is

$$\nabla_\alpha(\rho_0 u^\alpha) = 0, \tag{III.76}$$

with integral form

$$M_0 = \int_V \rho_0 u^\alpha dS_\alpha = \text{ constant}, \tag{III.77}$$

equivalent to Eq. (III.63). Its Newtonian limit is the mass conservation equation (III.25).

Conservation of energy

For a two-parameter equation of state, five variables determine the state of a perfect fluid; they can be taken to be ρ, P and three independent components of u^α. The dynamical evolution of the fluid is governed by the vanishing divergence of the stress-energy tensor,

$$\nabla_\beta T^{\alpha\beta} = 0, \tag{III.78}$$

and by conservation of baryons,

$$\nabla_\alpha(n u^\alpha) = 0. \tag{III.79}$$

The projection of the equation $\nabla_\beta T^{\alpha\beta} = 0$ along u^α yields an energy conservation law, while the projection orthogonal to u^α is the relativistic Euler equation.

The projection $u_\alpha \nabla_\beta T^{\alpha\beta} = 0$ similarly expresses energy conservation for a fluid element:

$$\begin{aligned}
0 &= u_\alpha \nabla_\beta T^{\alpha\beta} = u_\alpha \nabla_\beta [\rho u^\alpha u^\beta + P q^{\alpha\beta}] \\
&= -\nabla_\beta(\rho u^\beta) + P u_\alpha \nabla_\beta (g^{\alpha\beta} + u^\alpha u^\beta) \\
&= -\nabla_\beta(\rho u^\beta) - P \nabla_\beta u^\beta, \\
\Rightarrow \quad \nabla_\beta(\rho u^\beta) &= -P \nabla_\beta u^\beta. \qquad \text{(III.80)}
\end{aligned}$$

The equation means that the total energy of a fluid element decreases by the work,

$$P\, dV = PV\ \nabla_\beta u^\beta\, d\tau, \qquad \text{(III.81)}$$

it does on its surroundings in proper time $d\tau$.

The mass (energy) density is

$$\rho = \rho_0(1+u), \quad \text{or, with } c \text{ restored,} \quad \epsilon \equiv \rho c^2 = \rho_0 c^2 + \rho_0 u, \qquad \text{(III.82)}$$

with u the internal energy per baryon. Because $\rho_0 c^2$ is the dominant contribution to the energy density, energy conservation is just baryon conservation at leading order in c. Newtonian energy conservation is the order c^0 correction.

Relativistic Euler equation. The projection of the conservation of the stress-energy tensor orthogonal to u^α is

$$q^\alpha{}_\gamma \nabla_\beta T^{\beta\gamma} = 0, \qquad \text{(III.83)}$$

so that

$$\begin{aligned}
0 &= q^\alpha{}_\gamma \nabla_\beta [\rho u^\beta u^\gamma + P q^{\beta\gamma}] \\
&= q^\alpha{}_\gamma \rho u^\beta \nabla_\beta u^\gamma + q^{\alpha\beta} \nabla_\beta P + q^\alpha{}_\gamma p \nabla_\beta (u^\beta u^\gamma) \\
&= \rho u^\beta \nabla_\beta u^\alpha + q^{\alpha\beta} \nabla_\beta P + P u^\beta \nabla_\beta u^\alpha, \\
\Rightarrow \quad (\rho + P) u^\beta \nabla_\beta u^\alpha &= -q^{\alpha\beta} \nabla_\beta P. \qquad \text{(III.84)}
\end{aligned}$$

The TOV equation

First a summary, then a detailed derivation from the field equations:

For a spherical star in Schwarzschild coordinates (t, r, θ, ϕ), the metric takes the form

$$ds^2 = -e^{2\Phi(r)}dt^2 + \left[1 - \frac{2m(r)}{r}\right]^{-1} dr^2 + r^2(d\theta^2 + r^2 \sin^2\theta d\phi^2), \qquad \text{(III.85)}$$

where $\Phi(r)$ and $m(r)$, as well as the pressure $P(r)$, are determined by the Tolman-Oppenheimer-Volkoff (TOV) equations

$$\begin{aligned} \frac{dm}{dr} &= 4\pi r^2 \rho, & \text{(III.86)} \\ \frac{d\Phi}{dr} &= -\frac{1}{\rho + P}\frac{dP}{dr}, & \text{(III.87)} \\ \frac{dP}{dr} &= -\frac{(\rho + P)(m + 4\pi r^3 P)}{r(r - 2m)}, & \text{(III.88)} \end{aligned}$$

by integrating the above system from the center to the surface, $r = R_{\text{sph}}$, with conditions $m(0) = 0$, $P(0) = P_c$ and $\Phi(0)$ arbitrary. Here, P_c is the chosen value of central pressure. The arbitrariness in the initial value for $\Phi(r)$ is removed by matching the solution at the surface of the star to the analytic exterior solution

$$e^{2\Phi(r)} = 1 - \frac{2M}{r}. \qquad \text{(III.89)}$$

where $M := m(R)$ is the total mass.

Derivation

We begin with the general spherically symmetric metric,

$$ds^2 = -e^{2\Phi}dt^2 + e^{2\lambda}dr^2 + r^2 d\Omega^2 \qquad \text{(III.90)}$$

(see, for example, Sect. 5.6, p. 122, of Shapiro-Teukolsky). In this section we look at equilibrium models, and Φ and λ are then functions of r only. This means that the vector $\boldsymbol{t} = \boldsymbol{\partial}_t$, with components

$$t^\mu = \delta^\mu_0 \qquad \text{(III.91)}$$

is a Killing vector. For collapsing or oscillating stars, Φ and λ depend on r and t.

For our time-independent metric, the Einstein tensor has components

$$\begin{aligned} G^t{}_t &= e^{-2\lambda}\left(\frac{1}{r^2} - \frac{2}{r}\lambda'\right) - \frac{1}{r^2} & \text{(III.92)} \\ &= -\frac{1}{r^2}\frac{d}{dr}\left[r(1 - e^{-2\lambda})\right] \\ G^r{}_r &= e^{-2\lambda}\left(\frac{1}{r^2} + \frac{2}{r}\Phi'\right) - \frac{1}{r^2} & \text{(III.93)} \\ G^\theta{}_\theta &= G^\phi{}_\phi = e^{-2\lambda}[\Phi'' + (\Phi')^2 + \frac{1}{r}(\Phi' - \lambda') - \Phi'\lambda']. & \text{(III.94)} \end{aligned}$$

All other components $G^\mu{}_\nu$ vanish.

Spacetime outside a spherical star

In a vacuum, the field equations are

$$G^\mu{}_\nu = 0.$$

The G^t_t-equation,

$$G^t{}_t = -\frac{1}{r^2}\frac{d}{dr}[r(1-e^{-2\lambda})] = 0, \tag{III.95}$$

has the first integral

$$r(1-e^{-2\lambda}) = 2M, \text{ for some constant } M, \tag{III.96}$$

implying

$$e^{2\lambda} = \left(1-\frac{2M}{r}\right)^{-1}. \tag{III.97}$$

From the combination,

$$G^r{}_r - G^t{}_t = \frac{2}{r}\, e^{-2\lambda}(\Phi' + \lambda') = 0, \tag{III.98}$$

we have

$$\Phi' = -\lambda' \tag{III.99}$$

or

$$\Phi = -\lambda + k \qquad e^{2\Phi} = k\left(1-\frac{2M}{r}\right).$$

Reparametrizing the time by writing $\tilde{t} = \frac{1}{\sqrt{k}}t$, and changing the name of $\tilde{t}$ back to t, gives

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2 + r^2 d\Omega^2, \tag{III.100}$$

the *exterior Schwarzschild metric.* The geometry is asymptotically flat: For large r, the metric is the Minkowski metric:

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2. \tag{III.101}$$

For $r = 2M$ the components $g_{\mu\nu}$ are singular, so the form (III.100) provides a metric for a spacetime with a hole in it: $\infty > r > 2M, \ \ -\infty < t < \infty$. When we discuss black holes, we'll see that this is a coordinate singularity, like the poles in spherical coordinates. Changing to coordinates that are smooth at $r = 2M$ reveals what you know: The surface $r = 2M$ is an event horizon.

For large r, the metric (III.100) takes the post-Newtonian form

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1+\frac{2M}{r}\right)dr^2 + r^2 d\Omega^2. \tag{III.102}$$

For nearly Newtonian stars, g_{tt} determines the effect of the gravitational field on matter, and $-\dfrac{M}{r}$ is the Newtonian potential Φ. Because the trajectories of particles at large r are those of Newtonian particles about a mass M, one calls M the mass of the spacetime.

The equations, $G^\theta{}_\theta = 0,\ \ G^\phi{}_\phi = 0$, are automatically satisfied once $G^t{}_t = 0$ and $G^r{}_r = 0$ (not in general, just in this spherically symmetric case).

Stellar Interior

Equilibrium configurations of stars are accurately modeled as perfect fluids. For static, spherical stars, ρ and P depend only on r, while the 4-velocity u^α is along the Killing vector t^α :

$$u^\alpha = kt^\alpha.$$

But $t^\alpha t_\alpha = g_{tt} = -e^{2\Phi}$, so $u^\alpha u_\alpha = -1$ implies

$$u^\alpha = e^{-\Phi} t^\alpha.$$

Then, from

$$T^{\alpha\beta} = \rho u^\alpha u^\beta + P q^{\alpha\beta},$$

we have

$$T^t{}_t = -\rho \qquad T^r{}_r = P.$$

The field equation components are
$G^t{}_t = 8\pi T^t{}_t$:

$$\begin{aligned}
\frac{1}{r^2}\frac{d}{dr}[r(1-e^{-2\lambda})] &= 8\pi\rho \\
r(1-e^{-2\lambda}) &= 2\int_0^r 4\pi\rho r^2 dr =: 2m(r) && \text{(III.103)} \\
e^{2\lambda} &= \left(1-\frac{2m(r)}{r}\right)^{-1}. && \text{(III.104)}
\end{aligned}$$

Here $m(r)$ is a kind of mass within a radius r, and $m(r) = M$, the mass measured at infinity, for $r \geq R$.
$G^r{}_r = 8\pi T^r{}_r$:

$$\begin{aligned}
e^{-2\lambda}\left(\frac{2\Phi'}{r}+\frac{1}{r^2}\right)-\frac{1}{r^2} &= 8\pi P \\
(1-\frac{2m}{r})(\frac{2\Phi'}{r}+\frac{1}{r^2}) &= 8\pi P + \frac{1}{r^2} = \frac{8\pi P r^2+1}{r^2} \\
\frac{2\Phi'}{r}+\frac{1}{r^2} &= \frac{8\pi P r^2+1}{r(r-2m)} \\
2\Phi' &= \frac{8\pi P r^2+1}{r-2m} - \frac{1}{r} = \frac{8\pi P r^3+2m}{r(r-2m)} \\
\Phi' &= \frac{m+4\pi P r^3}{r(r-2m)} && \text{(III.105)}
\end{aligned}$$

Note that the Newtonian limit $(P \ll \rho,\ \ r \ll m)$ of (III.105) is

$$\Phi' = \frac{m}{r^2}\ ,$$

so that Φ is again the Newtonian potential.

The remaining field equation components, $G^\theta{}_\theta = 8\pi T^\theta{}_\theta,\ G^\phi{}_\phi = 8\pi T^\phi{}_\phi$, are identical to one another and are implied by (III.104), (III.105) and the equation of hydrostatic equilibrium $q^\alpha{}_\gamma \nabla_\beta T^{\gamma\beta}$, which we now obtain: Recall that $q^\alpha{}_\gamma \nabla_\beta T^{\gamma\beta} = 0$ has the form

$$u^\beta \nabla_\beta u_\alpha = -\frac{1}{\rho + P}\, q^\beta_\alpha \nabla_\beta P. \tag{III.106}$$

On the RHS, $q^\beta_\alpha \nabla_\beta P = (\delta_\alpha{}^\beta + u_\alpha u^\beta)\nabla_\beta P$. The second term vanishes because $\partial_t P = 0$. Then

$$q_\alpha{}^\beta \nabla_\beta P = \nabla_\alpha P.$$

Next, we have

$$\begin{aligned}
u^\beta \nabla_\beta\, u_\alpha &= e^{-\Phi} t^\beta \nabla_\beta (e^{-\Phi} t_\alpha) \\
&= e^{-2\Phi} t^\beta \nabla_\beta t_\alpha \ (\text{using } \ \mathrm{t}^\beta \nabla_\beta \Phi = 0) \\
&= -e^{-2\Phi} t^\beta \nabla_\alpha t_\beta \ \ (\nabla_\alpha t_\beta + \nabla_\beta t_\alpha = 0 \ \ - \ \text{Killing vector eq.}) \\
&= -\frac{1}{2} e^{-2\Phi} \nabla_\alpha (t^\beta t_\beta) \\
&= \frac{1}{2} e^{-2\Phi} \nabla_\alpha (e^{2\Phi}) \\
&= \nabla_\alpha \Phi \\
\nabla_\alpha \Phi = &- \ \frac{1}{\rho + P}\, \nabla_\alpha P
\end{aligned}$$

or

$$\Phi' = -\,\frac{1}{\rho + P}\, P' \ . \tag{III.107}$$

Eqs. (III.105) and (III.107) imply the equation of hydrostatic equilibrium - the TOV (Tolman-Oppenheimer-Volkov) equation:

$$\frac{dP}{dr} = -(\rho + P)\frac{m + 4\pi r^3 P}{r(r - 2m)} \ . \tag{III.108}$$

A spherical relativistic star is a solution to equations (III.104), (III.105), and (III.108) together with an equation of state; the numerical models that have been constructed usually involve equations of state of the simplest form

$$P = P(\rho) \ , \tag{III.109}$$

which are reasonably accurate for neutron stars. A general equation of state has the form $P = P(\rho, s, z_1, \ldots, z_n)$, with s the entropy per baryon and z_i the concentration of the ith particle species. Neutron stars and dwarfs are cold enough (KT $\ll$ Fermi energy) that they are nearly isentropic (s = constant), and their nuclear reactions have proceeded to completion, so each z_i is itself a function of ρ. That's why $P = P(\rho)$ is a good approximation. (At absolute zero of course, s is constant.)

One obtains a star by integrating eqs. (III.108) and (III.109) together with the defining equation for m. That is, one integrates the system

$$m(r) = \int_0^r \rho\, 4\pi r^2 dr, \quad \frac{dP}{dr} = -(\rho + P)\frac{m + 4\pi r^3 P}{r(r-2m)}\ , \quad P = P(\rho)\ . \tag{III.110}$$

One begins with a central density ρ_c and integrates up to the radius R at which P drops to zero (P is a decreasing function of r). This is the boundary of the star. The metric inside the star is then given by

$$e^{2\lambda} = \left(1 - \frac{2m}{r}\right)^{-1} \tag{III.111}$$

$$\Phi = \Phi(R) + \int_r^R \frac{1}{\rho + P}\frac{dP}{dr} \tag{III.112}$$

and outside by

$$e^{2\Phi} = e^{-2\lambda} = 1 - \frac{2M}{r}. \tag{III.113}$$

To restore G and c to these equations, one can multiply by the (unique) factors of G and c that allow each quantity have the desired conventional units. For example, in Eq. (III.111), both sides are dimensionless. To allow m and r to have dimensions of mass and length, multiply by the unique factor built from G and c that makes m/r dimensionless, namely G/c^2. That is, $\dfrac{G}{c^2}\dfrac{m}{r}$ is dimensionless, so the right side of (III.111) becomes $\left(1 - \dfrac{2Gm}{c^2 r}\right)^{-1}$.

In Eq. (III.113), Φ and λ are dimensionless. If they each are to have the conventional units L^2/T^2 of the Newtonian potential, the dimensionless forms are Φ/c^2 and λ/c^2. With G and c restored, Eqs. (III.112), (III.113) and (III.108), become

$$\Phi = \Phi(R) + \int_r^R \frac{1}{\rho + P/c^2}\frac{dP}{dr}, \quad \text{inside} \tag{III.114}$$

$$e^{2\Phi/c^2} = e^{-2\lambda/c^2} = 1 - \frac{2GM}{c^2 r}, \quad \text{outside} \tag{III.115}$$

$$\frac{dP}{dr} = -G(\rho + P/c^2)\frac{m + 4\pi r^3 P/c^2}{r(r - 2Gm/c^2)}\ . \tag{III.116}$$

Notice that, in the TOV equation, (III.116), each factor has its Newtonian form in the Newtonian hydrostatic equilibrium equation, $dP/dr = -G\rho m/r^2$, when terms involving $1/c^2$ are neglected.

Lecture IV Stellar Structure

IV.1 Properties of Stellar Equilibria and the Virial Theorem

Our equation (III.7) of hydrostatic equilibrium (HSE)

$$\frac{dP}{dr} = -\frac{Gm(r)\rho(r)}{r^2} = -\rho(r)g(r)$$

describes any static, spherically symmetric system (atmospheres, stars, planets). When a particle falls, the loss of potential energy is converted to increased kinetic energy of the particle. For a contracting star, some of the gravitational potential energy again goes into kinetic energy – into heating up the star – but some is radiated. An average of the HSE governs that distribution.

First, recall for a simpler system, a particle or a ring in circular orbit in a $1/r$ potential, that the kinetic energy is half as large as | potential energy|: $K = |U|/2$. The particle's total energy is then $E = K + U = -K$. When orbiting particles radiate energy in light or in gravitational waves, E becomes increasingly negative; because $K = |U|/2$, half of the loss of gravitational potential energy goes into increasing the kinetic energy of the particle and half is radiated: Energy radiated = total energy lost by the system = $|\Delta E| = |\Delta U|/2$. The result is also true for elliptical orbits when averaged over time – over the period T of the orbit:[6] $\langle K\rangle_T = |\langle U\rangle_T|$.

To generalize this to stars, we use HSE to relate the gravitational potential energy of the star to an integral over the pressure.

$$\int_0^R dr\, 4\pi r^3 \frac{dP}{dr} = -\int_0^R dr\, \frac{Gm(r)\rho(r)4\pi r^3}{r^2} \tag{IV.117}$$

where we multiplied both sides by $4\pi r^3$ and integrated. The RHS is

$$U_{\rm G} = -\int_0^M dm\, \frac{Gm(r)}{r} \tag{IV.118}$$

where $dm = 4\pi r^2\rho(r)dr$. Integrate the LHS by parts:

$$P(r)4\pi r^3|_0^R - 3\int_0^R dr\, 4\pi r^2 P(r) \tag{IV.119}$$

The first term vanishes because $P(R) = 0$. The second term is the volume average of P times V: $\langle P\rangle V$. So:

$$\langle P\rangle = -\frac{U_G}{3V} \tag{IV.120}$$

This is a very important result — one way of expressing the **virial theorem**.

[6] $2\langle K\rangle = \int_0^T dt\, \dot{\mathbf{r}}^2 = \underbrace{\mathbf{r}\cdot\dot{\mathbf{r}}|_0^T}_{0} - \int_0^T dt\, \mathbf{r}\cdot\ddot{\mathbf{r}} = \int_0^T dt\, \mathbf{r}\cdot\hat{\mathbf{r}}\frac{GM}{r^2} = \int_0^T dt\, \frac{GM}{r} = -\langle U\rangle$

We now use the virial theorem to generalize the $K = -U/2$ relation that governs a single particle in a bound orbit. For a non-degenerate gas at temperature T, the energy per particle associated with each degree of freedom is $\frac{1}{2}k_B T$. For n_f degrees of freedom, the total kinetic energy of a gas with N particles is then

$$K = \frac{1}{2} n_f N k_B T.$$

The number of degrees of freedom is associated with γ (adiabatic index, ratio of the specific heats): $\gamma = 1 + 2/n_f$, so the kinetic energy density is:

$$\frac{K}{V} = \frac{n k_B T}{\gamma - 1} = \frac{P}{\gamma - 1}, \tag{IV.121}$$

where $n = N/V$ is the number density of particles. So our Virial theorem becomes

$$(\gamma - 1)\frac{K}{V} = -\frac{U}{3V}. \tag{IV.122}$$

With $\gamma = 5/3$ for a monatomic gas we have $K = -U/2$, agreeing with the relation for a single orbiting particle to a self-gravitating system. When the gas particles have internal degrees of freedom or are relativistic (e.g. photons or electrons in a white dwarf near its upper mass limit or in a neutron star modeled with Newtonian gravity), $\gamma \neq 5/3$, and the generalized relation is Eq. (IV.121). What might happen for different values of γ? Clearly for $\gamma = 1$ this equation looks bad. But even $\gamma = 4/3$ is problematic, since that gives us $K = -U$. That means the total energy approaches 0, and the star becomes unbound. We will see that this happens for relativistic systems.

IV.2 Simple Stellar Models

Want to put all of these together. Make into 4 coupled first-order ODEs $P(r)$, $m(r)$, $T(r)$, $L(r)$. Need boundary conditions. Some are easy: $m(0) = L(0) = 0$ (no mass inside that). At the outside, $P(R)$ and $T(R)$ need to merge into the photosphere which is complicated. We will ignore that for the moment, assume $T(R) = P(R) = 0$.

Combine HSE and mass into:

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi G\rho$$

Second order in $P(r)$, $\rho(r)$. Assume a simple relation between these:

$$P = K\rho^\gamma = K\rho^{(n+1)/n}$$

This is a **polytrope** with index n, with $\gamma = (n+1)/n$. So we get:

$$\frac{1}{r^2}\frac{d}{dr}\left[\frac{r^2}{\rho}\frac{d}{dr}\left(K\rho^{(n+1)/n}\right)\right] = -4\pi G\rho$$

We can now make our additional boundary conditions $\rho(0) = \rho_c$, $d\rho/dr(0) = 0$. This sets the central density, and says that there is not a cusp of material. The outer boundary comes from having ρ go to 0, or $\rho(R) = 0$ and $m(R) = M$.

These models are overly simple, but can still be useful. Especially before computers. Let us work a bit on the math.

$$\left(\frac{n+1}{n}\right)\frac{K}{r^2}\frac{d}{dr}\left(r^2\rho^{(1-n)/n}\frac{d\rho}{dr}\right) = -4\pi G\rho$$

Let us simplify the units. $\rho(r) = \rho_c(D_n(r))^n$, where $D_n(r)$ is a function that goes between 0 and 1. So:

$$\left[(n+1)\frac{K\rho_c^{(1-n)/n}}{4\pi G}\right]\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dD_n}{dr}\right) = -D_n^n$$

The bit out in front has units of distance squared. So:

$$\lambda_n \equiv \left[(n+1)\frac{K\rho_c^{(1-n)/n}}{4\pi G}\right]^{1/2}$$

and normalize:

$$\xi \equiv \frac{r}{\lambda_n}$$

So we get:

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{dD_n}{d\xi}\right) = -D_n^n$$

This is the **Lane-Emden** equation for a polytrope. We have written it in terms of dimensionless variables $D_n(\xi)$ to make a physics problem into a math problem, but we must be careful to put the units back in before we give physics results.

Boundary conditions as before, but also stop the integration where $D_n(\xi) = 0$. This is the first 0 of the function, and defines the outer edge at $\xi = \xi_1$.

To compute the mass:

$$M = 4\pi \int_0^R dr\, \rho r^2 = 4\pi \int_0^{\xi_1} d(\lambda_n \xi)\, (\lambda_n \xi)^2 \rho_c D_n^n = 4\pi \lambda_n^3 \rho_c \int_0^{\xi_1} d\xi\, \xi^2 D_n^n$$

We don't necessarily have to solve for D_n and integrate to get this, since we can recognize that $\xi^2 D_n^n = -d/d\xi(\xi^2 dD_n/d\xi)$, so

$$M = -4\pi \lambda_n^3 \rho_c \xi_1^2 \frac{dD_n}{d\xi}|_{\xi_1}$$

Numerically this is useful, but there are a few analytic solutions. Namely, $n = 0$, 1, and 5. For $n = 1$ the solution is:

$$D_1(\xi) = \frac{\sin \xi}{\xi}$$

where we only do it up to the first zero, $\xi_1 = \pi$. And for $n = 5$ there is no finite radius:

$$D_5(\xi) = \left(1 + \frac{\xi^2}{3}\right)^{-1/2}$$

with $\xi_1 = \infty$. However, the total mass is finite. For $n > 5$ the mass is infinite.

For adiabatic monatomic gas, $\gamma = 5/3$ and $n = 1.5$. This also works for white dwarfs in some cases.

$n = 3$ is useful, since this is what happens for a star in radiative equilibrium. Add radiative and gas pressure, $P_g = \rho k_B T/\bar{m} = \beta P$, $P_r = aT^4/3 = (1-\beta)P$. Eliminate T in favor of β:

$$\frac{a}{3}\left(\frac{\beta P \bar{m}}{\rho k_B}\right)^4 = (1-\beta)P$$

So from here you can see how $P = K\rho^{4/3}$ comes out.

IV.2.1 Minimum Mass

Need central conditions extreme enough to sustain *pp* burning. Consider a collapsing cloud of mass M. Kelvin-Helmholtz contraction, so all of energy is from contraction (gravity) not fusion. Looks like an ideal gas:

$$P_c = \frac{\rho_c}{\bar{m}} k_B T_c$$

The contraction will be slow and close to HSE if the pressure is almost enough to balance the star. Equating the two pressures:

$$k_B T_c \approx \left(\frac{\pi}{6}\right)^{1/3} G\bar{m} M^{2/3} \rho_c^{1/3}$$

So $T_c \propto \rho_c^{1/3}$, which goes up during contraction. Contraction will continue until T is enough for fusion or electrons become degenerate — either way the center will be supported against further contraction. So it will not be a star if center is degenerate before fusion.

Assume that electrons have become degenerate. Then:

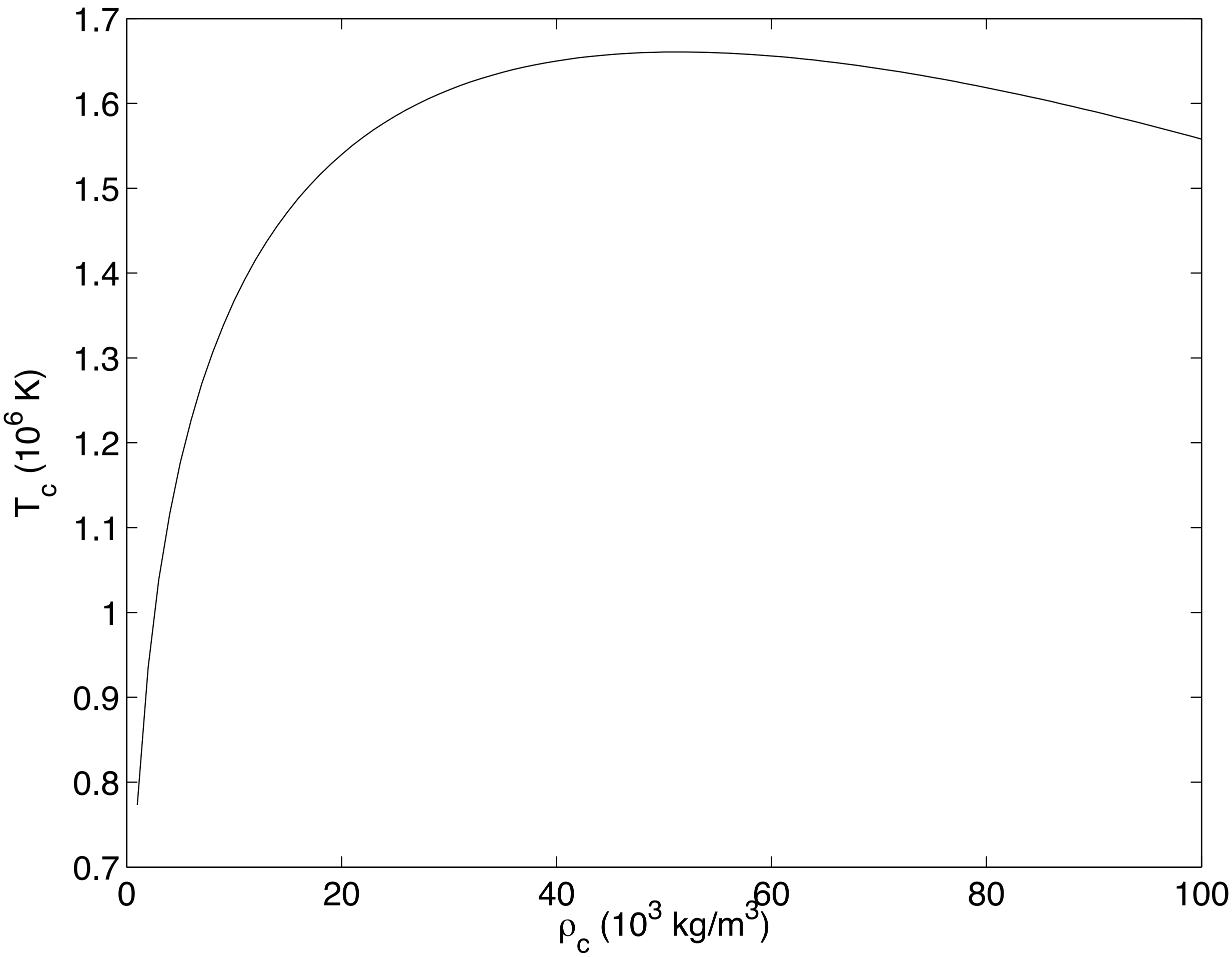
$$P_c = K_{\mathrm{NR}} n_e^{5/3} + n_i k_B T_c \approx K_{\mathrm{NR}} \left(\frac{\rho_c}{m_{\mathrm{H}}}\right)^{5/3} + \frac{\rho_c}{m_{\mathrm{H}}} k_B T_c$$

Set this equal to our P_c from before:

$$k_B T_c \approx \left(\frac{\pi}{6}\right)^{1/3} G m_{\mathrm{H}} M^{2/3} \rho_c^{1/3} - K_{\mathrm{NR}} \left(\frac{\rho_c}{m_{\mathrm{H}}}\right)^{2/3}$$

So this is the temperature when the electrons are degenerate but the ions are not. What is the maximum temperature that will be reached?

$$k_B T_c = A\rho_c^{1/3} - B\rho_c^{2/3}$$


1.7
1.6
1.5
1.4
1.3
1.2
1.1
1
0.9
0.8
0.7
T_c (10^6 K)
0
20
40
60
80
100
ρ_c (10^3 kg/m^3)

Can differentiate and find maximum. This is at $k_B T_c = A^2/4B$, and $\rho_c = (A/2B)^3$. Or:

$$k_B T_{c,\max} \approx \left(\frac{\pi}{6}\right)^{2/3} \frac{G^2 m_{\mathrm{H}}^{8/3}}{4K_{\mathrm{NR}}} M^{4/3}$$

Can then solve for $M_{\min}$ needed to have $T_c \geq T_{\text{ignition}}$. For a rough estimate, use $T_{\text{ign}} = T_{c,\odot}/10 = 1.5 \times 10^6$ K. This gives $M_{\min} = 0.05\, M_\odot$, which isn't bad. Real calculations say closer to 0.08 $M_\odot$.

IV.2.2 Maximum Mass

Things get tricky if pressure is from relativistic particles with $\gamma = 4/3$ (nearly unstable). Which will happen if radiation supplies most of the pressure.

$$P_g = \frac{\rho}{\bar{m}} k_B T_c = \beta P_c$$

and

$$P_r = \frac{a}{3} T_c^4 = (1-\beta) P_c$$

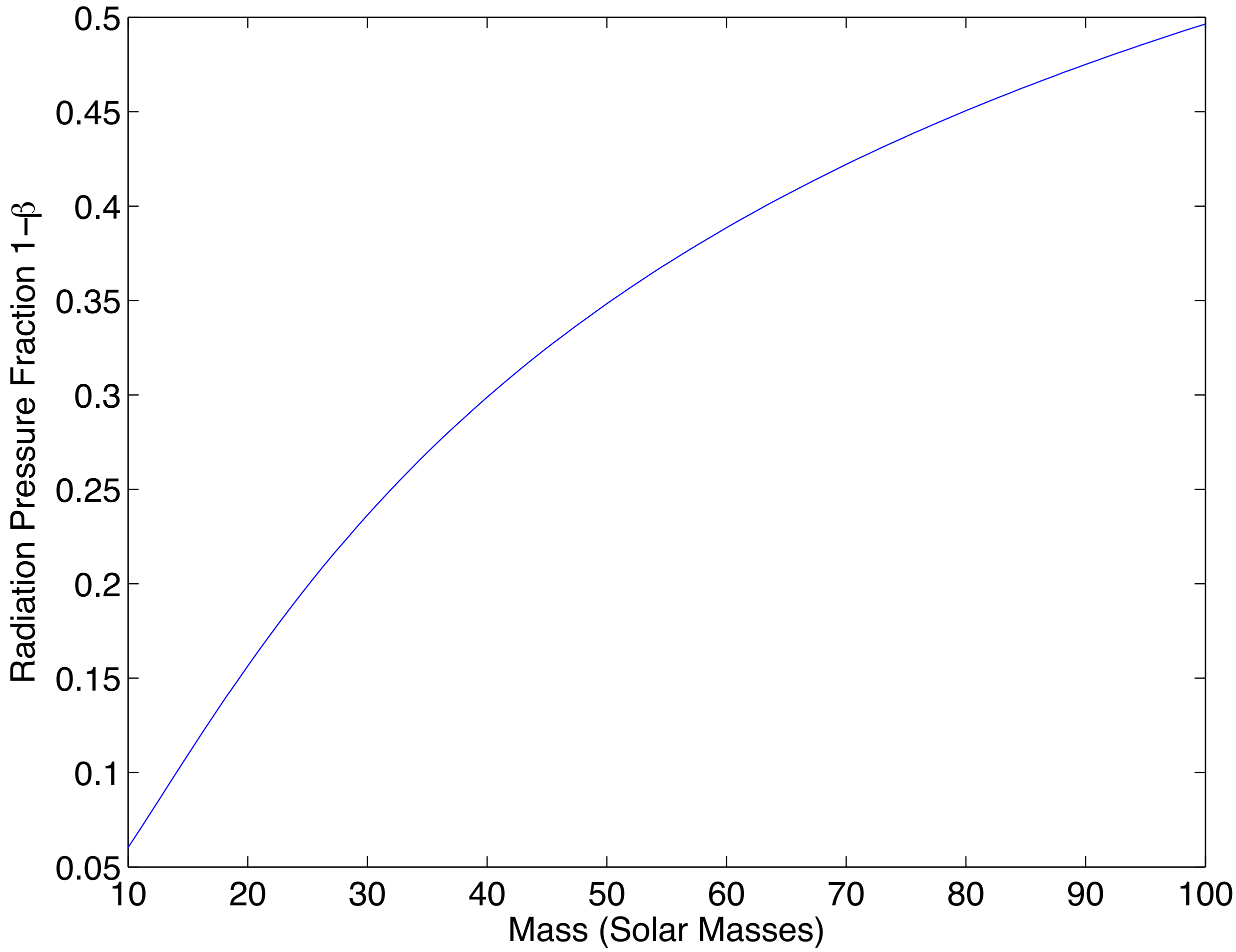
where β is the fraction of total pressure supplied by ions and electrons.

$$P_c = \left(\frac{3}{a} \frac{(1-\beta)}{\beta^4}\right)^{1/3} \left(\frac{k_B \rho_c}{\bar{m}}\right)^{4/3}$$

Equate this to pressure needed to support the star and get:

$$\left(\frac{\pi}{36}\right)^{1/3} G M^{2/3} = \left(\frac{3}{a} \frac{(1-\beta)}{\beta^4}\right)^{1/3} \left(\frac{k_B}{\bar{m}}\right)^{4/3}$$

Radiation pressure gets more important as the mass increases.



When $M > 100M_{\odot}$, $1 - \beta > 0.5$ and the star is very unstable. Even $> 50M_{\odot}$ is very rare, but then gain these stars do not live for a long time so they are hard to spot.

Lecture V Stellar Evolution

V.1.3 Low-Mass Stars

Main-sequence: core H fusion. If the star is $< 0.5\,M_\odot$ or so, will never fuse helium. May eventually become a red giant (degenerate He core, surrounded by H burning shell, surrounded by puffy envelope) and then a white dwarf, but this can take hundreds of billions of years.

V.1.4 Middle-Mass Stars

Main sequence lasts for Gyr. Eventually, only He in core. Contracts, increasing pressure and T but not enough for He to ignite. Becomes degenerate. Outside the core H burns in shell. Envelope puffs up, becomes red giant. Ascends the red giant branch (RGB).

H fusion continues to produce He. This "falls" into the core, making it contract further. Eventually, might get He fusion. Since this happens in a degenerate core (when $M < 1.5\,M_\odot$ or so), it will start as an unstable "flash" (raise T does not affect P), but the flash will take a long time to propagate through star so it will not really effect things too much. But extra energy will expand core, making things non-degenerate evetually. Moves to horizontal branch (hotter and smaller). This is basically a He-burning main-sequence, but it is much faster since the reaction is hotter and there is less fuel.

Eventually will exhaust He in core. Contracts again, looks a lot like RGB. Call this phase the asymptotic giant branch (AGB). Moves up again, things become somewhat unstable. Pulsations fling off outer layers, lead to planetary nebula (PN) and white dwarf (WD).

V.1.5 More Massive Stars

Core never becomes degenerate. So do not become much brighter on the RGB — mostly just become redder. Eventually ignite He, but it is a more gentle process. Move to horizontal branch (HB: helium burning main sequence). Exhaust He, then contract again up AGB. Get rid of outer layers, end up as PN and WD.

V.1.6 Massive Stars

Keep plowing through fusion, making more and more massive elements. Luminosity scales up steeply with mass: $L \propto M^3$ or M^4. Available fuel depends linearly on the mass. So timescale for evolution (nuclear timescale) is $\tau_{\rm nuc} \propto \mathrm{fuel/consumption\,rate} \sim M/L \sim M^{-2.5}$. Which means that massive stars burn through their fuel very quickly.

For instance, for a $25\,M_\odot$ star:

Stage	Timescale	$T/10^9$ K	ρ
H burning	7×10^6 yr	0.06	5×10^1
He burning	5×10^5 yr	0.23	7×10^2
C burning	600 yr	0.93	2×10^5
Ne burning	1 yr	1.7	4×10^6
O burning	6 mo	2.3	1×10^7
Si burning	1 day	4.1	3×10^7

This will be followed by (in general) a core-collapse supernova because further fusion is unable to generate any energy: no exothermic reactions are possible.

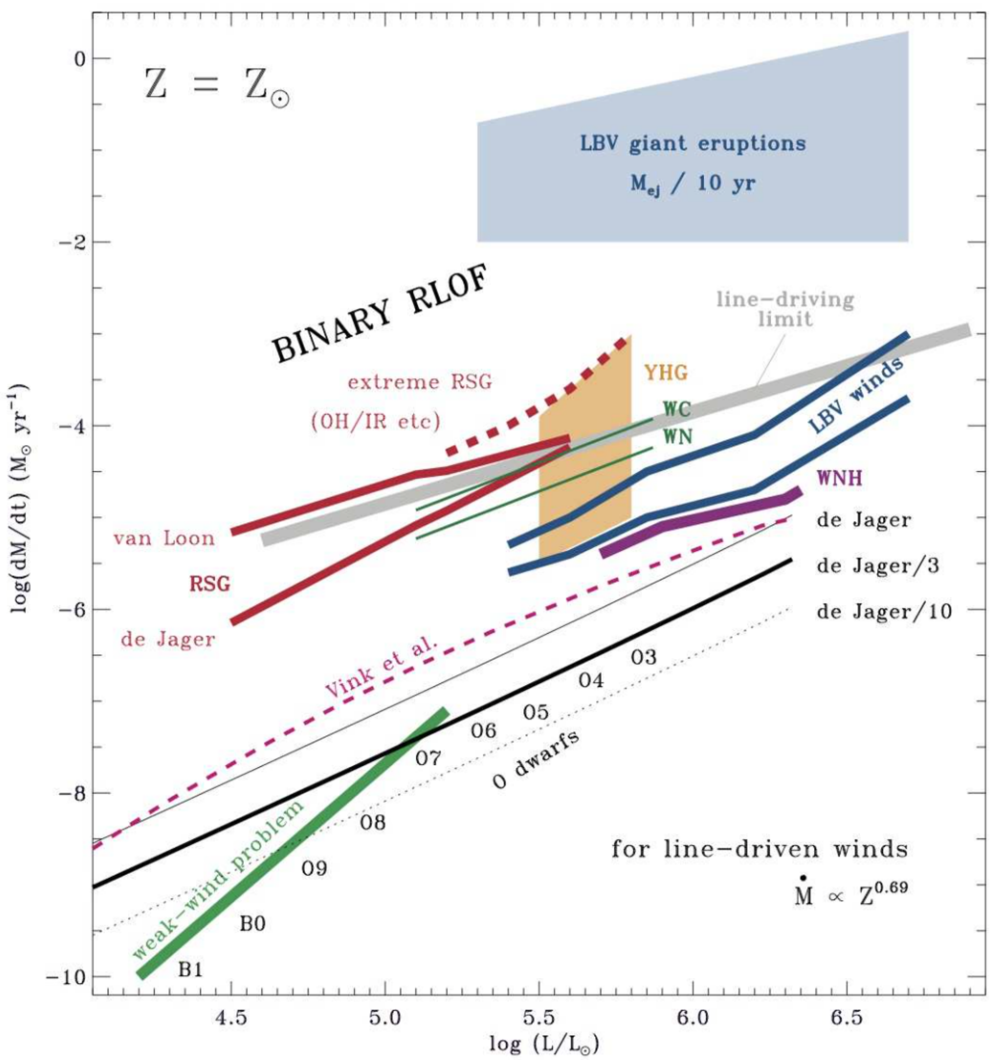
However, mass-loss also dominates the end products of these stars. The later stages of evolution are very large, so the outer regions are very loosely bound. Even in a $1\,M_\odot$ star this is significant. At $T = 5 \times 10^4$ K inside a star, the kinetic energy $(3/2)k_B T \sim 7$ eV. At a radius of $200\,R_\odot$ (AGB) the binding energy will be similar, and such particles can escape the star.

Overall mass-loss is complicated and we have to determine the results semi-empirically. For instance, there is an expression (Garmani & Conti 1984):

$$\dot{M} \approx -22 \times 10^{-8} \left(\frac{L}{10^3\,L_\odot}\right)^{3.7} \left(\frac{M}{M_\odot}\right)^{-3.1} \left(\frac{R}{10^2\,R_\odot}\right) M_\odot\,\mathrm{yr}^{-1} \tag{V.123}$$

This has a lot of free terms. But we can relate the mass to the radius and luminosity for different phases of evolution. In the end we find that mass-loss can be quite significant for high-mass stars.

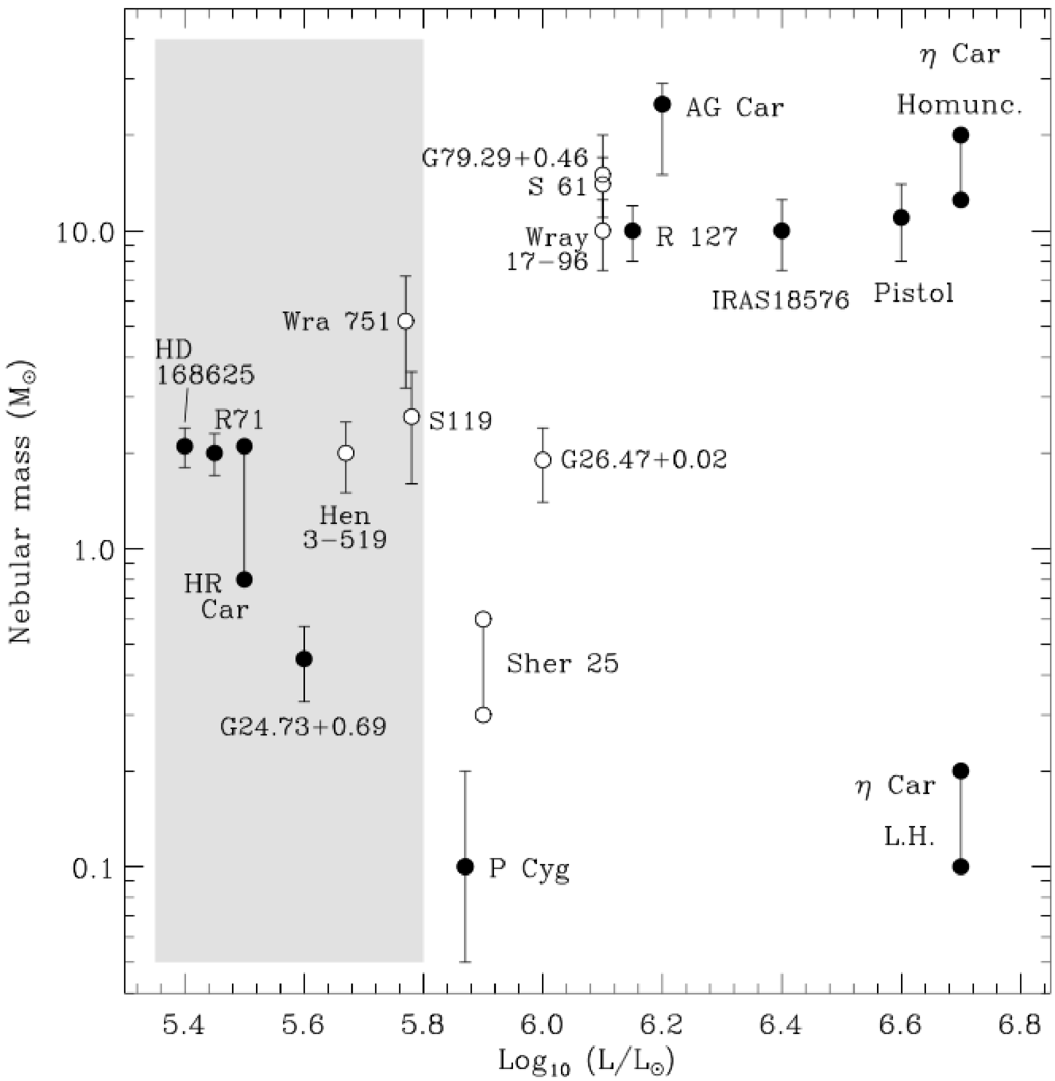
This plot, from Smith (2014, ARA&A), shows different prescriptions for mass-loss as well as typical observed values. Much mass-loss may not be steady, but may be through luminous blue variable (LBV) eruptions or clumpy winds, which make measuring and modeling much harder.



To complicate further, during the AGB phase we have large-amplitude radial pulsations driven by instabilities in the atmosphere, as well as a large increase in the luminosity.

The most massive stars that we know of are also very short-lived, rare, and far away. So hard to understand. But examples in the Milky Way are the Pistol Star and Eta Carina. Both of these are likely 10's of $M_\odot$ and have lost 10's of $M_\odot$ in significant outflows. So the mass of the star right before supernova is not necessarily very close to what it was when it was born. Binary interactions can further modify the result.

This plot from Smith (2014) shows the estimates of mass in shells around LBV stars. The stars are very luminous and the shells are very massive.



Very massive stars ($\gtrsim 100\,M_{\odot}$) will form. However, above $60\,M_{\odot}$ or so, pulsations occur driven by instabilities related to opacity changes and to nuclear power changes (κ and ϵ mechanisms) that drive a lot of mass-loss. Add to this the instability from a large amount of radiation pressure and they are not very stable.

Likely in the early universe at lower metallicity (fewer generations of stars) the mass-loss was less efficient. At very low metallicities the mass-loss prescriptions can change dramatically. That is because much of mass-loss is determined by very weakly bound sites in the outer atmosphere. Here parcels of gas are carried away by momentum exchange between the outgoing flux and gas due to opacity. If the gas is only H, then the opacity is low because there are very few atomic transitions. So we expect more massive stars early on. These most massive stars may not go through the normal channels to end up at core-collapse SN. Instead they may have a significant pair-instability mechanism, that can lead to violent pulsations and mass loss or even complete disruption through a supernova leading to no remnant (for a He core of $65\,M_\odot$, corresponding to an initial mass of $140\,M_\odot$).

The physics here is that radiation pressure is very significant. If the radiation field is sufficiently energetic such that $\gamma + \gamma \to e^- + e^+$ is favored, the pressure support will be reduced and the core will contract. Contraction will heat up the core, meaning more γ-rays and more pair production. This cascades until support is gone and the core collapses. The core has runaway thermonuclear reactions that release enough energy to disrupt the core, leaving nothing behind.

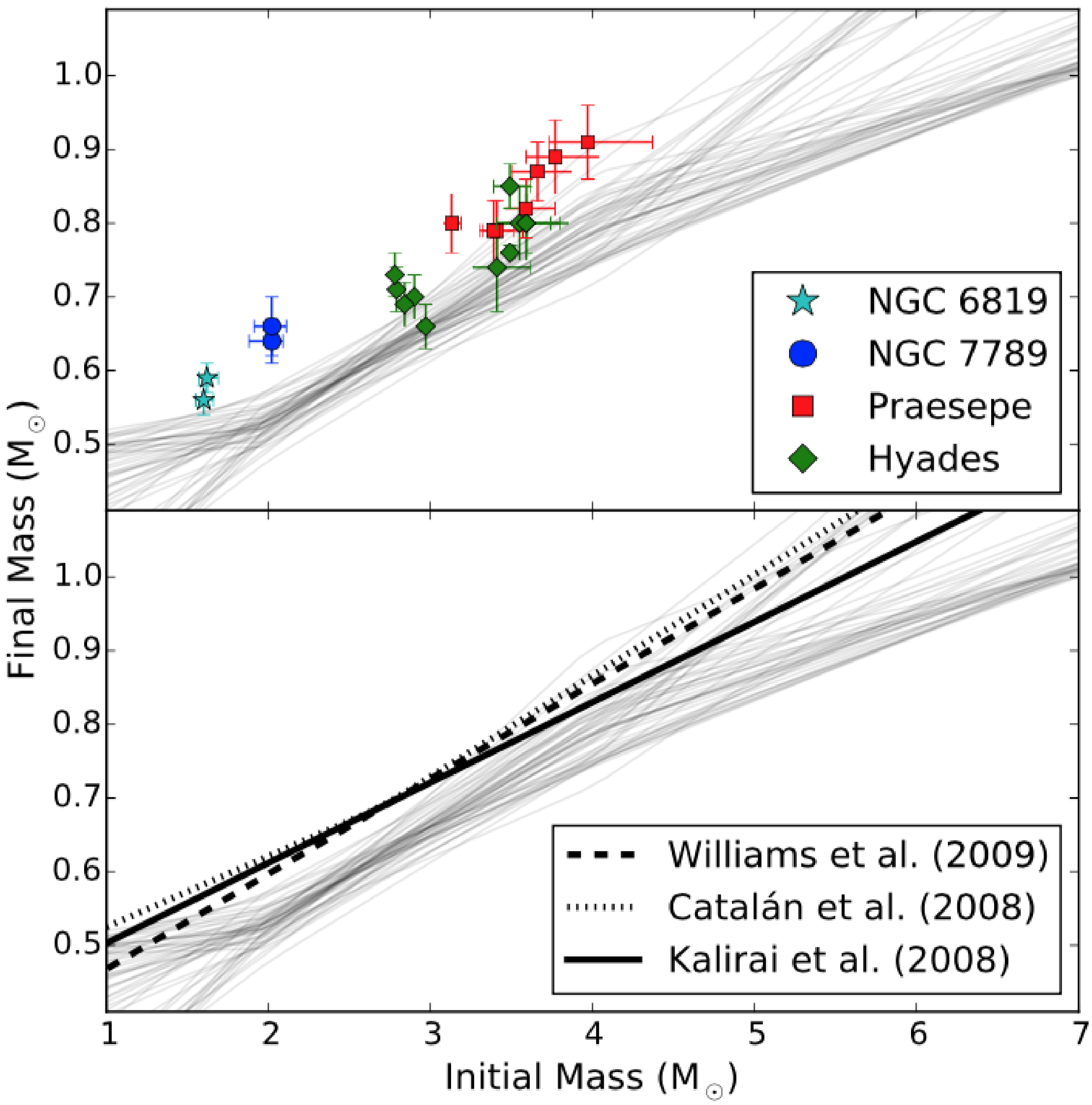
V.2 Diversion: How do we measure the mass of a star?

Methods:

1. Main-sequence fitting: requires knowledge that it is on the main sequence; requires estimate of foreground extinction (dust); requires assumption that it is a single star
2. Binaries: spectroscopic binaries have inclination degeneracy; eclipsing binaries (where we know inclination) are rare; can worry that binary evolution has altered single-star expectations
3. Asteroseismology: can be very precise, but only available for a small number of stars

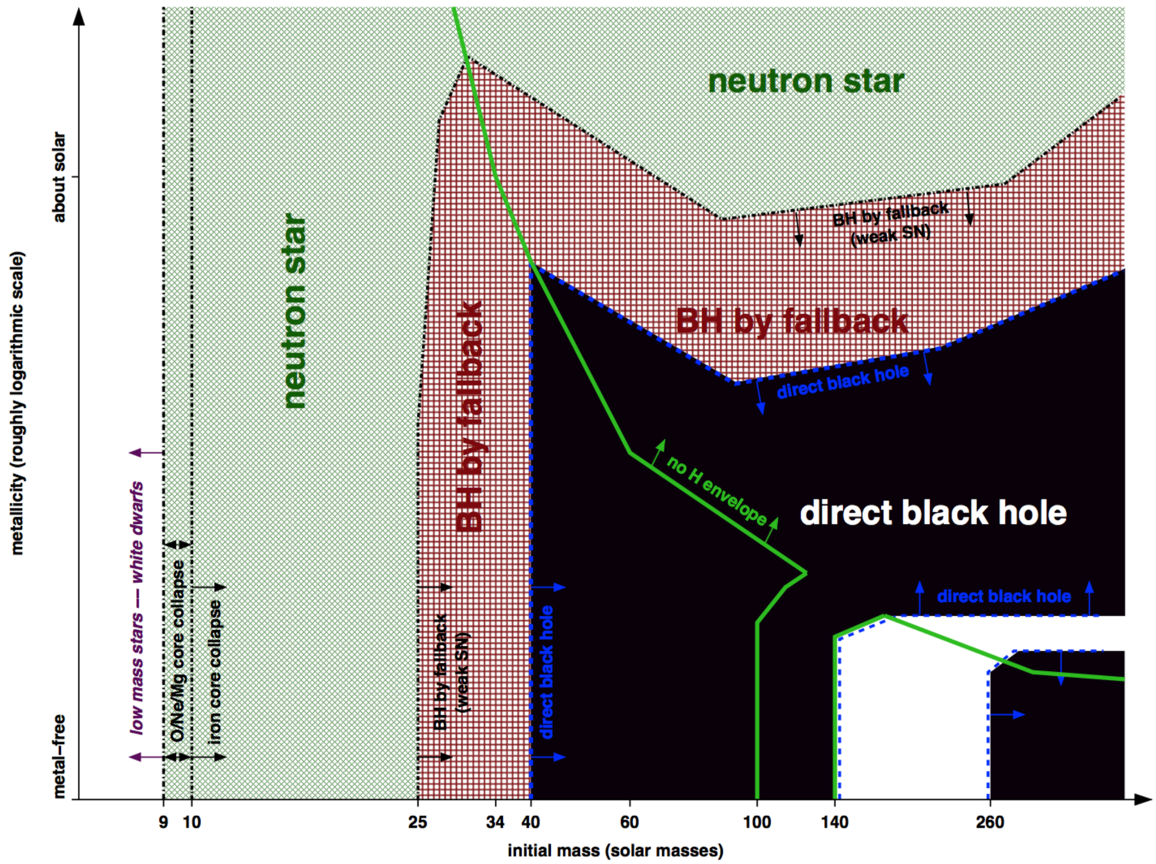
Lecture VI Stellar End Products

If the mass if $< 8\,M_\odot$ or so, degenerate core of the star ends up as a white dwarf.



Efforts to map the final WD product from the main-sequence mass are difficult. The main question is one of mass-loss: how much mass is lost in the main-sequence (small), RGB/AGB (considerably more) and PN formation (a lot) stages. In this plot from Andrews et al. (2015), the main differences between the different models (thin lines) are how they treat mass loss.

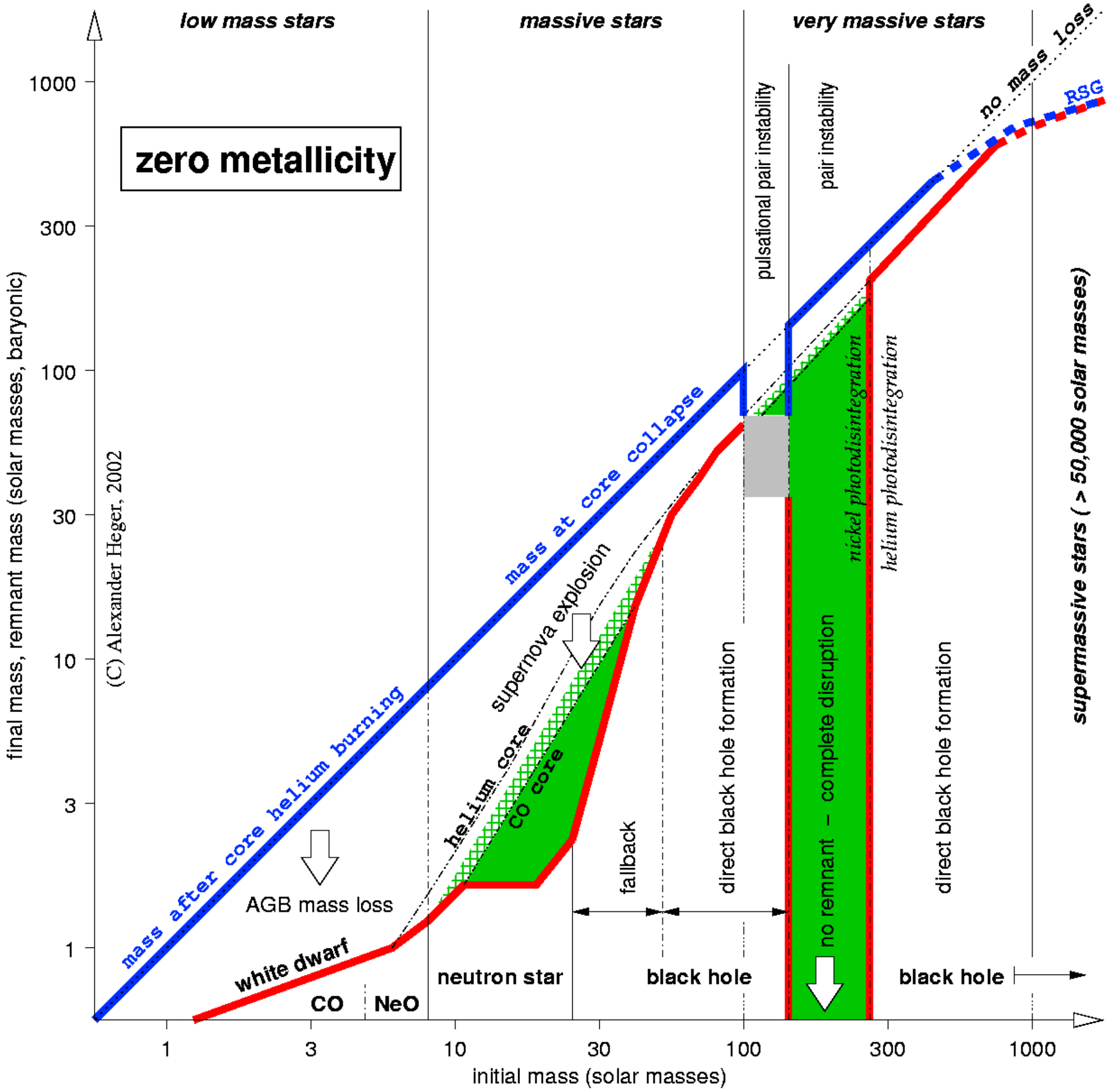
Extending to higher masses we can have counter-intuitive behavior from Heger et al. (2003):



At low masses the behavior is as expected: WDs transitioning to NSs. But eventually we need at least one more variable, which here is metallicity. That is because of the effects of mass-loss. There are possibly different regimes of black hole (BH) formation, both direct collapse and "fall-back", where there is an initial neutron star surrounded by a disk of material that drives it over its maximum mass limit. So there is the possibility of non-monatonic results, where more massive stars can have less massive products depending on metallicity. And there is also the regime of no product, in the pain-instabillity supernova gap.

Other uncertainties can come from the effects of metallicity, rotation, magnetic fields, etc. All of those can change the mass loss and are much harder to measure. So when inferring properties from small sub-samples can make systematic errors.

For example, see `http://2sn.org/firststars/`:



which may be relevant to the most massive stars produced in the early universe.

Lecture VII Core Collapse

VII.1.1 Advanced Burning

Carbon burning:

$$^4\mathrm{He} +^{12}\mathrm{C} \rightarrow^{16}\mathrm{O} + \gamma \tag{VII.124}$$

(which is good for us). Straightforward reaction w/o resonance or unstable meta-state. Then

$$^4\mathrm{He} +^{16}\mathrm{O} \rightarrow^{20}\mathrm{Ne} + \gamma \tag{VII.125}$$

Can have additional He captures to make other elements (Ne, Mg, Si), although this doesn't happen too much during He burning since the temperature isn't high enough.

These reactions bypass Li, Be, B. We do not see very much of these, and most of what we do see comes not from stars but from a cosmic ray hitting a heavy nucleus and splitting it (spallation).

Eventually hotter and hotter. Core of C and O builds up (if the star isn't massive enough, this ends up as WD). C burning starts at 5×10^8 K, with reactions like:

$$^{12}\mathrm{C} +^{12}\mathrm{C} \rightarrow^{20}\mathrm{Ne} +^4\mathrm{He} \tag{VII.126}$$

(or $^{23}\mathrm{Na}$ or $^{23}\mathrm{Mg}$). If the star is a little more massive (8–10 $M_\odot$) things can ened here with O/Ne/Mg WD.

Then Ne burning if $> 10^9$ K, making $^{24}\mathrm{Mg}$. Important next step is

$$^{16}\mathrm{O} +^{16}\mathrm{O} \rightarrow^{28}\mathrm{Si} +^4\mathrm{He} \tag{VII.127}$$

($> 2 \times 10^9$ K) followed by Si burning (3×10^9 K). The reactions mostly involve heavy nuclei + light particles made from breaking up the heavier ones, gets rather complicated.

This break-up happens when photons have enough energy to split apart nuclei. E.g.,

$$\gamma +^{28}\mathrm{Si} \rightarrow^{24}\mathrm{Mg} +^4\mathrm{He} \tag{VII.128}$$

Just like ionization, but at temperatures 10^6 times higher.

All this happens very quickly. And it needs a very massive star to keep going, where everything happens more quickly than in the Sun.

For a 15 $M_\odot$ star:

Stage	Timescale	Reaction	Product	$T/10^9$ K	ρ (cgs)	$L/L_\odot$	$L_\nu/L_{\nu,\odot}$
H burn	11 Myr	pp	He	0.035	5.8	28,000	1800
		CNO	He,N,Na				
He burn	2.0 Myr	3α	C	0.18	1390	44,000	1900
		$^{12}\mathrm{C} + \alpha$	O				
C burn	2000 yr	$^{12}\mathrm{C} +^{12}\mathrm{C}$	Ne,Na,Mg,Al	0.81	2.8×10^5	72,000	3.7×10^5
Ne burn	0.7 yr	$^{20}\mathrm{Ne} + \gamma$	O,Mg,Al	1.6	1.2×10^7	75,000	1.4×10^8
O burn	2.6 yr	$^{16}\mathrm{O} +^{16}\mathrm{O}$	Si,S,Ar,Ca	1.9	8.8×10^6	75,000	9.1×10^8
Si burn	18 d	$^{28}\mathrm{Si} + \gamma$	Fe,Ni,Cr,Ti,...	3.3	4.8×10^7	75,000	1.3×10^{11}
Fe collapse	1 s	neutronization	neutron star	> 7.1	$> 7.3 \times 10^9$	75,000	$> 3.6 \times 10^{15}$

Remember that overall luminosity is limited by Eddington limit, $L_{\rm Edd} = 3.2 \times 10^4\, L_\odot (M/M_\odot)$, so we are at $\approx 15\%$ of Eddington. Neutrinos have no such limit, so can be much higher.

Star with $> 10\, M_\odot$ (or so). Will go through all stages of nuclear burning in < 10 Myr. Eventually have Si burning making iron at $T = 3 \times 10^9$ K, surrounded by shells of lighter elements. Cannot get energy out of iron via fusion, so core contracts (just like RGB). Stabilized somewhat by degenerate electrons, but Si burning dumps increasing amounts of stuff on and electrons get increasingly relativistic. When the core is at $M_{\rm Ch} \approx 1.4\, M_\odot$, electrons have become ultra-relativistic and the core can no longer support itself.

VII.1.2 Onset of Collapse

During contraction T rises. If makes exothermic reactions possible, then T and pressure rise and collapse stops. But what if no exothermic reaction is possible? If only endothermic, reduces P, makes contraction into collapse. Once $kT > 1$ MeV (10^{10} K) you can also have direct neutrino production via:

$$\gamma + \gamma \to e^+ + e^- \to \nu_e + \bar{\nu}_e \tag{VII.129}$$

which further increases the neutrino luminosity and destabilizes the star.

Possible reactions are photodisintigration of nuclei and electron capture (inverse β decay). Photodisintigration: KE is used to unbind nuclei. Electron capture: KE of electrons is converted into KE of neutrinos (and lost). These both suck up energy very effectively, turning contraction into free-fall. At this point $\rho \approx 10^9\,{\rm g\,cm^{-3}}$, and free-fall happens with:

$$\tau_{\rm ff} = \sqrt{\frac{3\pi}{32 G\rho}} \approx 1\, ms \tag{VII.130}$$

VII.1.3 Photodisintigration

T rises enough such that photons have nuclear-scale energies. Takes a tightly-bound Fe nucleus and makes two or more loosely bound nuclei, absorbing binding energy. This can take many paths, but as an example:

$$\gamma +^{56} {\rm Fe} \leftrightarrow 13^4{\rm He} + 4n \tag{VII.131}$$

equilibrium between iron and helium + neutrons. This takes:

$$Q = (13 m_4 + 4 m_1 - m_{56})c^2 = 124.4\,{\rm MeV} \tag{VII.132}$$

So 1 kg of Fe can absorb 2×10^{21} erg (50 kton of TNT). Use Saha equation to determine relative fractions:

$$\mu_{56} = 13\mu_4 + 4\mu_1 \tag{VII.133}$$

with

$$\mu_A = m_A c^2 - k_B T \ln\left(\frac{g_A n_{Q,A}}{n_A}\right) \tag{VII.134}$$

and

$$n_{Q,A} = \left(\frac{2\pi m_A k_B T}{h^2}\right)^{3/2} \tag{VII.135}$$

which gives:

$$\frac{n_4^{13} n_1^4}{n_{56}} = \frac{g_4^{13} g_1^4}{g_{56}} \frac{n_{Q,4}^{13} n_{Q,1}^4}{n_{Q,56}} e^{-Q/k_B T} \tag{VII.136}$$

The g factors can be complicated, but we will assume $g_1 = 2$, $g_4 = g_{56} = 1$. From this we get that roughly 75% of the Fe is dissociated when $\rho = 10^9\,\mathrm{g\,cm^{-3}}$ and $T = 10^{10}$ K.

For higher temperatures still:

$$\gamma +^4 \mathrm{He} \leftrightarrow 2p + 2n \tag{VII.137}$$

Overall, in collapse of $1.4\,M_\odot$, absorb 4×10^{51} erg in breaking Fe and 1×10^{52} erg in breaking He, for a total of $E_{\text{photo}} \approx 1.4 \times 10^{52}$ erg. This is $L_\odot \times 10^{11}$ yr. Easy to see how this could lead to collapse.

VII.1.4 Electron Captures

Neutron can decay on its own (β decay):

$$n \rightarrow p + e^- + \bar{\nu}_e \tag{VII.138}$$

with half-life of 10.25 min. This produces electrons and neutrinos with total energy of $(m_p - m_n)c^2 = 1.3\,\mathrm{MeV}$, so the max electron energy is $1.3\,\mathrm{MeV}$. If electrons with that energy cannot be produces, neutrons cannot decay. For instance, if all of the low-energy spots are filled by other electrons (in a dense gas of degenerate electrons with $E_F > 1.3\,\mathrm{MeV}$) this cannot happen.

Moreover, if electrons with $E > 1.3\,\mathrm{MeV}$ are around, they can capture onto protons to form neutrons:

$$e^- + p \rightarrow n + \nu_e \tag{VII.139}$$

This can happen even if the protons are in nuclei. For instance, *neutronization* starts when:

$$e +^{56} \mathrm{Fe} \rightarrow^{56} \mathrm{Mn} + \nu_e \tag{VII.140}$$

is favorable, at $\rho > 1.2 \times 10^9\,\mathrm{g\,cm^{-3}}$. This happens when $E_F = m_e c^2 + 3.7\,\mathrm{MeV}$. The Mn would normally decay back in 2.6 hr, but here instead it will capture again to make ^{56}Cr. And so on as the density goes up past $10^{10}\,\mathrm{g\,cm^{-3}}$.

This speeds up further when $\rho > 10^{11}\,\mathrm{g\,cm^{-3}}$. Almost all of the energy in neutrinos is lost. So the pressure support goes away quickly. How much energy? Core has $\sim 10^{57}$ electrons, which could make 10^{57} neutrinos. Each capture will take an electron with $E \approx 10\,\mathrm{MeV}$, appropriate for $\rho > 2 \times 10^{10}\,\mathrm{g\,cm^{-3}}$. So total energy is:

$$E_{\text{cap}} \approx 10^{57} \times 10\,\mathrm{MeV} = 1.6 \times 10^{52}\,\mathrm{erg} \tag{VII.141}$$

which is similar to that from photodisintigration. But in this case it is carried from the star in a burst of neutrinos. If they could get out immediately, the burst would take $\sim$ ms. But in fact when the core density is $> 10^{11}\,\mathrm{g\,cm^{-3}}$ the mfp becomes comparable to the size of the core, a few km. They will get out, but it will take a few seconds.

VII.1.5 And Then...

The collapse will proceed on the free-fall timescale. What will stop it? It will stop when the bulk density is comparable to the nuclear density. For a nucleus with A nucleons, $R \approx r_0 A^{1/3}$ with $r_0 = 1.2$ fm. So $\rho_{\rm nuc} = 3m_n/4\pi r_0^3 = 2.3 \times 10^{14}\,{\rm g\,cm^{-3}}$. Once we are that this stage we need new physics (neutron degeneracy, nuclear forces). The collapse will stop when the density is a few times this as strong nuclear force comes in, and creates a "bounce". This propels a shock wave through the material, leading to a supernova.

Supernovae are observed to have 10^{51} erg of KE and 10^{49} erg of optical energy (over the first few years). Where does this come from? Gravitational binding energy:

$$E_G \sim \frac{GM^2}{R_{\rm core}} = 3 \times 10^{53}\,{\rm erg} \left(\frac{M}{M_\odot}\right)^2 \frac{10\,{\rm km}}{R} \tag{VII.142}$$

Which is orders of magnitude more than we see. We only see a small fraction of this, and we don't quite know exactly how the energy is partitioned. But this is plenty of energy compared to photodisintigration or electron capture. Most of the energy in fact comes via neutrinos, either right during collapse or later, as the neutron star cools. This happens over the diffusion timescale, $R^2/c\bar{l}$, and each flavor of neutrinos will carry $\sim E_G/6$.

On Feb 23, 1987, two neutrino detectors recorded excesses. They identify neutrinos via:

$$\bar{\nu}_e + p \to n + e^+ \tag{VII.143}$$

and if the positron has enough energy, it will be faster than the local speed of light in water, so it will emit Čerenkov radiation. That can be detected. Only ~ 1 in 10^{15} neutrinos is expected to be detected.

Saw about 20 neutrinos over $\sim 10\,$s. Expect that this is the diffusion timescale, which makes sense if $R \approx 100$ km and $\bar{l} = 10^{-4}R$. This came from SN 1987A in the LMC, implying an energy of about $(0.3 - 0.5) \times 10^{53}\,$erg for $\bar{\nu}_e$.

Energies of the neutrinos is consistent with $T_{\rm Eff} \approx 5 \times 10^{10}\,$K. Compare to internal temperature, using the mfp:

$$T_{\rm Eff} \approx \left(\frac{\bar{l}}{R}\right)^{1/4} T_I \tag{VII.144}$$

implies an internal temperature of $10^{11-12}\,$K.

Lecture VIII The Neutrino Mechanism

As we have discussed, when the core of a massive star reaches the Chandrasekhar mass, it can collapse. The collapse of the core to radii $\sim$ 30 km releases a huge amount of energy – typically per nucleon

$$E \approx \frac{GMm_p}{R} \approx 62 \left(\frac{M}{1.4\,M_\odot}\right)\left(\frac{R}{30\,\mathrm{km}}\right)^{-1} \text{ MeV} \tag{VIII.145}$$

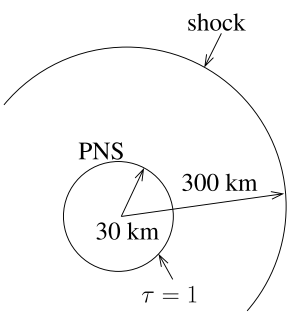
, which is much greater than the 7 MeV that typically binds the nuclei together. As a result, the iron is photo dissociated to alpha particles, i.e., helium nuclei.

The collapse core reaches nuclear densities and bounces to a radius of $r \sim 30$ km, which drive a shock wave into the in falling material and we have a hot protoneutron star. This shock wave moves out to about 300 km where it stalls – balanced by ram pressure and in falling material.

In the protoneutron star, it is so hot that everything dissociates to nucleons. The density is also so high that it is energetically favorable to combine protons and electrons to make neutrons in a process known a neutronization.

$$p + e \rightarrow n + \nu_e, \tag{VIII.146}$$

where the ν_e would normally escape to infinity. However, now the density of the protoneutron star is so high that the neutrinos don't immediately escape to infinity, but must diffuse out. So a picture of the system can be visualize as:



We can estimate the neutrino photosphere, i.e., the $\tau = 1$ surface by consider the cross section:

$$\sigma_\nu = 10^{-45}\left(\frac{E_\nu}{m_e c^2}\right)^2 A^2\,\mathrm{cm}^2, \tag{VIII.147}$$

where E_ν is the energy of the electron neutrino. Now the electrons are degenerate and have Fermi energy

$$E_F = 1\left(\frac{\rho}{10^6\,\mathrm{g\,cm^{-3}}}\right)^{1/3}\mu_e^{-4/3}\,\mathrm{MeV}, \tag{VIII.148}$$

where $\mu_e = A/Z$ is the nucleon to electron number densities. For a density of 10^{12} and $\mu_e = 2$, we find $E_F \approx 40$ MeV. The mean free path of these neutrinos is then

$$\lambda_{\mathrm{mfp}} = \frac{1}{n\sigma_\nu} = \frac{Am_p}{\rho\sigma_\nu} = 1\left(\frac{\rho}{10^{12}\,\mathrm{g\,cm^{-3}}}\right)^{-5/3}\left(\frac{A}{2}\right)^{-1} \text{ km} \tag{VIII.149}$$

The diffusion time is then

$$t_{\text{diff}} = \frac{R^2}{\lambda_{\text{mfp}} c}. \tag{VIII.150}$$

For nuclear density $\rho \sim 10^{14}\,\mathrm{g\,cm^{-3}}$ and $R \sim 30$ km, gives $\lambda_{\text{mfp}} = 50$ cm and $t_{\text{diff}} = 6$ s, which is close to the right answer of 10 s. The photosphere is at $\tau = 1 = n\sigma_\nu R \to \lambda_{\text{mfp}} \sim R$. This gives $\rho_{\tau=1} = 1.3 \times 10^{11}\,\mathrm{g\,cm^{-3}}$.

So we know that the diffusion time is ~ 10 s and the energy of the collapse is

$$E = \frac{GM^2}{R} \sim 10^{53}\,\mathrm{ergs} \to L = \frac{E}{t_{\text{diff}}} \sim 10^{52}\,\mathrm{ergs\,s^{-1}}. \tag{VIII.151}$$

So we can tap $\sim 1\%$ of the neutrino energy to get the energy of an observed SN, which is 10^{51} ergs.

To see if this is possible, let's look at how core collapse SN might proceed. The neutrinos from the photosphere illuminate the region behind the shock heating it up and adding to its energy. This region is known as the **gain region**.

The material that is falling into the gain region from the shock is order $0.1\,M_\odot\,\mathrm{s^{-1}}$. Assuming that is comes in through 4π, we have

$$\dot{M} = 4\pi\rho r_s^2 v \tag{VIII.152}$$

For $v = \sqrt{GM_{\text{PNS}}r_s}$, we can show that $\rho \approx 10^{10}\,\mathrm{g\,cm^{-3}}$ giving a mass of the gain region of $M_g = 4\pi r_s^3\rho/3 \approx 0.01\,M_\odot$. The amount of optical depth is then

$$\tau = n\sigma r_s = 6.3 \times 10^{-5}\left(\frac{r_s}{100\,\mathrm{km}}\right)\left(\frac{\rho}{10^{10}\,\mathrm{g\,cm^{-3}}}\right)\left(\frac{E_\nu}{m_e c^2}\right)^2 A \tag{VIII.153}$$

Now we can estimate average energy of the neutrino from the luminosity:

$$L_\nu = 10^{52}\,\mathrm{ergs\,s^{-1}} = 4\pi R^2\sigma T_e^4 \to T_e \approx 3\,\mathrm{MeV} \tag{VIII.154}$$

This gives an average neutrino energy of $E_\nu \approx 4k_B T_e \approx 12$ MeV. The factor of 4 comes from Fermi-Dirac statistics. It would be 3 for Bose-Einstein statistics, i.e., photons. In any case, we find

$$\tau = 0.04 - 0.08\left(\frac{r_s}{100\,\mathrm{km}}\right)\left(\frac{\rho}{10^{10}\,\mathrm{g\,cm^{-3}}}\right)\left(\frac{T_e}{3\,\mathrm{MeV}}\right)^2 \tag{VIII.155}$$

This corresponds to a total heating rate of $\dot{E} = L_\nu\tau$ for $\tau \ll 1$ so in principle there is both plenty of heating and plenty of coupling to get SN energies of 10^{51} ergs. In fact, it should be possible to get energies of order 10^{52} ergs, but the best 3-d calculations get at most 10^{51} ergs. The reasons for this is that the stuff that is falling through that gain region is falling onto the star and does not stay there forever. If it would stick around for 10 seconds they this is a possibility, but it does not. Rather the time it stays in the gain region is called the residence time

$$t_{\text{res}} = \frac{M_g}{\dot{M}} = \frac{r_s}{v} = 0.1s. \tag{VIII.156}$$

So the total energy absorbed is

$$E_{\rm SN} = L_\nu \tau t_{\rm res} \approx 4 - 10 \times 10^{-3} L_\nu, \tag{VIII.157}$$

If we say L_ν starts out a bit larger initially – say few $\times 10^{52}\,{\rm ergs\,s^{-1}}$, then we can almost get up to the 10^{51} ergs required for a SN explosion.

This neutrino mechanism is by far the most popular story by which SN explosion occur. This is because it is the greatest reservoir of energy and the fact that neutrinos were detected from SN 1987A. However, its coupling and hence efficiency is low. Other mechanisms include:

- Rapid spin and magnetic despinning – requires PNS to be born with large magnetic fields
- Jets from accreting material - Possible, but unclear how jets form in these environments – though this appear to be the case for long-GRBs
- Thermonuclear explosions - most recent idea, not yet well explored.

Lecture IX Supernova Explosions

Collapse of iron core of massive star.

10^{53} erg released, which is mostly the gravitational energy before collapse.

Of this:

- 1% goes into the kinetic energy of the explosion (10^{51} erg, so this is known as 1 foe).
- 0.01% goes into photons
- 99% goes into neutrinos

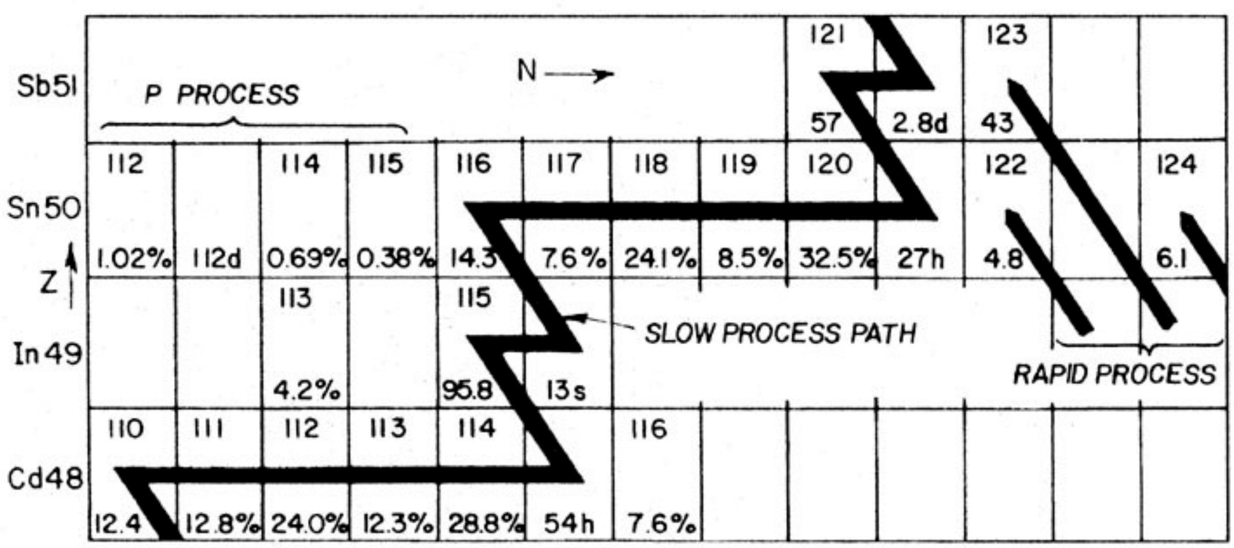
Late-time lightcurve of supernova depends mostly on what was surrounding the star (thick envelope or not) — the initial kinetic energy + radioactive decay does a good job of reproducing the energetics. But you have to be careful about observing at a specific wavelength because detailed processes (e.g., dust formation) can come into play and change the appearance.

IX.1.6 Chemistry

Sun produces mostly He from fusion. Massive stars can produce up to Fe, but still limits (i.e., little Li produced). How to make the rest? SNe.

Most of this happens through the *r-process* (rapid). Heavy elements + many neutrons $\rightarrow$ very heavy, unstable nuclei. Then decay to something stable (but still heavy).

The *s-process* (slow) can also occur, but that is mostly in post-main sequence evolution, where there is repeated n capture then α or β decay.



Core-collapse SNe produce a lot of α elements in addition, coming from α capture onto products of He burning (so mass is multiple of 4): Ne, Mg, Si, S, Ar, Ca, Ti. Type Ia supernovae produce a lot of Fe peak elements (V, Cr, Mn, Fe, Co, Ni) from explosive nucleosynthesis of C/O, but have much lower neutron flux around so don't get r-process.

Lecture X Long GRBs

Nice review paper: Piran (2004, Reviews of Modern Physics, 76, 1143).

1960s: Vela satellites are sent up to monitor nuclear test-ban treaty by looking for γ-ray flashes. Saw bursts of γ-rays from above rather than below.

Durations of bursts: 0.01 s–minutes, with structure down to <ms.

It was a major question as to where GRBs came from. Solar system? Galactic? Extragalactic? See fluence (integral of flux over time) of up to $10^{-4}\,\mathrm{erg\,cm^{-2}}$. Photon energies are typically MeV, stretching from keV to GeV. With unknown distance had unknown L (10^{32} erg up to 10^{52} erg), so didn't know mechanism.

Saw that bursts had very uniform distribution over the sky. Suggests that has to be isotropic. Could still be local (Oort cloud), but more likely distant (halo of MW or extragalactic).

Can try to figure out something about what the origin is even without knowing how far away sources are. Assume that they are homogeneous: same source density everywhere. If n bursts per volume per time and space is Euclidean, number with fluence $> S_0$:

$$N(S_0) = \frac{4}{3}\pi n r(S_0)^3 \tag{X.158}$$

with $r = \sqrt{E/4\pi S_0}$. So:

$$N(S_0) = \frac{4}{3}\pi n \left(\frac{E}{4\pi S_0}\right)^{3/2} \tag{X.159}$$

So $N(S_0) \propto S_0^{-3/2}$. We can plot the actual distribution of soures and we see it follows this at high S_0, but at low S_0 we don't see enough GRBs. Something is contributing to a paucity of sources. Maybe this is because of cosmology? (this can also be seen through a $V/V_{\max}$ test).

We see two classes of GRBs. Short/hard (< 1 s, higher energies) and long/soft (> 3 s, lower energies). Here we consider long/soft.

After the GRB, we see afterglow emission. It progresses from X-rays (keV) to optical, IR, and then radio. The afterglows enabled accurate positions to be found, which then led to actual redshifts and host galaxies. With redshifts, estimate energy $> 10^{52}$ erg. Optical declines $\propto t^{-\alpha}$ with $\alpha = 1.2$, but it often changes to $\alpha = 2$ across all wavelengths (achromatic) after a few days. Radio afterglow shows some very short timescale variability ($\sim$ days).

X.2 GRB Physics

General model is relativistic, jetted explosion coming from a massive star.

Can significantly lower E is we take the the emission to come from an anisotropic jet with bulk Lorentz factor Γ. We only see emission from $\theta < 1/\Gamma$. However, Γ decreases with time through the afterglow. Eventually it will reach the physical size of the jet θ_j. This is called a "jet break."

We correct the energy $E_\gamma = (\theta_j^2/2)E_{\gamma,\mathrm{iso}}$. Estimate the angle from the time until the achromatic break (also need to assume the local density):

$$\theta_j = 0.16(n/E_{k,\mathrm{iso},52})^{1/8} t_{b,\mathrm{days}}^{3/8} \tag{X.160}$$

This uses the total kinetic energy rather than just the γ-ray energy. Find jet breaks of few to 10°. Thus we end up with energies near 10^{51} erg, pretty uniformly. This also revises the rates to $1/3 \times 10^5$ yr per galaxy, roughly $1/3000$ the rate of SNe. Roughly tracks cosmic history of star formation.

We infer relativistic expansion from radio scintillation: need small source sizes to have refraction by ISM. Infer 10^{17} cm after 2 weeks, which would infer $v = 20c$. Instead $\Gamma > 100$ makes it work (actual size is $\sim \Gamma^2 ct$, but apparent size is $\sim \Gamma ct$).

Actual radiation (at all wavelengths) is synchrotron.

Association with star-forming galaxies and in some cases contemporaneous supernovae shows that these are coming from collapse of massive stars: “collapsar” model. This explains how to get so much energy out in so little time. Something like accretion onto BH in center of SN.

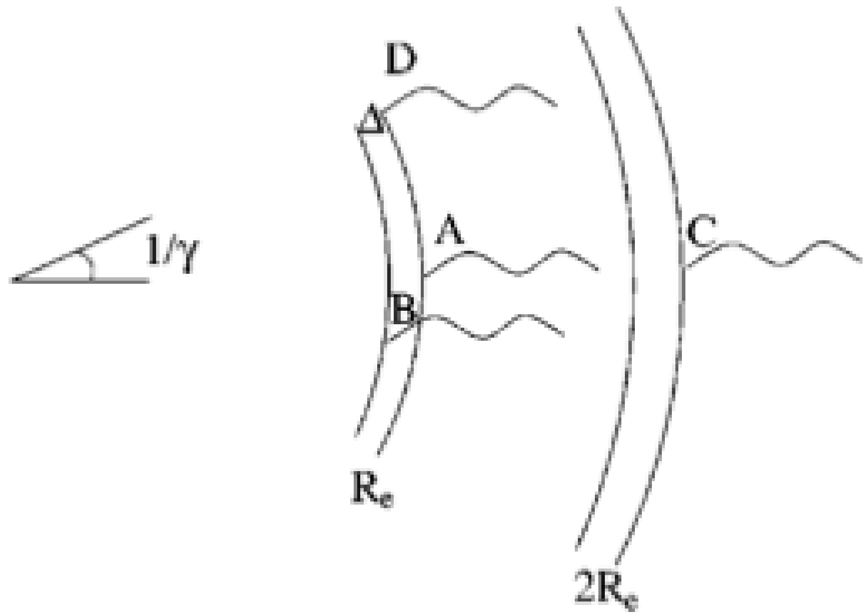
Relativistic motion also solves “compactness problem:” expect source to be optically thick to high-energy photons. Fluctuate with δt, so source is $< c\delta t$ in size. Would infer 10^{15} optical depth to $\gamma + \gamma \to e^+ + e^-$.

Instead, bulk motion with Γ. So the energy of the photons is reduced by Γ, so fewer can pair-produce. Size of inferred source is also smaller, $c\delta t\Gamma^2$ (observed time for emission from R is $R/2c\Gamma^2$). Makes it so that $\tau < 1$.

Emission from a shell is also influenced by Γ. For off-axis emission it comes at $R\theta^2/2c$, but since $\theta \sim 1/\Gamma$, comes at $R/2c\Gamma^2$. This is the same as the previous timescale.

X.2.1 Prompt Emission

Internal shocks in expanding shells. Duration of burst T from size of total jet, and substructure δt from the size of each shell. What is the cause of all this? People considered two models: external or internal shocks.



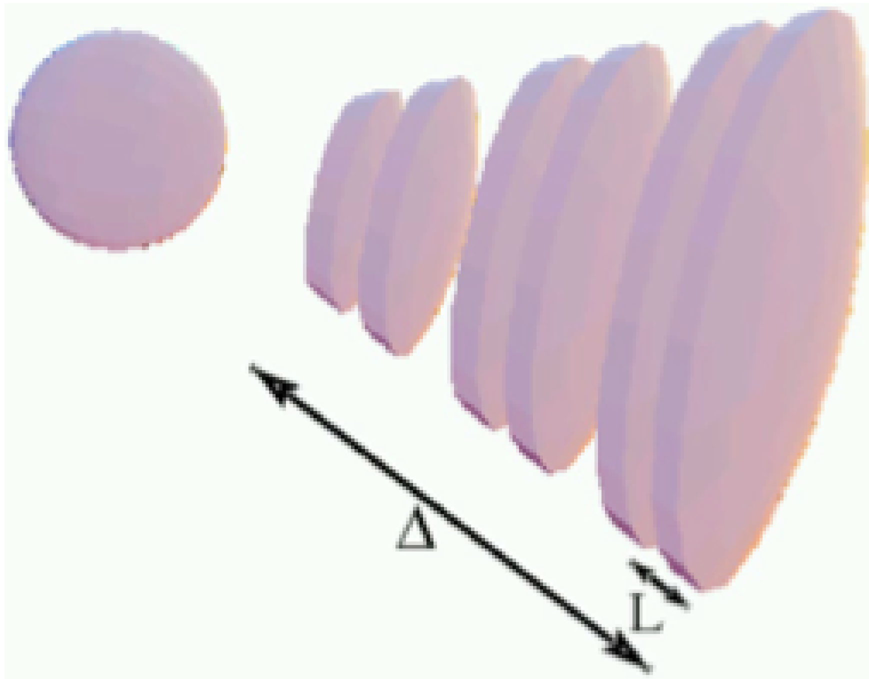
A model showing shell that emits from plowing into external medium and generates shock.

For the **external shock** model:
From Piran (2004): "Consider a quasispherical relativistic emitting shell with a radius R, a width Δ and a Lorentz factor Γ ... Consider now photons emitted at different points along the shock. Photons emitted by matter moving directly towards the observer (point A) will arrive first. Photons emitted by matter moving at an angle $1/\Gamma$ (point D) would arrive after $t_{ang} = R/2c\Gamma^2$. This is also t_R, the time of arrival of photons emitted by matter moving directly towards the observer but emitted at $2R$ (point C). Thus $t_R \approx t_{ang}$."

From each shell will spread out emission over $t_\Delta = \Delta/c$.

Range of angles will smear out emission over t_{ang}. If $t_\Delta < t_{ang}$, then burst will be smooth in time with width t_{ang}. So the whole thing will not have substructure. Instead we need $t_\Delta = \Delta/c > t_{ang}$: angular smoothing should be much smaller than the thickness of the emitting region. Then $t_\Delta \sim T$, and $t_{ang} \sim \delta t$.

For the **internal shock** model:
Again, from Piran (2004): "Consider an 'inner engine' emitting a relativistic wind active over a time $t_\Delta = \Delta/c$ (where Δ is the overall width of the flow in the observer frame). The source is variable on a scale L/c. Internal shocks will take place at $R_s \sim L\Gamma^2$. At this place the angular time and the radial time satisfy $t_{ang} \sim t_R \sim L/c$. Internal shocks continue as long as the source is active, thus the overall observed duration $T = t_\Delta$ reflects the time that the 'inner engine' is active. Note that now $t_{ang} \sim L/c < t_\Delta$ is trivially satisfied. The observed variability time scale, δt, reflects the variability of the source L/c, while the overall duration of the burst reflects the overall duration of the activity of the 'inner engine.'"



Based on most observables the **prompt** emission is caused by internal shocks. However, the **afterglow** is caused by external shocks. So we have the internal/external model for long GRBs.

Internal shock has a faster shell overtake a slower one at:

$$R_{int} = c\delta t \Gamma^2 = 3 \times 10^{14}\,\mathrm{cm}\Gamma_{100}^2 \tilde{\delta t} \tag{X.161}$$

$\tilde{\delta t}$ is the time difference between when the shells were emitted. This timescale must be less than the timescale for external shocking. If Γ is too high this will happen too late.

Consider two shells with m_r and m_s with $\Gamma_r > \Gamma_s \gg 1$. In a collision the bulk Lorentz factor of the result is:

$$\Gamma_m \approx \sqrt{\frac{m_r\Gamma_r + m_s\Gamma_s}{m_r/\Gamma_r + m_s/\Gamma_s}} \tag{X.162}$$

We have $\mathcal{E}$ is the energy in the local frame and $E = \Gamma_m \mathcal{E}$ the energy in the observer frame. But E is the difference in total energies before and after collision:

$$E = m_r c^2(\Gamma_r - \Gamma_m) + m_s c^2(\Gamma_s - \Gamma_m) \tag{X.163}$$

So we can look at the conversion efficiency: how much of the original KE gets turned into shock energy:

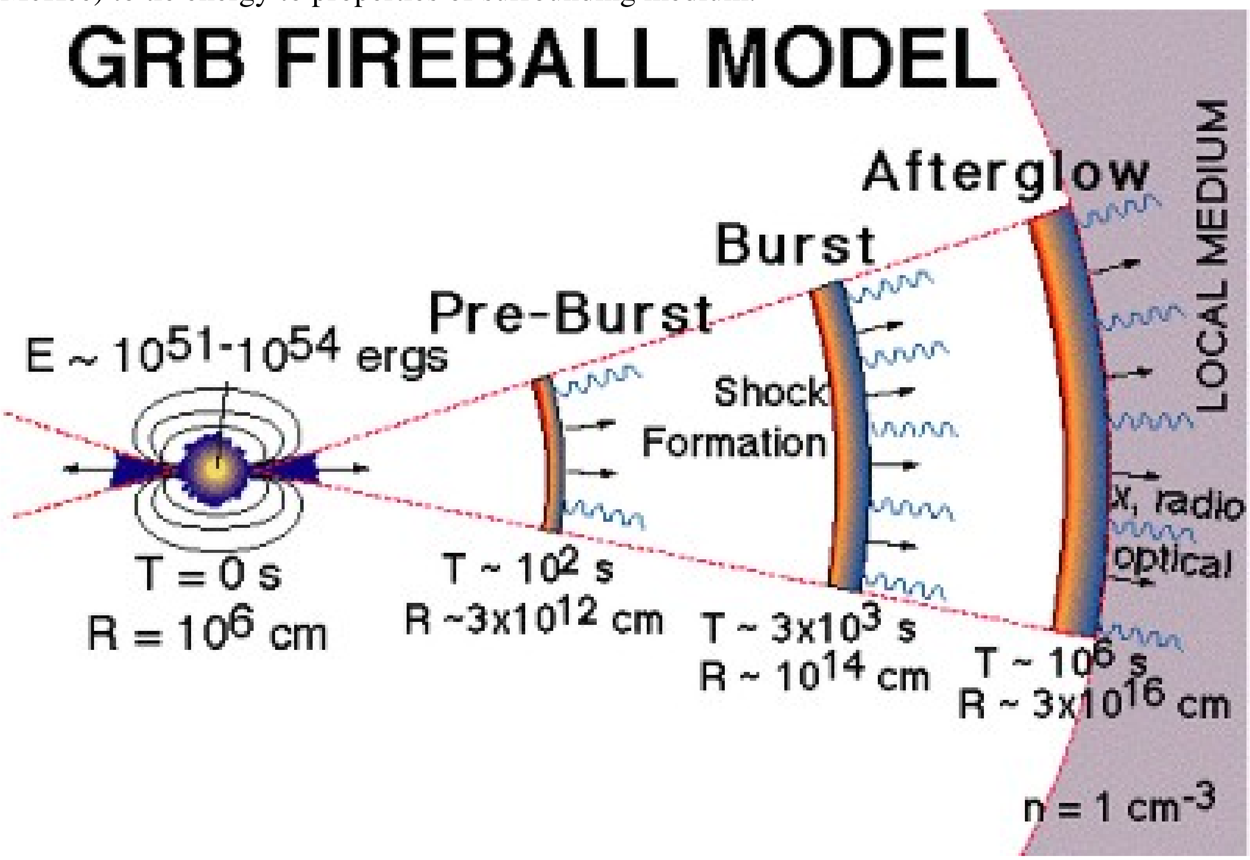
$$\epsilon = 1 - \frac{(m_r + m_s)\Gamma_m}{m_r\Gamma_r + m_s\Gamma_s} \tag{X.164}$$

To have high ϵ needs $\Gamma_r \gg \Gamma_s$ and comparable masses.

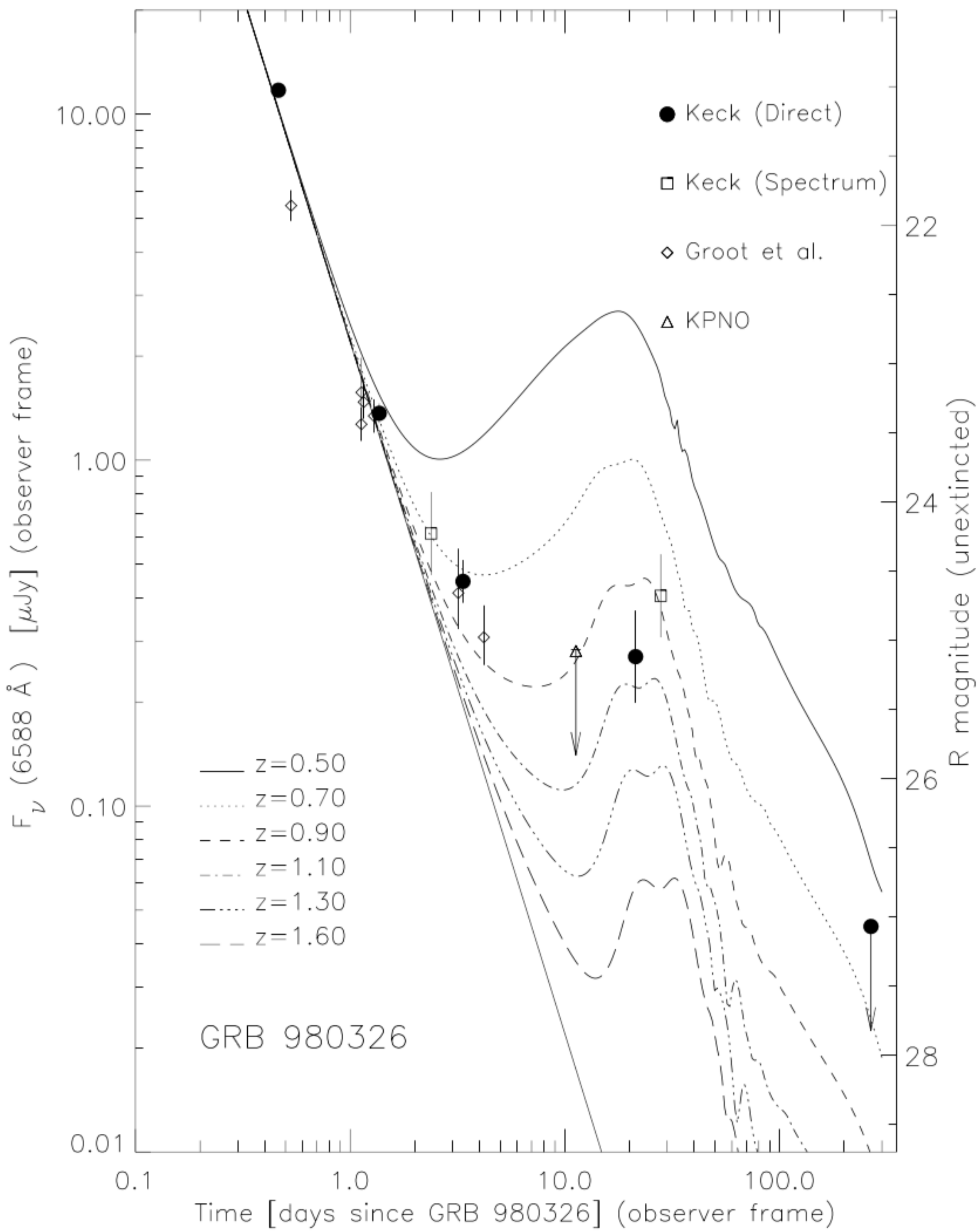
There are still questions about the actual acceleration mechanism, the nature of the central engine, the efficiency, and baryon loading. The last is that when the jet sweeps up $\Gamma m_j c^2$ it should slow down. But it doesn't. So the region above the jet should be pretty empty. We think that this is evacuated by the progenitor star.

X.2.2 Afterglow

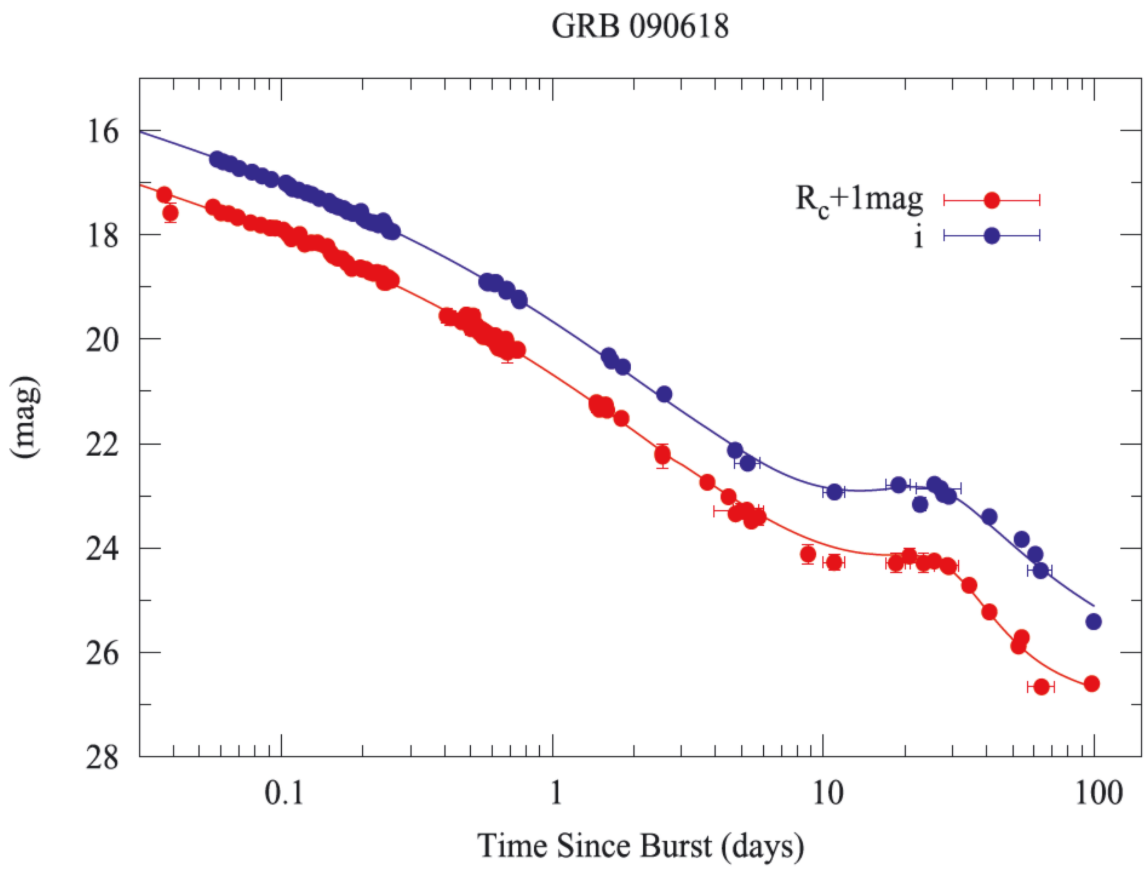
Relativistic jet collides with external medium. Progression in energy is from cooling and from optical depth considerations: at the beginning is optically thick at low energies. This $\tau = 1$ energy moves down from X-ray to optical to radio. Use relativistic variant of Sedov model (Blandford-McKee) to tie energy to properties of surrounding medium.



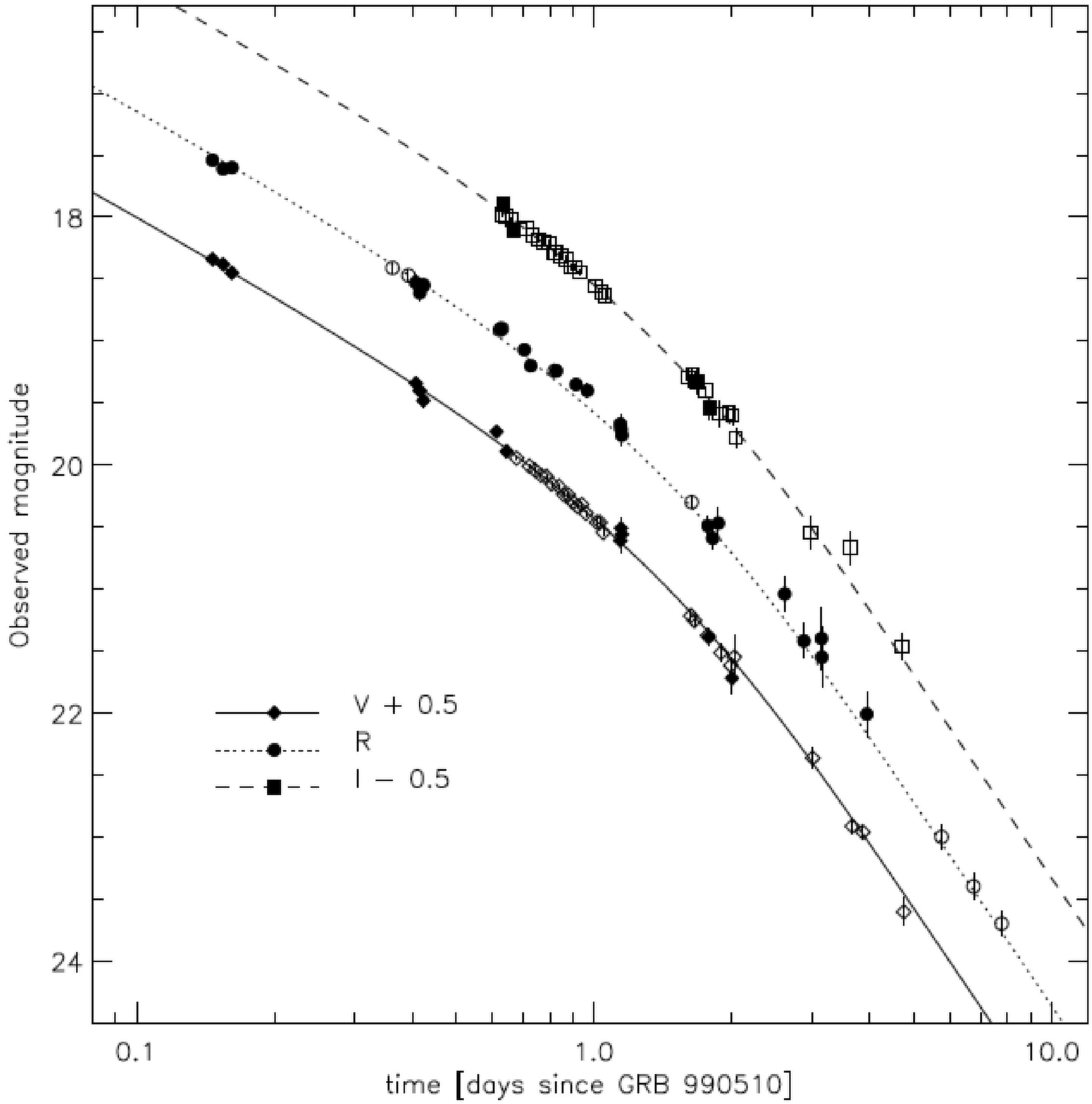
Blast wave evolution from
`https://swift.gsfc.nasa.gov/about_swift/objectives/environment.html`



An optical SN-like bump superimposed on the afterglow of GRB 980326. Models of SN 1998bw at different redshifts are shown. The color and light curve of the bump was found to be consistent with 1998bw at redshift of unity. From Bloom et al. (1999).



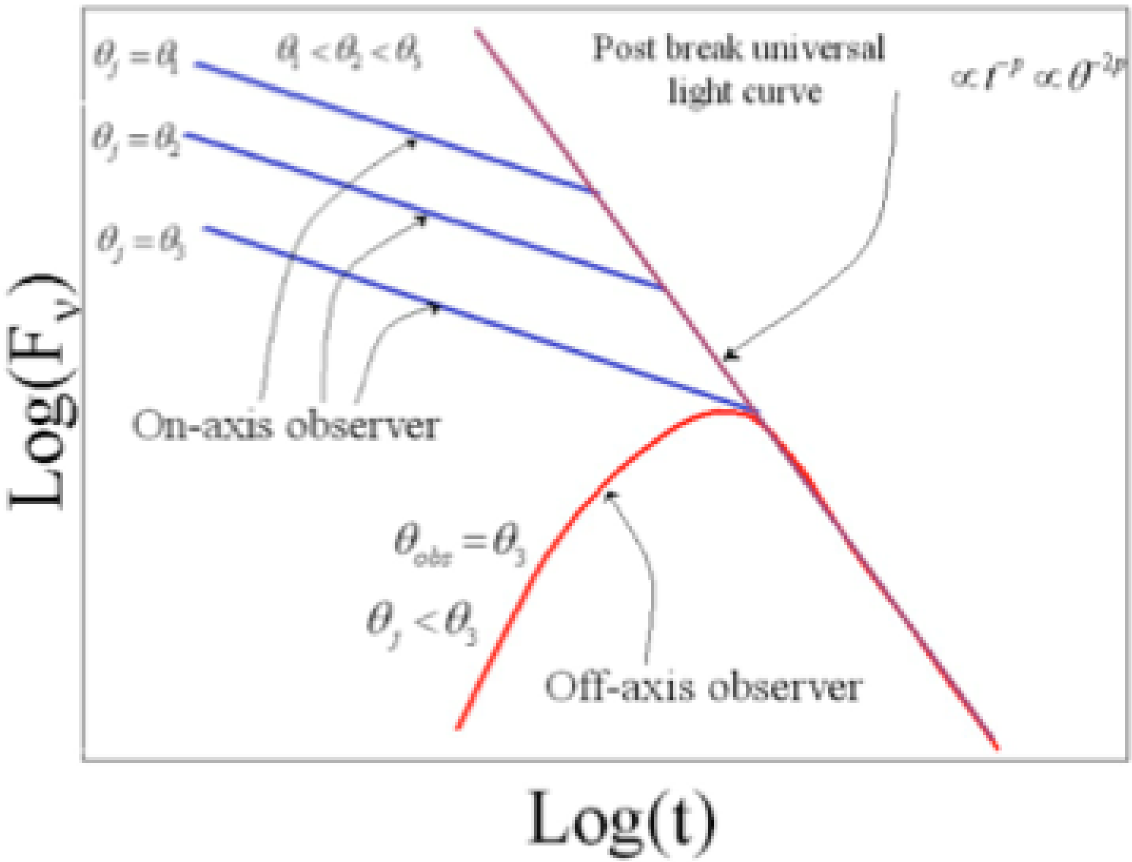
Late SN bump is visible at ~ 30 days after the GRB in the optical light curve of GRB 090618. From Cano et al. (2011).



Observed light curves of the optical afterglow of GRB 990510 in three filters (Harrison et al. 1999). Achromatic steepening of the light curve is seen.

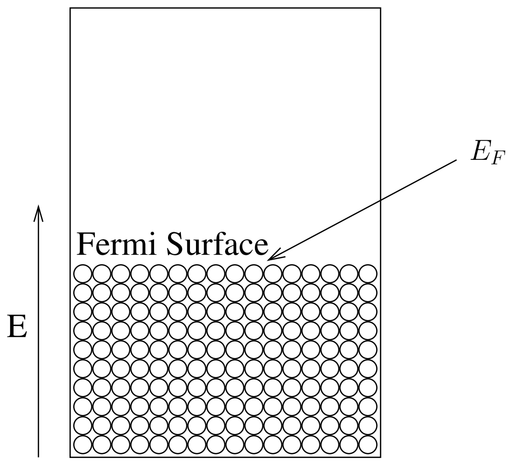
X.2.3 Orphan Afterglows

Can we see afterglow without GRB? Mostly no. This would be if we are at $\theta_{\rm obs} > \theta_j$: no prompt emission, but will see afterglow once $\Gamma < 1/\theta_{\rm obs}$. This is a major target of next-generation transient facilities. We *have* detected optical afterglow before we realized it was a GRB (work by Brad Cenko and Alex Urban), but eventually we got the GRB signal from satellites and realized it was pretty normal.


$\theta_j = \theta_1$
$\theta_1 < \theta_2 < \theta_3$
Post break universal light curve
$\propto t^{-p} \propto \theta^{-2p}$
$\theta_j = \theta_2$
$\theta_j = \theta_3$
Log(F$_\nu$)
On-axis observer
$\theta_{obs} = \theta_3$
$\theta_j < \theta_3$
Off-axis observer
Log(t)

Lecture XI Type Ia supernova

As you compress material, the electrons are forced closer together, i.e., they start to occupy similar states. But electrons are fermions and so Pauli's exclusion principle holds. As a result, electron fills up higher and higher energy states up to the Fermi energy E_F, which depends on the density:



This gives an energy density that produces a pressure, i.e., energy density is roughly $n_e E_F$. This is called degeneracy pressure or Fermi pressure. This can be fitted using

$$P = K\rho^{\Gamma} \quad \begin{cases} K = \frac{10^{13}}{\mu_e^{5/3}} & \Gamma = \frac{5}{3} \quad \text{for } \rho \ll 10^6\,\mathrm{g\,cm^{-3}} \\ K = \frac{1.24\times10^{15}}{\mu_e^{4/3}} & \Gamma = \frac{4}{3} \quad \text{for } \rho \gg 10^6\,\mathrm{g\,cm^{-3}} \end{cases} \tag{XI.165}$$

The difference in Γ is due to electrons being non-relativistic vs relativistic. The transition occurs at $\rho = 10^6\,\mathrm{g\,cm^{-3}}$. This in turn allows us to derive a mass-radius relation. To do this we use the equation for hydrostatic balance

$$\frac{dP}{dr} = -\frac{GM\rho}{r^2} \to -\frac{P_c}{r} = -\frac{GM^2}{r^5} \tag{XI.166}$$

where we have approximated $dP/dr = P_c/r$ and $\rho = M/r^3$. Now this gives us $P_c = GM^2/r^4$. Plugging in our fits for non-relativistic Fermi pressure, we find

$$K\left(\frac{M}{R^3}\right)^{5/3} = \frac{GM^2}{R^4} \tag{XI.167}$$

This gives a scaling $R \propto M^{-1/3}$, so as the mass increases, the star shrinks.

Now if we do this same exercise for relativistic electrons:

$$K\left(\frac{M}{R^3}\right)^{4/3} = \frac{GM^2}{R^4} \to M^{2/3} = \frac{K}{G} = 1.2\,M_\odot \tag{XI.168}$$

Note that we don't seem to find a relation between M and R. In fact, formally the radius approaches zero as M approaches a mass where the electron are relativistic. This mass is called the Chandrasekhar limit or Chandrasekhar mass and is estimated to be $1.4M_\odot$.

This results from an equation of state like $P \propto \rho^{4/3}$, which is characteristic for pressure which are due to relativistic particles. In fact, these systems either collapse or explode. The other case there this pressure is important is radiation pressure which precludes stars with $M \gg 100\, M_\odot$.

Let use go ahead and estimate the binding energy of this star – non relativistic.

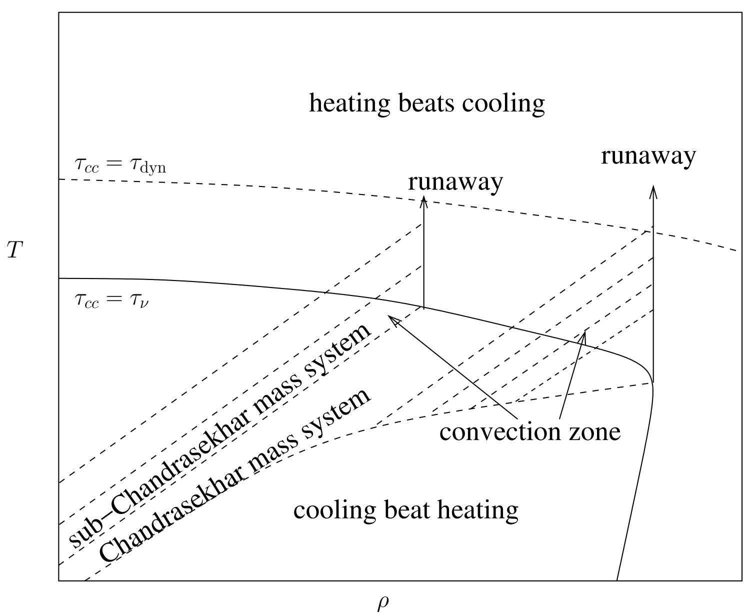
$$E = \int PdV = \int P\frac{dV}{dM}dM = \int c_s^2 dM = \int K\frac{M^{2/3}}{R^2}dM = 2.2\times 10^{50}\left(\frac{M}{1M_\odot}\right)^{7/3} \quad \text{(XI.169)}$$

For $M = 1.4M_\odot$, the binding energy is 0.5×10^{51} ergs.

Now if I burn $\sim 1M_\odot$ of C (1/2 to Ni and 1/2 to Si), we find that energy released is 2×10^{51} ergs greater than the binding energy. So burning all the material will unbind the star.

When this burning happens explosively, this results in what is called a type Ia supernova, which is the optically brightest supernova that we see. The "I" means no hydrogen lines, which suggests a star without hydrogen. The "a" means strong silicon lines which suggests lots burning of alpha elements.

The challenge is to get these stars to explode. To understand how this works consider a plot of a star in T-ρ space



To start these stars burning, we have to either compress things to very high densities or heat them up significantly. Once it cross the cc= ν cooling curve, then carbon burning will heat up the star, which causes more carbon burning which drives more heating. This runaway in the temperature will build up a convection zone above the burning region.

The formation of the convection zone occurs because there is no way for the heat to escape – the time it take the heat to escape if 1 Myr while the total burning time is at most 10,000 years. This convection zone is quite substantial. For a 1.4 solar mass star, you have have 1.1-1.2 solar masses in the convection zone.

Eventually burning region hits $\tau_{cc} = \tau_{dyn}$. At this stage, the carbon burning time hits the dynamical time and dynamical burning can occur. This dynamical burning will proceed by either

- deflagration
- detonation

Deflagation is like burning paper. A local region burns and raises it temperature significantly. Heat is conducted or radiate to its neighbors, which heat them up and cause them to burn. On the other hand, detonation occurs via a shock. Dynamite or TNT is consumed this way. A shock wave compresses, burns, and repowers itself with the burning material.

At this point, it is useful to discuss the various ways we can set up these stars to explode. There are two different models for SN Ia.

- Single degenerate model: A single white dwarf accretes material from its companion non-degenerate star.
- double degenerate model: The more massive white dwarf in a double white dwarf systems either accretes He or C/O from it less massive companion or tidally disrupts it companion.
- double detonation model: A helium layer on the surface triggers a detonation in the CO material by driving a shock into the material.

Up until 2010, the single degenerate model was the most popular model for SN Ia, but a few major problems remain with this model. these issues include

1. How do these WD gain mass – most of the material accreted is H or He. In order for the WD to have a change to increase mass, the accretion rate has to be larger than $10^{-7} M_\odot\,\mathrm{yr}^{-1}$. But in order for these stars to not prematurely explode, it has to be less than $10^{-6} M_\odot\,\mathrm{yr}^{-1}$. These systems should appear as supersoft x-ray sources, but the number observed in our galaxy and in other galaxies is much to low to account for the observed SN Ia rate.
2. How do these SD explode – to make the observed Si, you need a deflagration, but to make the observed Ni, you need a detonation. Hybrid models of explosion can get around this, but it is unclear what the physics behind it is.

Early double degenerate models suppose that 2 WDs merge to produce a super-Chandrasekhar mass WD. Recent numerical simulations of these systems find that these type of merger paradoxically gives SN that are less bright – too little Ni. Other models including by your truly, suppose that sub-Chandrasekhar mass systems can also explode, but it is not clear if they do and how they can get hot enough.

Lecture XII Light Curves of SNIa

Type Ia supernova is a perfect jumping off point to study light curves from transient events. The physics for radioactively power light curves are the same no matter what the situation. let's begin by stating some properties of SN Ia.

- Very bright optical explosion, $\sim 10^{43}$ ergs/s at maximum light. Galaxies are similar in brightness $\sim 10^{42} - 10^{43}$ ergs/s so hence then can outshing an entire galaxy.
- Reach maximum light at 10-15 days. Decays exponentially afterwards. The timescale is similar to the timescale of radioactive decay of ^{5}6Ni
- Powered by radioactive decay of ^{5}6Ni via the process: ^{5}6Ni $\rightarrow^{5}$ 6Co, with $\tau \approx 9$ d; ^{5}6Co $\rightarrow^{5}$ 6Fe, $\tau \approx 114$d
- They are what are called standardizable candles, not standard candles. Their width and peak correlate.
- Can be used the measure luminosity distance.
- dominant source of Fe in the universe.
- Occurs in both young and old populations. This is unlike core-collapse SN, which only occurs in young populations.

A few SN Ia light curves are shown in Figure 4. They all have the same general structure – rising in the beginning followed by falling at the end. They have all been zeroed at maximum light, so you might notice that the dimmer it is, the quicker it rises and falls. This correlation between the brightness and the overall length make them really standardizable candles and the fact that they are standardizable and bright make them crucially important in cosmology as they allow us to measure the luminosity distance. To see how this works consider the bottom plot in Figure 4. Here, by adjusting their overall width and normalization, we can see that they all belong to the same family.

The relation between the maximum brightness and the speed of the supernova is knowns as the "stretch-width" relation or the "Phillips" relation. Roughly this relation is empirical and is given by

$$M_{\rm B,max} = -21.726 + 2.698\Delta m_{\rm B,15}, \tag{XII.170}$$

where $M_{\rm B,max}$ is the absolute B-band magnitude and $\Delta m_{\rm B,15}$ is the decline in the B band luminosity at 15 days after maximum light. Figure 5 shows graphically the definition of $\Delta m_{\rm B,15}$

The origin of this Phillips relation is though to be due to the physics of radiative transfers and energy deposition in SN Ia. Both of these processes crucially are due to the same thing, the Fe-group elements that supply the opacity and the radioactivity. In addition, this physics is generally applicable to almost any transient explosion.

The first crucial point is that the radiation is not a result of the original explosion, but from latent radioactive heat. To see why this is so, let us begin by considering if the gas is heated up during

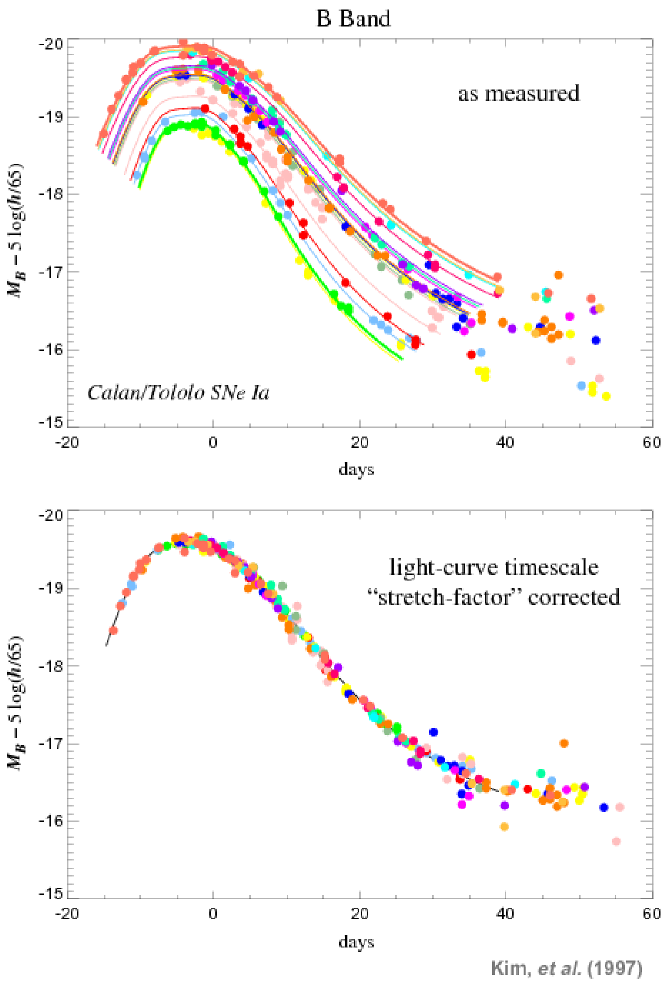


Figure 4: A bunch of SN Ia light curves uncorrected (top) and corrected (bottom)

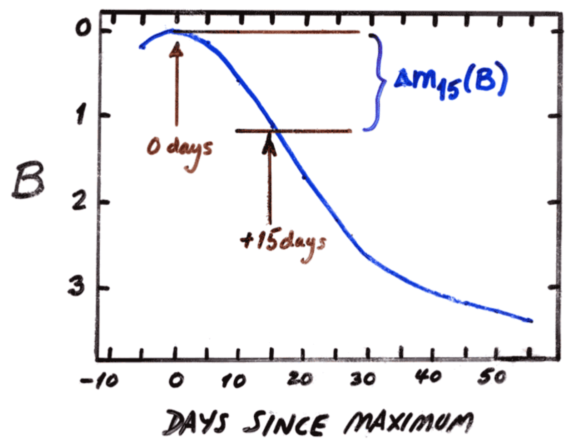


Figure 5: Definition of $\Delta m_{\mathrm{B},15}$

the SN explosion. For a 1000 km WD, the temperature of a gas heated to a thermal energy of 10^{51} ergs is

$$aT^4 = \frac{E}{V} = \frac{E}{4\pi R^3/3} \to T \approx 10^{10} K \left(\frac{E}{10^{51}\,\mathrm{ergs}}\right)^{1/4} \left(\frac{R}{1000\,\mathrm{km}}\right)^{-3/4}, \tag{XII.171}$$

where I assume (rightly) that radiation energy is the dominant energy. Now as this material expands, $P \sim aT^4 \sim \rho^{4/3} \to T \propto 1/r$. So in other words

$$T(t) = T_i \frac{R}{vt}, \tag{XII.172}$$

where $T_i \approx 10^{10}$ K is the initial temperature, R is the initial radius, and $v \approx 10^4$ km/s is the expansion velocity. At the maximum light is reached at $10 - 15$ days ($t \approx 10^6$ s), then the temperature of this material is roughly 1000 K, which is very cold and certainly would not emit in the optical.

So to understand the Phillips relation, let's calculate the time to maximum light. The time to maximum light is going to be related to the width, but I'll assume this relation is simple for now. To calculate the time to maximum light, let's consider the diffusion of photons in an expanding medium. In any material the rate $\mathcal{R}$ at which photons scatter is

$$\mathcal{R} = n\sigma c \tag{XII.173}$$

where n is the number density of scatterer (atoms) and σ is the cross section. The mean free path is then

$$\lambda = \frac{c}{\mathcal{R}} = \frac{1}{n\sigma}. \tag{XII.174}$$

Now consider a uniform sphere of radius R and mass M. In order for a random photon to escape, it must make N steps where

$$N = \frac{R^2}{\lambda^2}, \tag{XII.175}$$

where we assume a random walk. The total distance traveled and escape time are then

$$L = N\lambda = \frac{R^2}{\lambda^2}\lambda \quad \text{and} \quad t_{\rm esc} = \frac{L}{c} = \frac{R^2}{\lambda c}, \tag{XII.176}$$

respectively. The resulting effective speed is then

$$v_{\rm esc} = \frac{R}{t_{\rm esc}} = c\frac{\lambda}{R} = \frac{c}{\tau}, \tag{XII.177}$$

where $\tau = n\sigma R$ is called the optical depth. Estimating $n = M/R^3 m_p$, and $R = v_{\rm exp} t_{\rm exp}$, where $v_{\rm exp}$ and $t_{\rm exp}$ are the expansion velocities and times, we have

$$t_{\rm esc} = \frac{R^2}{R^3}\frac{M}{m_p}\frac{\sigma}{c} = \frac{M}{m_p}\frac{\sigma}{c v_{\rm exp} t_{\rm exp}}. \tag{XII.178}$$

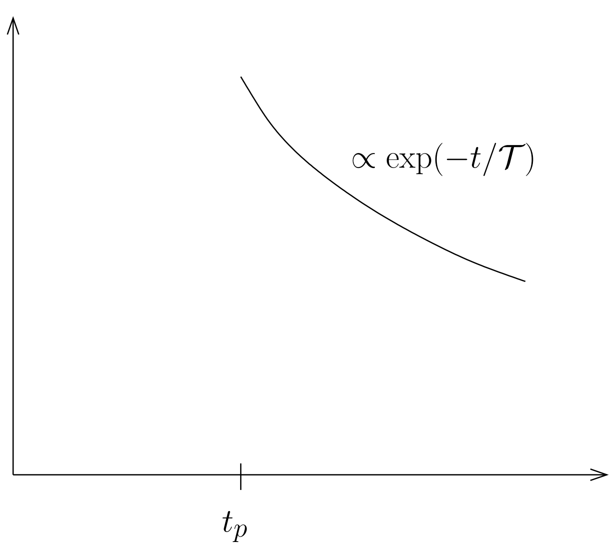
Now when $t_p = t_{\rm esc} = t_{\rm exp}$ then the photons (produced by radioactive decay) can escape before adiabatic expansion dilutes them. This gives

$$t_p = \sqrt{\frac{M}{m_p}\frac{\sigma}{c v_{\rm exp}}} \approx 19 \left(\frac{M}{1\,M_\odot}\right)^{1/2} \left(\frac{\kappa}{0.04\,{\rm cm^2\,g^{-1}}}\right)^{1/2} \left(\frac{v}{10^4\,{\rm km\,s^{-1}}}\right)^{-1/2} {\rm d}, \tag{XII.179}$$

where $\kappa = \sigma/m_p$ is the Rosseland mean opacity. The number is promising as it is similar to the timescale for SN Ia.

Interestingly, at this time $v_{\rm esc} = v_{\rm exp} = c/\tau \to \tau = c/v_{\rm exp}$. As the expansion velocity is always less than the speed of light, this means that τ is generally between 10-30. I should warn you that alot of people will make statements like the peak time is determined by then the supernova is optically thin or calculate max light for when $\tau = 1$. Don't believe them. The real condition is when $v_{\rm exp} = v_{\rm esc}$.

Now that we have determined t_p, we need to figure out what happens next. Right now t_p is a meaningless timescale, but it does allow us to determine the next phase. In particular, light that is produced as a result of nuclear decay after this time will immediately escape. So roughly if $L \sim \dot{E} \sim \exp(-t/\mathcal{T})$, we know that for $t > t_p$, $L \propto \exp(-t/\mathcal{T})$, where $\mathcal{T}$ is some nuclear half-life.



Now we need to figure out what happens for $t < t_p$. Let's begin with a simple brain dead model. Suppose we have a fixed temperature photosphere, $R \propto v_{\text{exp}} t_{\text{exp}}$, then we have

$$L = 4\pi R^2 \sigma T_e^4 \to L \propto t^2, \tag{XII.180}$$

but this is quite unjustified. Why is it constant T_e in particular? However, because it suggest for $t < t_p$, L is rising, this would show that t_p is around where L peaks.

So let's do something more sophisticated. Suppose we look from infinity at an expanding supernova. You will only detect the photons from a depth of ΔR, where

$$v_{\text{esc}} = \frac{c}{\tau(\Delta R)} = v_{\text{exp}} = \frac{c}{n\sigma \Delta R} = \frac{c}{\frac{M\sigma}{R^3 m_p}\Delta R} \to \Delta R = \frac{cR^3}{\frac{M}{m_p}\sigma v_{\text{exp}}} \tag{XII.181}$$

The associate amount of material that powers the supernova is then

$$\Delta M = M \frac{c}{v_{\text{exp}}} \frac{m_p}{\sigma} \frac{R^2}{M} = M \frac{c v_{\text{exp}}}{M\kappa} t^2 \approx 0.003 M_\odot \left(\frac{v}{10^4\,\text{km}\,\text{s}^{-1}}\right)\left(\frac{\kappa}{0.04}\right)^{-1}\left(\frac{t}{1\,\text{d}}\right)^2 \tag{XII.182}$$

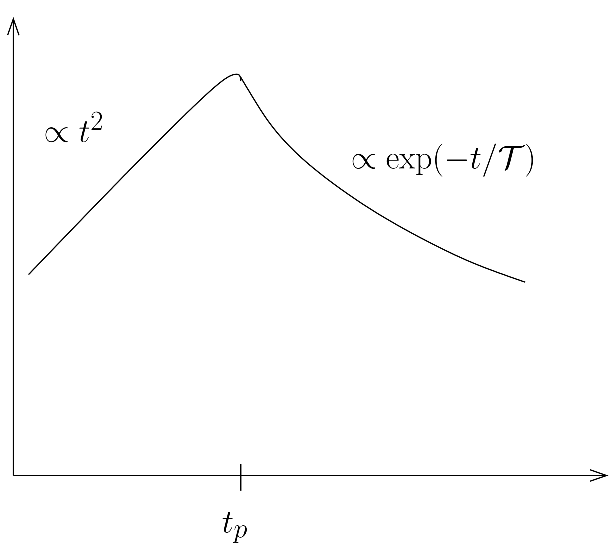
, which is the mass shell that powers what you see at early times. Now if we assume $t \ll \mathcal{T}$, i.e., early times, then the nuclear decay rate per gram is roughly constant, i.e., $\dot{q} = \text{constant}$. So we have

$$L = \Delta M \dot{q} \propto t^2 \tag{XII.183}$$

as before.

Let's now go ahead and calculate the photosphere. Well if we are looking at early times then $\Delta R \ll R$, which implies that the $R_{\text{ph}} \approx R = v_{\text{exp}} t_{\text{exp}}$. So we have if $L \propto t^2$ and $R \propto t$ then T_e is constant!!!

This is amazing and is the result of $\Delta R \propto R$, and $t \ll \mathcal{T}$. But in any case, we can go ahead and fill in the light curve, and gives t_p is the time of maximum light.



Now at maximum light, the luminosity is roughly

$$L_{\max} = M\dot{q}_0 \exp\left(-\frac{t_p}{\mathcal{T}}\right) \tag{XII.184}$$

which encodes our version of the Phillips relation. To see this, we need to emphasize an implicit assumption we have made up to this point, which is the M supplying the radioactive decay is the same M that is providing the opacity. It turns out that this is indeed the case as the M supplying the radioactive decay are iron group elements, i.e., ^{5}6Ni, and supplies most of the opacity, i.e., Fe group elements have alot of lines at these temperatures.

But this by itself is not enough. let's go and take the $\log$ of equation (XII.184) which gives

$$\log L_{\max} = \log M - \frac{t_p}{\mathcal{T}} \tag{XII.185}$$

Noting that $t_p \propto M^{1/2}$, we have

$$\log L_{\max} = 2\log t_p - \frac{t_p}{\mathcal{T}} \tag{XII.186}$$

This is a Phillips relation in disguise. It might seem that as I increase t_p that I have a positive contribution from the first term and a negative contribution from the second term. But in fact the first term usually dominates. This seems odd as it is a logarithmic contribution compared to the second contribution, but in fact this is the case.

In Figure 6, I show the above relation for different values of $\mathcal{T}$ (units are in days). The period of time we are interested in is for t_p between 10-15 days. Here, it is obvious that for $\mathcal{T} \gtrsim 10$ days, then there is a positive correlation between t_p and $\log L_{\max}$ which is a proxy for magnitude. This is due to the fact that while $\log t_p$ is normally a weak contribution, $t_p/\mathcal{T}$ is even smaller for $t_p \ll \mathcal{T}$.

XII.2 Early time light curves

Will maximum light is usually the easiest thing to observe, the advent of automated transient finders is allowing the capture of the early timescale observation of supernova. This is important as it is enormously powerful for constraining SNIa and other transients. For instance:

1. Repeated observations over an area where a SN occurred can constrain the type of projenitors – most useful for very nearly Ia's as progenitor are too dim for most Ia's.
2. The early time light curve can constrain the mechanism of explosion and the size of the exploding object.

In particular, if the explosion is powered by a shock then it is known that shocks accelerate as it moves up a density gradient. In particular for a adiabatic, radiation pressure ($\gamma = 4/3$) dominated shock, the velocity of the shock as it moves to lower density is (see Piro, Chang, & Weinberg 2010)

$$v_s = v_{s,0}\left(\frac{\rho}{\rho_0}\right)^{-0.1858}, \tag{XII.187}$$

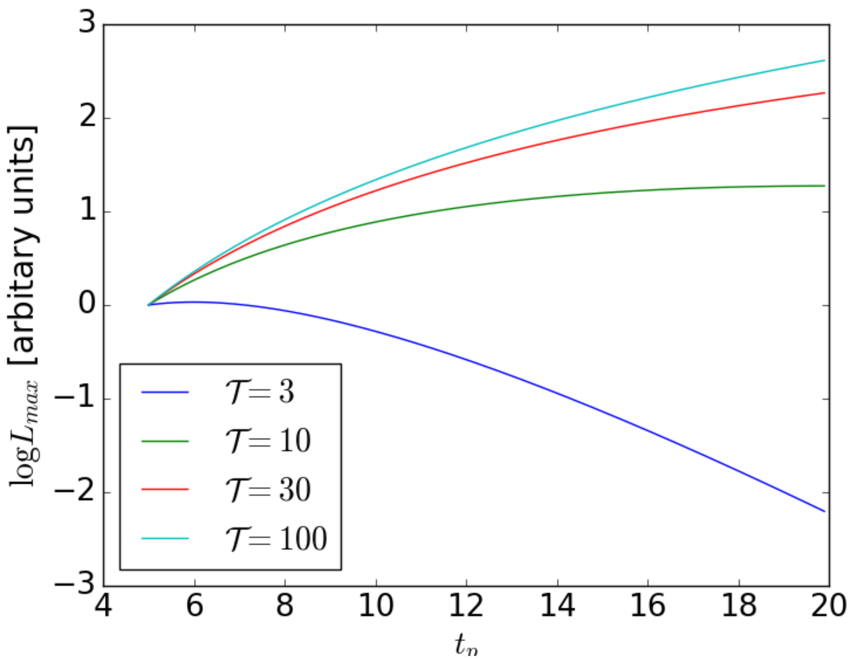


Figure 6: Plot of a simple phillips relation from theory

where $v_{s,0}$ and $\rho_{s,0}$ is the characteristic shock velocity and density. As this shock moves toward the surface of the WD, it produces a very bright x-ray flash with a characteristic time of ≈ 1 sec and a luminosity of $L \sim 10^{40}$ ergs/s, though at very short times it can be much higher than this. This is call a shock breakout.

This x-ray shock breakout is fairly short and low luminosity, but for larger objects, e.g., core-collapse supernova, it is far more promising. For instance SN2008D was discovered by the swift satellite while it was monitoring another object in the galaxy NGC2770. During the observation SN2008D went off and because the XRT was pointed at it, it was able to capture its rise and fall. The discovery image is shown in Figure 7.

You may ask for instance what why does this shock breakout result in x-rays. In fact this is a property of most shock breakout in that it produces flashes of high energy radiation and the timescale and energetics depends on the size of the object and the structure of the atmosphere in which the shocks are propagating. To get a flavor of how this works consider the breakout of a shock through a WD atmosphere, where I will need to tell you that $v_{s,0} = 10^9$ cm/s and $\rho_{s,0} = 10^6\,\mathrm{g\,cm^{-3}}$. Now as this propagates to the surface, $v_s = v_{\rm esc}$ occurs at a density of $\rho = 10^{-3}\,\mathrm{g\,cm^{-3}}$, where $v_{\rm esc} = c/\tau$ is the effective speed of escaping photons and τ is the optical depth measure from the surface. This gives a shock velocity of nearly $v_s \approx 3\times10^{10} = c$. You may be disturbed by this, but in fact $v_{\rm esc} \to \Gamma v_{\rm esc}$. I have dropped the Lorentz factor for computational ease.

In any case the energy density of the shock is

$$\epsilon = \rho v_s^2 = aT^4 \to T \approx 10^8\,\mathrm{K}, \tag{XII.188}$$

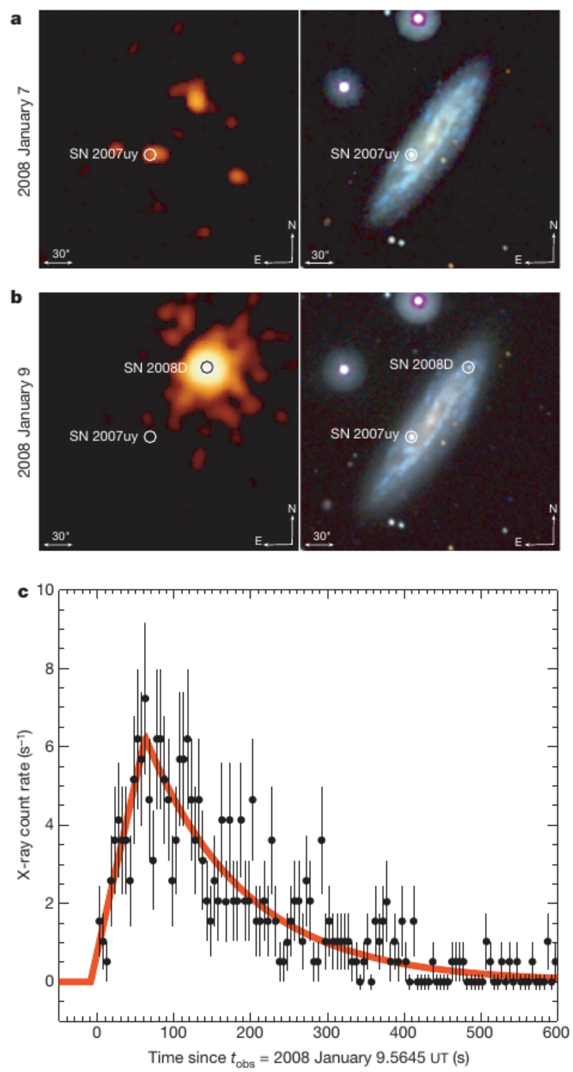


Figure 7: Plot of a simple phillips relation from theory

which is about 10 keV. The total energetics is also similarly argued. By noting

$$E = 4\pi R^2 \Delta R \epsilon = 4\pi R^2 H \epsilon, \tag{XII.189}$$

where we have used the height of the atmosphere as an estimate for the size of the shell. To get H, we recall the equation of hydrostatic balance and assume an initial isothermal atmosphere

$$\frac{dP}{dz} = -\rho g \rightarrow \frac{P}{H} = \frac{\rho k_B T}{m_p H} = \rho g \tag{XII.190}$$

for $g \approx 10^9$ appropriate for a WD, we find $H \approx 10^4$ cm for $T \approx 10^5$ K. This gives $E \approx 10^{41}$ ergs.

In the HW, you will do a simple analysis of the luminosity evolution. A sophisticated analysis of the luminosity evolution yields a very shallow power law shown in top panel of Figure 8 where the power law scales like $L \propto t^{-0.14}$. The effective temperature drops more steeply $T_e \propto t^{-0.44}$. In any case, one assume that the photosphere radiates like a blackbody and so

$$I_\lambda(T_e) = \frac{2hc^2}{\lambda^5} \frac{1}{\exp(hc/\lambda k_B T_e) - 1} \tag{XII.191}$$

where $I_\lambda(T_e)$ is the specific intensity and λ is the wavelength and integrate the flux in a band, $F(t) = \int_{\lambda_1}^{\lambda_2} I_\lambda(T_e(t)) d\lambda$, where λ_1 and λ_2 are the limits of the band to get the optical luminosity in some band, i.e.,

$$L_{\text{band}} = 4\pi R(t)^2 F(t). \tag{XII.192}$$

This is also shown in the top plot of Figure 8 and in Figure 9 by the dotted lines.

Note that the optical luminosity has a light curve that rises and then falls even though the bolometric (totally EM flux) is a power law without any such features. This is a warning that it is not always the case the transitions and peaks in light curves have any meaning, i.e., radiation escapes or optical thinness, but is just the fact that we normally detect just a narrow sliver of EM radiation.

As mentioned in the previous class these models are really useful to measure the properties of the progenitor. In Figure 9, we show one such measurement that was made for the early time light curve of SN 2011 fe. This type Ia explosion occurred on M101, a nearby galaxy and was the nearest Type Ia since the time of Tycho Brahe. In any case, the light curve was captured by the Palomar Transient Factory at 12 hours after the explosion – the black dots. As you can see they rule out an initially stellar radii of the object to be $< 0.1\,R_\odot$ compared to the models of Piro et al. and Rabinak & Waxman (2011). Additionally, images taken 8 hours before, a mere 4 hours after explosion suggest that it was really dim, ruling out shock breakout of even a $0.02\,R_\odot$, which is about 2x the size of earth.

Interestingly, it also puts a constraint on possible companions stars as well. For instance, if the Ia was the result of Roche-lobe overflow accretion from a companion star, the shock wave from the exploding star will impact the companion and given a favorable viewing angle, would shock the material and produce radiation that is visible. This is shown by the solid line in Figure 9 and sets a constraint on size of the companion star. In any case, it also gives a constraint on the companion, which if were to be believe would rule out almost any Roche-filling system.

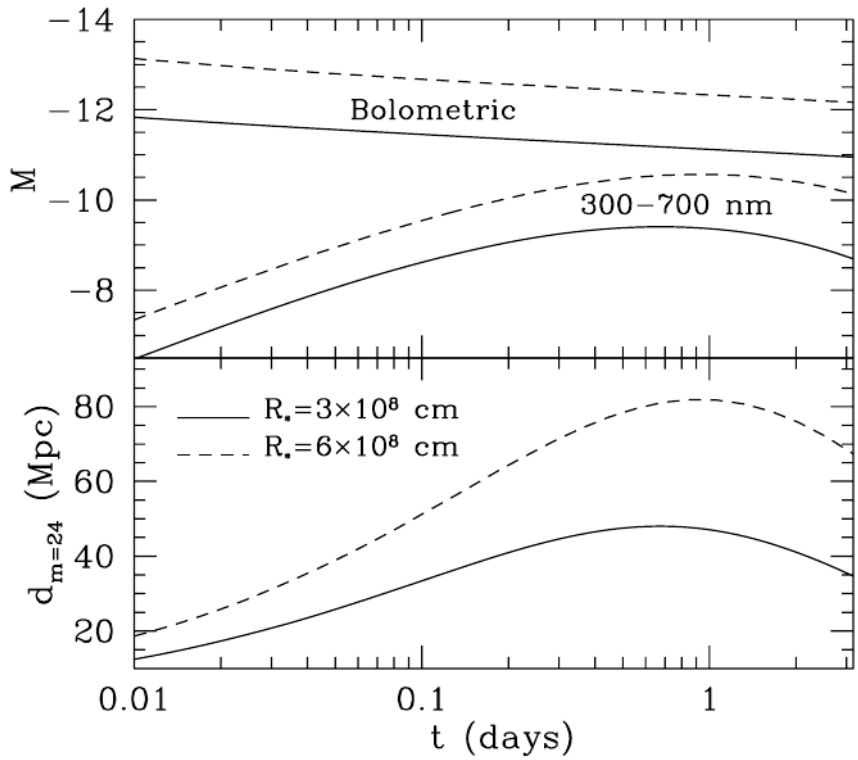


Figure 8: Plot of bolometric and band flux from shock heated expanding envelope

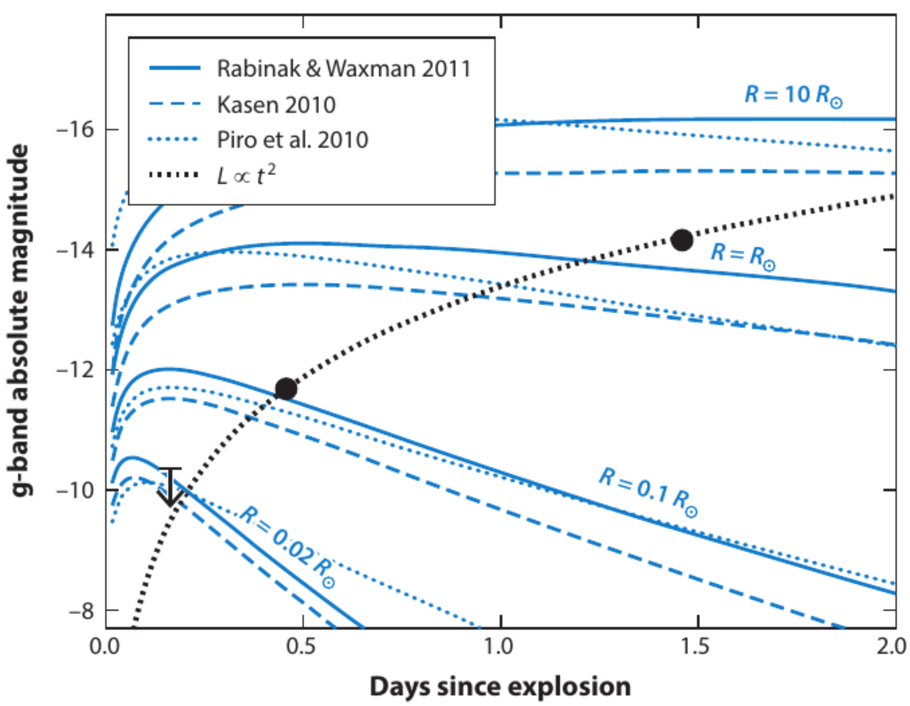


Figure 9: Constraints on early time emission compared to models.

XII.3 Late Time Radio Emission of SN Ia

Talking about radio emission from SN Ia's is a funny thing. This is in part because there is no radio emission from Ia's. We will talk about why this is significant in a bit. In any case, radio emission is increasingly an important part of constraining properties of the region around explosive events. It is also an important part of GRB research, but this is for another time.

In any case, radio emission from many astrophysical phenomenon is due to synchrotron radiation. Which is just relativistically boosted cyclotron radiation. To see how this works consider an electron moving in a magnetic field. From freshman physics we know

$$F = ma = m_e \frac{v^2}{r} = e\frac{v}{c} \times B \rightarrow \omega_L = \frac{v}{r} = \frac{eB}{m_e c}, \tag{XII.193}$$

where ω_L is the Larmor or cyclotron frequency. Now for a relativistic electron, just replace $\omega_L \rightarrow \omega_g = \omega_L/\gamma$, where γ is the Lorentz factor.

This accelerating electron will emit. Nonrelativistically it will emit at a frequency of $\nu = \omega_L/2\pi$, but relativistically this radiation will be beamed and be in a cone of angle $\sim 1/\gamma$. So as an electron moves in a circle, you will see a pulse when the electron is moving within $1/\gamma$ of you line of sight. Time dilation gives you another factor of γ. Length contraction gives yet another factor of γ, so all together, the frequency you observe is

$$\nu = \gamma^3 \nu_g = \gamma^2 \nu_L, \tag{XII.194}$$

where $\nu_g = \omega_g/2\pi$ and $\nu_L = \omega_L/2\pi$. This is great. Now the amount of power that an electron will emit is

$$\frac{dE}{dt} = \frac{4}{3}\sigma_T c \beta^2 \gamma^2 U_B, \tag{XII.195}$$

where $U_B = B^2/8\pi$ is the energy density of the magnetic field. Look in Jackson to see how this is derived or take my word for it.

The expansion of a SN is a shock that plows into the circumstellar medium (CSM). As this shock moves through the CSM, the shock will accelerate electron to high energies. This distribution of particles is most definitely not thermal and gives a high energy tail, namely

$$\frac{dN}{dE} = N_0 E^{-p}, \tag{XII.196}$$

where E is the energy of the electrons. In this case, general arguments give you $p \approx 2 - 3$. These accelerated electrons will radiate via synchrotron radiation with $\nu = \gamma^2 \nu_L \propto E^2$, i.e., their frequency depends on energy. Let see how this ensemble will radiate.

In particular, the intensity of radiation will be

$$j_{\nu(E)} d\nu = \frac{dE}{dt}\frac{dN}{dE} dE \tag{XII.197}$$

Noting that

$$\nu = \gamma^2 \nu_L \rightarrow E = \sqrt{\frac{\nu}{\nu_L}} m_e c^2, \tag{XII.198}$$

we take the derivative $dE/d\nu$ to find

$$\frac{dE}{d\nu} = \frac{1}{2}\frac{m_e c^2}{\sqrt{\nu\nu_L}}, \tag{XII.199}$$

which gives

$$j_{\nu(E)} = \frac{dE}{dt}\frac{dN}{dE}\frac{dE}{d\nu} = \frac{4}{3}\sigma_T c\beta^2\gamma^2 U_B(-p)N_0 E^{p-1}\frac{1}{2}\frac{m_e c^2}{\sqrt{\nu\nu_L}} \propto \nu^{(1-p)/2} \tag{XII.200}$$

Note that for $p \approx 2-3$, this gives a falling spectrum.

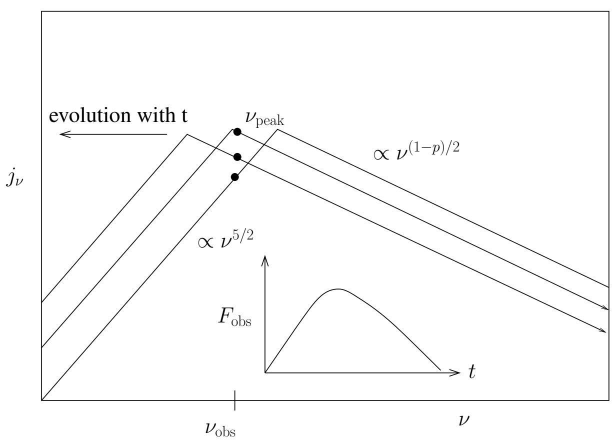
Now this emission is assumed to be optically thin, but it turns out that at low frequencies this radiation can be absorbed. Absorption and re-emission gives rise to a blackbody radiation. To see how this works consider the rayleigh-jeans tail of the Planck function

$$I_\nu = \frac{2k_B T}{c^2}\nu^2 \tag{XII.201}$$

for $h\nu \ll k_B T$. Now in this case the T is the temperature of the synchrotron electrons, i.e., $k_B T = E$. Using our calculation for E from above we see that

$$I_\nu \propto \nu^{1/2}\nu^2 = \nu^{5/2}, \tag{XII.202}$$

which is rising with time. Given these two limits one can then draw the spectra.



We are left with one last caveat, and that is the position of the peak. In this case, one can guess it is the boundary between optically thin and optically thick, i.e., $\tau = 1$. To estimate where this has to be, we need to know where

$$\tau = 1 = \kappa_\nu \rho r, \tag{XII.203}$$

where κ is the opacity. To determine κ, we use detailed balance, i.e., in equilibrium, emission of radiation is equal to absorption of the local radiation field. Namely

$$j_\nu = I_\nu\alpha_\nu \to \alpha_\nu = \frac{j_\nu}{I_\nu} \propto \frac{\nu^{(1-p)/2}}{\nu^{5/2}} = \nu^{-(p+4)/2} \tag{XII.204}$$

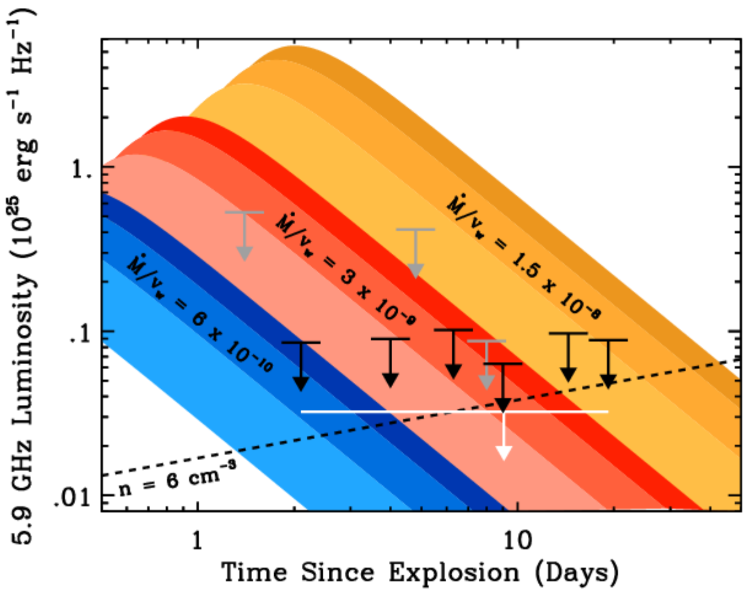


Figure 10: Light curves from shock interaction + radio limits

where α_ν is the absorption coefficient. The absorption coefficient is related to the opacity, so we can estimate

$$\tau = 1 \propto \alpha_{\nu_{\text{peak}}} \rho r, \tag{XII.205}$$

so that we see that the peak frequency is related to the density and size of the local environment.

Now you may wonder what does any of this have to do with the CSM. Well in many scenarios for Ia, the companion was some kind of degenerate star that fills it Roche lobe. If it were an evolved star, then these stars would have given off powerful winds that pollute the CSM. For a constant velocity wind at a constant mass loss rate, we can find

$$4\pi r^2 \rho v_w = \dot{M} \to \rho \propto \frac{\dot{M}}{v_w} r^{-2} \tag{XII.206}$$

Using this result, we find

$$\nu_{\text{peak}}^{-(p+4)/2} \rho r \propto 1 \to \nu_{\text{peak}} \propto r^{-2/(p+4)} \propto t^{-2/(p+4)} \tag{XII.207}$$

For $p \approx 2 - 3$, we find $\nu_{\text{peak}} \sim t^{-0.3}$, which means that the peak moves to lower and lower frequencies as a function of time. If we look at a particular frequency starting in the optically thick regime, we will see an initial rise follow by a fall. And the amplitude constrains $\dot{M}/v_w$. Alas, no such radio detection has been observed in an Ia. In Figure 10, we show the strong constraints from SN 2011fe. As you can see the observations constrains $\dot{M}/v_w$ as low as 6×10^{-10} in units of $M_\odot\,\text{yr}^{-1}/100\,\text{km}\,\text{s}^{-1}$, which appear to exclude any sort of non-degenerate Roche filling companion, though caveats remain.

Lecture XIII Equation of state of degenerate matter

XIII.0 White Dwarfs: Elementary Estimates

XIII.0.1 Degeneracy pressure

Cold matter is stable against collapse, because two fermions cannot be in the same quantum state. If N electrons (or N identical fermions of any kind) are in a volume V, their average momentum is as large as if each occupied a volume V/N. (In a collection of bosons in its ground state, each boson has the ground state energy for a box of volume V) The spacing between electrons is $l = (V/N)^{1/3} = n^{-1/3}$, where n is the particle density, $n = N/V$. An electron confined to a box of side l has a minimum momentum given by the uncertainty relation,

$$p \sim \frac{\hbar}{l} = \hbar n^{1/3}.$$

An ordinary gas is too hot for the minimum momentum allowed by the uncertainty relation to be important; the particles are widely separated, and their average thermal momentum, of order $\sqrt{mkT}$, is much larger than this minimum momentum. As a result, the translational kinetic energy per particle is the thermal energy $E = \frac{3}{2}kT$, and their pressure is very close to thermal pressure of an ideal gas, $P = nkT$. In a liquid or solid, however, the pressure is provided by the resistance of electrons against being crushed to a length smaller than their ground state spacing. In the ground state, their kinetic energy is not zero; the minimum momentum given above implies a minimum kinetic energy,

$$E = \frac{p^2}{2m} = \frac{\hbar^2}{m} n^{2/3},$$

and the corresponding minimum pressure is

$$P = n\langle p_x v_x\rangle = \frac{1}{3} n\langle pv\rangle = \frac{p^2}{3m} n \sim \frac{\hbar^2}{m} n^{5/3}. \tag{XIII.1}$$

In a metal, the outer electrons are free to move, but they are still in or near a ground state, with the spacing between electrons equal to atomic spacing. For what density n is this minimum pressure comparable to the kinetic pressure $P = nkT$? The degeneracy energy must be comparable to kT:

$$E = \frac{\hbar^2}{m} n^{2/3} = kT. \tag{XIII.2}$$

At room temperature, with $m = m_e$, we have,

$$\begin{aligned} n^{2/3} &= \frac{kTm_e}{\hbar^2} = \frac{\left(\frac{1}{40}\text{eV}\right)[.511\text{ MeV}/(3\times 10^{10}\text{cm/s})^2]}{(6.48\times 10^{-22}\text{MeV-s})^2} && \text{(XIII.3)} \\ &= 3\times 10^{13}\text{cm}^{-2} \Rightarrow && \text{(XIII.4)} \\ \ell &= n^{-1/3} = 2\times 10^{-7}\text{cm}. && \text{(XIII.5)} \end{aligned}$$

This is well above atomic spacing, so metals are degenerate. At atomic spacing, $\ell = 5 \times 10^{-9}$ cm, matter is degenerate when

$$\begin{aligned} kT &< \frac{\hbar^2}{m_e} n^{2/3} = \frac{\hbar^2}{m_e \ell^2} \sim 10\text{eV}, &\text{(XIII.6)}\\ \Rightarrow T &< 10^5 K. &\text{(XIII.7)} \end{aligned}$$

10 eV is the kinetic energy of electrons confined to an atomic-size volume. (It is on the order of the binding energy – e.g., 13.6 eV for hydrogen). In particular, the Sun and Jupiter have electrons at atomic spacing (each has density of about 1 g cm^{-3}) so the Sun, with average temperature far above 10^5 K is not degenerate, while Jupiter, with temperature well below 10^5 K, is.

XIII.0.2 White Dwarfs

The following estimates supplement David's summary in XI.1 .

XIII.0.2.1 Structure: Mass-Radius Relation

A white dwarf is the final state of a star whose *initial* mass is less than about 4 $M_\odot$. At the end of its evolution, the star blows off its outer envelope of hydrogen, and the core that remains contracts and eventually cools to a dead ball of He, or of He and C, depending on the star's initial mass. (The most massive stars that end as white dwarfs leave dwarfs with the heavier elements O, Ne and Mg.) Because the nuclear reactions have turned off, the dead star is held apart by its degeneracy pressure. The size of such a star turns out to *decrease* as its mass increases: adding baryons increases the gravitational attraction enough that more baryons are packed in a smaller total volume. This relation between mass and radius can be found from our equations of hydrostatic equilibrium and the equation of state of a degenerate gas. Here we'll again obtain an estimate based on the averaged form of the equation of hydrostatic equilibrium with dP/dr approximated by $-P/R$, $m(r)$ by $M \sim \rho R^3$, with M the total mass.

The rest you've already seen in Eq. (XI.167)

$$\begin{aligned} \frac{P}{R} &= \frac{GM\rho}{R^2}, \qquad P = \frac{\hbar^2}{m_e} n^{5/3} &\text{(XIII.8)}\\ \rho &= m_p n = \frac{M}{R^3} \Rightarrow n = \frac{M}{m_p R^3} \\ (XIII.8) \Rightarrow P &= \frac{GM^2}{R^4} = \frac{\hbar^2}{m_e}\left(\frac{M}{m_p R^3}\right)^{5/3} &\text{(XIII.9)}\\ \frac{GM^2}{R^4} &= \frac{\hbar^2}{m_e}\frac{M^{5/3}}{m_p^{5/3} R^5} \end{aligned}$$

$$R = \frac{\hbar^2}{G m_e m_p^{5/3}} \cdot \frac{1}{M^{1/3}}. \qquad \text{(XIII.10)}$$

With the right numerical factors for a Helium dwarf,

$$R = 1.4 \frac{\hbar^2}{G m_e m_p^{5/3}} M^{-1/3}$$

or

$$\frac{R}{R_\odot} = 0.014 \left(\frac{M_\odot}{M} \right)^{1/3}. \tag{XIII.11}$$

(When $M = M_\odot$, have $R = .014 R_\odot$). So we obtain the relation $R \propto M^{-1/3}$, valid when the star is

- Dense and cold enough for degeneracy: $M > M_{\rm Jupiter} = \frac{1}{1000} M_\odot$.
- Not so massive that it collapses: $M < 1.5 M_\odot$.

XIII.0.2.2 Chandrasekhar limit

> *The star has to go on radiating and radiating and contracting and contracting until, I suppose, it gets down to a few km. radius, when gravity becomes strong enough to hold in the radiation, and the star can at last find peace. Dr. Chandrasekhar had got this result before, but he has rubbed it in in his latest paper; and, when discussing it with him, I felt driven to the conclusion that this was almost a* reductio ad absurdum *of the relativistic degeneracy formula.* A. S. Eddington (1935), published version of comments that followed a talk by Chandra on the upper mass limit.

Notice that Chandrasekhar and Eddington recognized the possibility of collapse to a black hole resulting from the upper limit on a mass supported by degeneracy pressure, but they failed to make the connection between collapse and supernovae. Baade and Zwicky had proposed in the previous year that supernovae were the result of collapse to a neutron star, but didn't relate collapse to the upper mass limit. The connection between the limiting mass of a degenerate core (or a white dwarf) and the collapse to a neutron star did not appear in print until 1939 articles by Gamow and by Chandrasekhar.

For a non-relativistic gas, $v = p/m$. As the gas becomes more relativistic, the energy per particle, $m\gamma v$, rises, but the velocity is limited by the speed of light: $v < c$. This limit on velocity implies a limit on the pressure per unit density: $p = m\gamma v$ implies $P/\rho = \frac{1}{3} pvn/(m\gamma n) < \frac{1}{3} c^2$. But the gravitational attraction has no bound. Therefore, as one increases the mass, the gravitational attraction inevitably overcomes the pressure, and this sets an *upper limit on the mass of white dwarfs*:

$$P = \frac{1}{3} pvn = \frac{1}{3} pcn < \frac{1}{3} \hbar n^{1/3} cn = \frac{1}{3} \hbar c n^{4/3}. \tag{XIII.12}$$

Repeating the argument from Eq. (XI.168), we again use

$$n = \frac{M}{m_p R^3}, \text{ and}$$

equation (XIII.12) to write

$$\begin{aligned} P &= \frac{GM^2}{R^4} < \hbar c \frac{M^{4/3}}{m_p^{4/3} R^4} \qquad \text{(XIII.13)} \\ &\quad R^4\text{'s cancel} \Longrightarrow \\ M &< \left(\frac{\hbar c}{G}\right)^{3/2} \frac{1}{m_p^2} = 1.9 M_\odot \end{aligned}$$

Again we have neglected numerical constants, and a more precise upper limit is

$$M < 0.78 \left(\frac{\hbar c}{G}\right)^{3/2} \frac{1}{m_p^2} = 1.4 M_\odot. \qquad \text{(XIII.14)}$$

This mass is the Chandrasekhar limit. As the mass of a cold star increases, its radius decreases according to the mass-radius relation (XIII.10), until its electrons become relativistic. When the electrons become relativistic, the pressure they contribute rises more slowly than the gravitational attraction, until, for $M > 1.4M_\odot$, the electrons would have to travel faster than light to support the star. If accreting matter drives a white dwarf, or the dead core of a star close to this upper mass limit, it collapses.

XIII.1 Some Thermodynamics: Newtonian and Relativistic

We will follow Chapter 2 of S&T, with emphasis on Sections 2.1-2.3.

XIII.1.1 Gibbs Condition for Phase Equilibrium

The Gibbs condition relates numbers N_i of particles of different species when physical, chemical or nuclear reactions relate the species. Thermodynamic variables are defined for equilibrium states, but we will need to assume that one can define the thermodynamic variables P, T, S and V for a system with any values of the N_i, even though chemical equilibrium has not been established. That is, one must be able to define equilibrium configurations in which the reactions are frozen out. This is often the case. For example, in a mixture of reacting gases, the number of molecules that, at any instant, are reacting is so small, even in a fast reaction, that they do not appreciably affect the thermodynamic parameters of the mixture (see, e.g., Pippard, *Classical Thermodynamics*, pp. 106-7).

More precisely, one assumes the existence of a function

$$E(S, V, N_i), \tag{XIII.15}$$

and one defines temperature T, pressure P and chemical potentials μ_i as the partial derivatives

$$T = \frac{\partial E}{\partial S}, \quad P = -\frac{\partial E}{\partial V}, \quad \mu_i = \frac{\partial E}{\partial N_i}, \tag{XIII.16}$$

so that

$$dE = TdS - PdV + \sum \mu_i dN_i. \tag{XIII.17}$$

When the reactions reach equilibrium, the number N_i of particles of each species are related, and the system lies on a submanifold of the larger space of configurations $\{(S, V, N_i)\}$.

A fluid element in a star is defined to move with the fluid, so that no fluid flows across its boundary. Then energy is transferred to its surroundings only by heat flow and work, not by the outflow or inflow of the particles N_i. The surrounding star imposes a fixed temperature and pressure on the fluid element. We thus model the local thermodynamic equilibrium of a fluid element as a system

(i) at fixed T and P, for which

(ii) the only energy exchange with its surroundings is in the form of heat ΔQ and work $P\Delta V$.

Condition (ii) implies that if the system absorbs heat δQ from its surroundings, and does work PdV on its surroundings, then

$$dE = \delta Q - PdV. \tag{XIII.18}$$

The change in any system's entropy satisfies the inequality $dS \geq \frac{\delta Q}{T}$ (in which both δQ and dS may be negative if heat is lost to the surroundings), with equality holding for an equilibrium process. Thus

$$dE \leq TdS - PdV. \tag{XIII.19}$$

In particular, under our assumption that reactions proceed slowly enough to let us define the function $E = E(S, V, N_i)$, then from Eq. (XIII.17), we conclude

$$\sum \mu_i dN_i \le 0, \tag{XIII.20}$$

with

$$\sum \mu_i dN_i = 0, \tag{XIII.21}$$

when chemical equilibrium is established.

The Gibbs potential,

$$G = E - TS + PV,$$

satisfies, in general,

$$dG = -SdT + VdP + \sum \mu_i dN_i. \tag{XIII.22}$$

For our system, assumption (i) implies $dG = \sum \mu_i dN_i$, and (XIII.20) becomes

$$dG \le 0. \tag{XIII.23}$$

Thus, on the way to chemical equilibrium, G decreases, and chemical equilibrium is a state of minimum G, with

$$dG = \sum \mu_i dN_i = 0, \tag{XIII.24}$$

for any set of changes dN_i away from chemical equilibrium that are allowed by the chemical reactions. In other words, for fixed T and P, G is an extremum at chemical equilibrium.

The restrictions imposed on the particle numbers N_i depend on the reactions that are in equilibrium. We begin with an example and then consider the general case. The reaction

$$H + e^- \leftrightarrow H^- \tag{XIII.25}$$

is important near the surface of the sun: Light emitted by this reaction is a key part of the Sun's continuous spectrum. Changes in N_H, N_e and N_{H^-} produced by the reaction satisfy

$$dN_H = dN_e = -dN_{H^-}.$$

At equilibrium, Eq. (XIII.21) implies

$$\begin{aligned} 0 &= \mu_H dN_H + \mu_e dN_e + \mu_{H^-} dN_{H^-} = (\mu_H + \mu_e - \mu_{H^-}) dN_{H^-} \\ &\Rightarrow \mu_H + \mu_e - \mu_{H^-} = 0 \end{aligned} \tag{XIII.26}$$

To describe a general reaction, let c_i be the coefficient of any reactant ($c_H = c_e = 1$ in (XIII.22)), $-c_i$ the coefficient of any product ($c_{H^-} = -1$). Then the chemical potentials at equilbrium satisfy

$$c_i \mu_i = 0. \tag{XIII.27}$$

If there are k species, each reaction defines a vector $\mathbf{c}$ in a k-dimensional space, $\mathbb{R}^k$. Independent reactions correspond to linearly independent vectors. If there are l independent reactions, then l

linearly independent relations among the μ_i leave $k-l$ independent chemical potentials — and $k-l$ independent conserved functions of particle numbers.

For example, at a time in the early universe when there is equilibrium among $e^+, e^-, \nu_e, \bar\nu_e, \nu_\mu, \bar\nu_\mu, \mu^+, \mu^-$, and a set of baryons and anti-baryons, reactions conserve only baryon number, charge, and lepton number. Then all chemical potentials will be linear combinations of, say, μ_n, μ_p and μ_e. If the expansion time is short compared to the weak interaction timescale, equilibria will not be established between muonic leptons $(\mu^\pm, \mu_\mu, \bar\nu_\mu)$ and electron leptons $(e^\pm, \nu_e, \bar\nu_e)$, and there will be four independent chemical potentials μ_n, μ_p, μ_e and μ_μ.

XIII.1.2 Extensive Quantities and Euler's Theorem

The quantities G, E, S, V and N_i are all extensive, all proportional to the size of the system. That is, when one considers two systems that differ only by overall size, all extensive quantities scale in the same way. Consequently,

$$E(\lambda S, \lambda V, \lambda N_i) = \lambda E(S, V, N_i). \tag{XIII.28}$$

Theorem (Euler). Let $f(x^1, \ldots, x^k)$ satisfy

$$f(\lambda \mathbf{x}) = \lambda f(\mathbf{x}). \tag{XIII.29}$$

Then

$$f = x^i \frac{\partial f}{\partial x^i}. \tag{XIII.30}$$

Proof. Left as an exercise.
From Euler's theorem we have

$$E = TS - PV + \mu_i N_i, \tag{XIII.31}$$

implying

$$G = \mu_i N_i. \tag{XIII.32}$$

XIII.1.3 Relativistic Thermodynamics

In the formalism so far presented, only one real change arises in going from the Newtonian limit to relativistic thermodynamics: The relativistic energy density includes the mass

$$E = M_0 + E_{\text{Newtonian}} \quad (\text{taking } c = 1).$$

Energy, entropy, volume, and particle numbers remain well defined, if one specifies that they are to be measured by a comoving observer, an observer whose 4-velocity agrees with the 4-velocity u^α of the fluid. One can imagine measurements made by an inertial observer instantaneously at rest relative to a fluid element.

A set of intensive, local thermodynamic quantities that characterize a fluid is listed below (all measured by a comoving observer):

$$\begin{aligned}
T &= \text{temperature} \\
P &= \text{pressure} \\
\epsilon &= \text{energy density} \\
n &= \text{baryon density} \\
\rho_0 &= nm_B \simeq \text{rest mass density (when antibaryons are not present)} \\
n_i &= \text{density of } i^{th} \text{ species of particle} \\
Y_i &= \frac{n_i}{n}
\end{aligned}$$

The energy density ϵ is denoted by ε in Shapiro-Teukolsky, and S-T write $\varepsilon = \rho c^2$.

For a fluid element of volume V, conserved baryon number N, energy E, entropy S, we have

$$\epsilon = \frac{E}{V}, \quad s = \frac{S}{N}, \quad n = \frac{N}{V}, \quad Y_i = \frac{N_i}{N}, \tag{XIII.33}$$

N conserved $\Rightarrow dN = 0$. Then, from

$$dE = TdS - PdV + \sum \mu_i dN_i, \tag{XIII.34}$$

we have

$$d\left(\frac{E}{N}\right) = Td\left(\frac{S}{N}\right) - Pd\left(\frac{V}{N}\right) + \mu_i d\left(\frac{N_i}{N}\right) \tag{XIII.35}$$

$$d\left(\frac{\epsilon}{n}\right) = Tds - Pd\left(\frac{1}{n}\right) + \mu_i dY_i. \tag{XIII.36}$$

Equivalently,

$$d\epsilon = nTds + (\epsilon + P)\frac{dn}{n} + n\mu_i dY_i. \tag{XIII.37}$$

With a neutral system and enough reactions in equilibrium that only electron lepton number,muon lepton number, and baryon number are conserved, one can take as independent variables s, n, and two Y_i's:

$$\epsilon = \epsilon(s, n, Y_1, Y_2). \tag{XIII.38}$$

More generally

$$\epsilon = \epsilon(s, n, Y_1, \ldots, Y_m), \tag{XIII.39}$$

and

$$P = -\frac{\partial(\epsilon/n)}{\partial(1/n)} = n\frac{\partial \epsilon}{\partial n} - \epsilon, \tag{XIII.40}$$

$$T = \frac{\partial(\epsilon/n)}{\partial s} = \frac{1}{n}\frac{\partial \epsilon}{\partial s} \tag{XIII.41}$$

$$\mu_i = \frac{1}{n}\frac{\partial \epsilon}{\partial Y_i} \tag{XIII.42}$$

Eqs. (XIII.31) and (XIII.32) imply

$$\epsilon = nTs - P + \mu_i n_i, \qquad g := \frac{G}{N} = \mu_i Y_i \tag{XIII.43}$$

XIII.2 From Kinetic Theory and Stat Mech

In statistical mechanics, an ideal gas is described by a distribution function $\mathfrak{f}(x, p)$ that gives the number of particles per unit volume of phase space, so that

$$n = \int \mathfrak{f}(x, p) d^3p \tag{XIII.44}$$

is the number of particles per unit volume of physical space. The expectation value of a quantity $Q(x, p)$ is then

$$\langle Q(x) \rangle = \frac{\int Q\, \mathfrak{f}\, d^3p}{\int \mathfrak{f}\, d^3p}$$

or

$$n \langle Q \rangle = \int Q\, \mathfrak{f}\, d^3p. \tag{XIII.45}$$

The distribution function has dimension $1/$(volume in phase space), and S&T write it as

$$\mathfrak{f} = \frac{g}{h^3} f, \tag{XIII.46}$$

with h Planck's constant and f dimensionless. Here g counts the spin degrees of freedom: $g = 2S + 1$ for massive particles and $g = 2$ for photons (spin is either along or opposite to photon momentum). For a collection of free particles at a fixed temperature T, f has the familiar form

$$f = \frac{1}{e^{(E-\mu)/kT} \pm 1}, \text{ with the sign } \begin{cases} +, & \text{fermions} \\ -, & \text{bosons} \end{cases} \tag{XIII.47}$$

When the temperature is high and the particle density low, this becomes the Maxwell-Boltzmann distribution, $f = e^{(\mu-E)/kT}$.

Given a distribution function, we can compute the energy density ϵ by writing $\epsilon = n\langle E \rangle$, with $E = p^2c^2 + m^2c^4$ the energy per particle,

$$\epsilon = n \langle E \rangle = \int \sqrt{p^2c^2 + m^2c^4}\, \mathfrak{f}\, d^3p. \tag{XIII.48}$$

To compute the pressure, insert a wall and note that the pressure on the wall is the change in momentum per unit time from particles colliding with the wall: For a wall perpendicular to the x-axis, the change in momentum of a particle is $2p_x$, when $v_x > 0$, and half the particles have $v_x > 0$. (Particles with $v_x < 0$ move away from the wall.) For $v_x > 0$, nv_x particles collide per unit area per unit time. The pressure is then

$$P = n\langle v_x 2p_x \rangle \frac{1}{2} = n\langle v_x p_x \rangle = \frac{1}{3} n\langle v_x p_x + v_y p_y + v_z p_z \rangle = \frac{1}{3}\langle vp \rangle.$$

Using $p = m\gamma v$, $E = m\gamma c^2$ to write $v = pc^2/E = pc^2/\sqrt{p^2c^2 + m^2c^4}$, we have

$$P = \frac{1}{3} \int \frac{p^2 c}{\sqrt{p^2 + m^2c^2}}\, \mathfrak{f}\, d^3p. \tag{XIII.49}$$

XIII.3 EOS of a Completely Degenerate Fermi Gas

Because the temperatures of white dwarfs and neutrons stars are far below the Fermi temperature of their electrons and nucleons, respectively, they can be accurately approximated as supported by the pressure of degenerate fermions. In neutron stars, however, the interactions between nucleons are important; the ideal gas distribution function (XIII.47) ignores interactions and gives a poor approximation to the neutron degeneracy pressure. In white dwarfs, on the other hand, interactions between elections are a small correction, and we can use the ideal gas distribution function at zero temperature to obtain the electron degeneracy pressure and infer the equation of state.

In the limit $T \to 0$, the distribution function becomes a step function, with all states below $\mu(T = 0)$ occupied and no particles with higher energy. That means $\mu(T = 0)$ is the Fermi energy, $\mu(T = 0) = E_F$.

$$f = \begin{cases} 1, & E < E_F \\ 0, & E > E_F \end{cases} \tag{XIII.50}$$

We begin with the number density of electrons, n_e. Defining the Fermi momentum p_F by $E_F^2 = p_F^2 c^2 + m_e^2 c^4$, we have

$$\begin{aligned} n_e &= \int \mathfrak{f}(x,p) d^3p = \frac{2}{h^3}\int_0^{p_F} 4\pi p^2 dp = \frac{8}{3h^3}\pi p_F^3 \\ &= \frac{1}{3\pi^2}\left(\frac{p_F}{\hbar}\right)^3. \qquad (h \to \hbar \text{ here}) \end{aligned} \tag{XIII.51}$$

We next find the pressure and energy density. Note first that the only dimensionful constants here are $\hbar, c$ and the electron mass m_e. Because these have three independent dimensions, we can construct exactly one quantity with a given dimension from these constants. In particular, S&T use the unique combination $m_e c$ with dimension of momentum to replace p_F by the dimensionless version

$$x := \frac{p_F}{m_e c}. \tag{XIII.52}$$

The combinations with dimensions of length and energy are

$$\lambda_e := \frac{\hbar}{m_e c}, \quad \text{and } m_e c^2. \tag{XIII.53}$$

Because n_e has dimension L^{-3} and depends only on $\hbar, m_e, c$ and p_F, it must have the form $\lambda_e^{-3} F(x)$, and it does: Eq. (XIII.51) is

$$n_e := \frac{1}{3\pi^2 \lambda_e^3} x^3. \tag{XIII.54}$$

Equivalently, we can adopt units with $\hbar = m_e = c = 1$. In these units, $p_F = x$ and $n_e = F(x)$; we return to conventional units as usual by multiplying n_e by the combination of $\hbar, m_e, c$ with dimension L^{-3}.

Because P and ϵ each have dimension of energy density, they can be written in the forms

$$P = \frac{m_e c^2}{\lambda_e^3}\phi(x), \qquad \epsilon = \frac{m_e c^2}{\lambda_e^3}\chi(x). \tag{XIII.55}$$

Our job is to find $\phi(x)$ and $\chi(x)$.

From Eq. (XIII.49), we have

$$P = \frac{1}{3}n\langle pv\rangle = \frac{1}{3}\frac{2}{(2\pi\hbar)^3}\int_0^{p_F}\frac{p^2c}{\sqrt{p^2+m^2c^2}}4\pi p^2 dp = \frac{m_e c^2}{\lambda_e^3}\frac{1}{3\pi^2}\int_0^x \frac{x^4}{\sqrt{1+x^2}}dx, \tag{XIII.56}$$

so

$$\phi(x) = \frac{1}{3\pi^2}\int_0^x \frac{x^4}{\sqrt{1+x^2}}dx. \tag{XIII.57}$$

Similarly,

$$\epsilon = n\,\langle E\rangle = \frac{2}{(2\pi\hbar)^3}\int_0^{p_F}\sqrt{p^2c^2+m^2c^4}\;4\pi p^2 dp = \frac{m_e c^2}{\lambda_e^3}\frac{1}{\pi^2}\int_0^x x^2\sqrt{1+x^2}dx \tag{XIII.58}$$

$$\chi(x) = \frac{1}{\pi^2}\int_0^x x^2\sqrt{1+x^2}dx. \tag{XIII.59}$$

S&T give ϕ and χ in closed form, but the integrals themselves more clearly show the behavior of the functions. The Newtonian regime has $x \ll 1$, and the highly relativistic regime has $x \gg 1$. In the Newtonian case, $\sqrt{1+x^2}$ becomes 1 and we have

$$\phi(x) = \frac{1}{15\pi^2}x^5 \qquad \text{(Newtonian)} \tag{XIII.60}$$

implying $P \propto n^{5/3}$ as anticipated. In the ultrarelativistic case, the integrals are dominated by the contribution with $x' \gg 1$ (writing the integrand in terms of x' instead of x to be clear), and we can replace $\sqrt{1+x'^2}$ by x'. Then

$$\phi(x) = \frac{1}{12\pi^2}x^4 \qquad \text{(ultrarelativistic)} \tag{XIII.61}$$

again agreeing with $P \propto n^{4/3}$ from our previous uncertainty-principle argument.

The EOS $P = P(n)$ at all densities follows from the expression for $P(x)$ by using Eq. (XIII.54) to replace x by $\lambda_e(3\pi^2 n_e)^{1/3}$. Explicitly,

$$P = \frac{1}{24\pi^2}\left[3\sinh^{-1}x - x(3-2x^2)\sqrt{1+x^2}\right]\Bigg|_{x=\lambda_e(3\pi^2 n_e)^{1/3}}. \tag{XIII.62}$$

At zero temperature, the first law of thermodynamics implies

$$\frac{d\epsilon}{\epsilon+P} = \frac{dn}{n}. \tag{XIII.63}$$

That means one can compute $\epsilon(n)$ from $P(n)$ without explicitly computing ϵ as the expectation value (XIII.58). Checking that Eqs. (XIII.63) and (XIII.58) are consistent will be an exercise in the next problem set. Note that the thermodynamic relation (XIII.63) relies *only* on a temperature much smaller than the Fermi energy ($kT \ll E_F$), an excellent approximation for both white dwarfs and neutron stars.

In contrast, as noted earlier, the ideal degenerate Fermi gas is a poor approximation for neutron stars. We now look at its main errors for white dwarfs.

XIII.4 Electrostatic Correction: Electron Binding Energy

S&T now discuss the two primary corrections to the ideal Fermi gas EOS for white dwarfs. The first is the Coulomb binding energy of the electrons. The second (next two sections) arises from the changing fraction of neutrons to protons in nuclei as the density of the dwarf increases. The Coulomb binding energy changes the total energy E per electron and so changes the functions $\epsilon(n)$ and $P(n)$ that constitute the equation of state.

We can estimate the size of the correction by comparing the Coulomb energy $Ze^2/r \sim e^2 n^{1/3}$ to the Fermi energy. In the Newtonian regime, $E_F = \dfrac{p_F^2}{2m_e} = \dfrac{(3\pi^2)^{2/3}\hbar^2 n_e^{2/3}}{2m_e}$ giving

$$\begin{aligned}\frac{E_c}{E_F} &\sim \frac{2Ze^2 m_e n_e^{1/3}}{(3\pi^2)^{2/3}\hbar^2 n_e^{2/3}} = \frac{2}{(3\pi^2)^{2/3}}\frac{Z}{a_0 n_e^{1/3}},\\ &= Z\left(\frac{6\times 10^{22}\mathrm{cm}^{-3}}{n_e}\right)^{1/3} = Z\frac{n_e^{-1/3}}{2.6\times 10^{-8}\mathrm{cm}}\end{aligned} \tag{XIII.64}$$

Here $a_0 = \hbar^2/(m_e e^2)$ is the Bohr radius.

To estimate a typical white dwarf number density, use the mass-radius relation, Eq. XIII.11, in the form $(R_\odot/R)^3 = (0.014)^{-3} M/M_\odot$.

$$\begin{aligned}\frac{\rho}{\rho_\odot} &= \frac{M}{M_\odot}\left(\frac{R_\odot}{R}\right)^3 = (0.014)^{-3}\left(\frac{M}{M_\odot}\right)^2\\ \rho &\approx 5\times 10^5 \left(\frac{\mathrm{M}}{\mathrm{M}_\odot}\right)^2 \mathrm{g/cm}^3\end{aligned} \tag{XIII.65}$$

For a dwarf with 2 nucleons per electron (e.g., a He or He-C dwarf),

$$n_e = \frac{\rho}{2m_p} = 1.5\times 10^{29}\left(\frac{M}{M_\odot}\right)^2 \mathrm{cm}^{-3}, \tag{XIII.66}$$

giving $n_e \approx 2\times 10^{28}\mathrm{cm}^{-3}$ and a spacing between nuclei $n_e^{-1/3} \approx 4\times 10^{-10}\mathrm{cm} = 0.07 a_0$ for a typical dwarf with $M \sim M_\odot/2$: Our estimate (XIII.64) of the fractional Coulomb correction to the energy is then of order 0.03.

Now for the actual computation of the Coulomb binding energy of the electrons: Assign Z electrons to each nucleus, uniformly distributed in a sphere of volume $1/n_N$, n_N the number density of nuclei. The radius of r_0 the sphere is given by $\frac{4}{3}\pi r_0^3 = \frac{1}{n_N} = \frac{Z}{n_e}$, implying

$$r_0 = \left(\frac{3Z}{4\pi n_e}\right)^{1/3}. \tag{XIII.67}$$

The uniform charge distribution gives within a radius r the charge $q = -Ze\frac{r^3}{r_0^3}$. The work to bring a shell of charge $dq = 3\frac{r^2}{r_0^3}dr$ in from infinity is $\dfrac{q\,dq}{r}$, and we have

$$E_{e-e} = \int_0^{Ze} \frac{q}{r} dq = \frac{3}{5}\frac{Z^2e^2}{r_0}. \tag{XIII.68}$$

We subtract from this the work done by the ion (the point nucleus),

$$E_{e-i} = -\int_0^r \frac{Ze}{r} dq = -\frac{3}{2}\frac{Z^2e^2}{r_0}, \tag{XIII.69}$$

to obtain a binding energy per electron

$$\frac{E_c}{Z} = \frac{E_{e-e} + E_{e-i}}{Z} = -\frac{9}{10}\left(\frac{4}{3}\pi\right)^{1/3} Z^{2/3}e^2 n_e^{1/3}. \tag{XIII.70}$$

The contribution to the pressure $P(n_e)$ is given by Eq. (XIII.40)

$$P_c = n_e^2 \frac{d}{dn_e}(E_c/Z) = -\frac{3}{10}\left(\frac{4}{3}\pi\right)^{1/3} Z^{2/3}e^2 n_e^{4/3}; \tag{XIII.71}$$

For relativistic electrons, this just lowers the coefficient of $n_e^{4/3}$ by the fractional amount

$$-\frac{2^{5/3}}{5}\left(\frac{3}{\pi}\right)^{1/3} \alpha Z^{2/3} \approx -0.01 \tag{XIII.72}$$

for $Z = 2$, where $\alpha = e^2/\hbar c$ is the fine structure constant. Adding the details has given us a result consistent with but somewhat smaller than our estimated 3% correction to energy per electron for nonrelativistic electrons.

We have assumed uniform density, and the error in that assumption is again of order α - the equilibrium electron distribution n_e minimizes an energy that has an order α error. Now changing $n_e(x)$ by order α changes the degeneracy energy by order α^2 for the following reason (which you've encountered in first-order perturbation theory in QM): An eigenstate (here the ground state) is an extremum (here an absolute minimum) of $\langle\psi|H|\psi\rangle$, and changing the wave function by an amount ϵ therefore changes the energy only at order ϵ^2.

There is also a correction to the electrostatic correction itself: The correction is of order alpha, and the correction to that correction is then of order α^2. S&T compute the correction to n_e in the last part of Sect. 2.4.

XIII.5 Inverse β-decay: The Ideal, Cold, n-p-e Gas.

The second correction to the ideal Fermi gas for the white dwarf EOS comes from change in composition as a function of density, changing Y_e the number of electrons per baryon. To understand how this works, we start first with the simplest system of protons, electrons and neutrons and then, in the next section, look at the real case, where the nuclei are not primarily protons. The n-p-e system is not a bad approximation for the interior of neutron stars, so we'll get a decent estimate of the proton/neutron ratio as a function of density in neutron stars and an intuitive understanding of why that ratio increases as the density increases in both white dwarfs and neutron stars.

We want the equilibrium concentrations of n, p and e for the reactions

$$n \to p + e + \bar{\nu}_e, \quad p + e \to n + \nu_e. \tag{XIII.73}$$

The neutrinos leave the star without reacting: Only at the end of collapse, when the neutron star is forming, or in the merger of two neutron stars (or a neutron star and a black hole) is the cross section for neutrino capture high enough to give a mean free path less than a km. Then the number density of neutrinos remains unchanged at 0 and the equilibrium is determined by
(1) the Gibbs condition (XIII.24), $\sum \mu_i dN_i = 0$, with $dN_\nu = 0$, namely

$$\mu_p + \mu_e - \mu_n = 0; \tag{XIII.74}$$

(2) charge neutrality

$$n_e = n_p. \tag{XIII.75}$$

Now μ_i is the energy needed to add a particle of the ith species. Here, at zero temperature, all energy levels are filled up to the Fermi level, none above, and an additional particle is therefore added at its Fermi energy:

$$\mu_i = E_F^i = \sqrt{p_F^2 + m_i^2} \qquad (c = 1). \tag{XIII.76}$$

Neutrons can be created only if the sum of the Fermi energies of an electron and a proton is greater than the rest mass of a neutron. That is, at low density, $\mu_e = m_e,\ \mu_p = m_p,\ \mu_n = m_n$, and the relation (XIII.74) can't be satisfied – The Gibbs condition

$$0 = \mu_e dN_e + \mu_p dN_p + \mu_n dN_n = (\mu_e + \mu_p - \mu_n) dN_e$$

holds only if there is no reaction, $dN_e = 0$.

As one increases the density of an p-e gas, neutrons first form when $\mu_e + \mu_p = m_n$. Here $\mu_n = m_n$ because the neutron density is zero. Now charge neutrality together with the relation (XIII.51) $n \propto p_F^3$ implies $p_F^e = p_F^n$. The equilibrium condition becomes

$$\sqrt{{p_F^e}^2 + m_e^2} + \sqrt{{p_F^e}^2 + m_p^2} = \sqrt{{p_F^n}^2 + m_n^2}. \tag{XIII.77}$$

(This is S&T (2.5.5), written with p_F instead of x to make the physics clearer.) When neutrons first form, the electrons must have p_F^e of order m_e, because the mass difference between neutron and (proton + electron) is of order m_e. The protons are still nonrelativistic $p_F^p \ll m_p$, since $m_e \ll m_p$, so $E_F^p = m_p$ and we can write the condition for neutrons to first form as

$$E_F^e \equiv \sqrt{{p_F^e}^2 + m_e^2} = m_n - m_p : \tag{XIII.78}$$

The electron Fermi energy must be equal to the mass difference between neutron and proton. The critical density is then

$$n_e = \frac{1}{3\pi^2}\left(\frac{p_F^e}{\hbar}\right)^3 = \frac{1}{3\pi^2\hbar^3}[(m_n - m_p)^2 - m_e^2]^{3/2} = \frac{1}{3\pi^2\lambda_e^3}\left[\left(\frac{m_n - m_p}{m_e}\right)^2 - 1\right]^{3/2} \tag{XIII.79}$$

$$= 7 \times 10^{30}\text{cm}^{-3},$$

$$\rho = m_p n_p = m_p n_e = 1.3 \times 10^7 \text{g/cm}^3, \tag{XIII.80}$$

This number density is an order of magnitude above that (XIII.66) of a typical dwarf but seven orders of magnitude below nuclear density.

Exercise. Check the consistency of our assumption $E_F^p = m_p$: Taylor expand $E = \sqrt{p^2 + m^2}$ to obtain the Newtonian energy correction to the rest mass, and find the fractional error $(E_F^p - m_p)/m_p$ in using m_p. Use the value of $p_F^p = p_F^e$ from (XIII.78) or (XIII.79). (Don't forget to either restore c or use energy units for m.)

For densities intermediate between the critical density for neutron formation and densities high enough that protons and neutrons are relativistic the Gibbs condition gives the ratio S&T (2.5.17). With $Q := m_n - m_p$, the expression is

$$\frac{n_p}{n_n} = \frac{1}{8}\left\{\frac{1 + \frac{4Q}{m_n x_n^2} + 4\frac{Q^2 - m_e^2}{m_n^4 x_n^4}}{1 + 1/x_n^2}\right\}^{3/2} \tag{XIII.81}$$

and we'll go over the somewhat lengthy derivation. The high density limit of this equation is the large x limit: $n_p/n_n = 1/8$. It turns out to be easy to find this high-density ratio directly from charge neutrality and the form of $\mu = E_F$ for ultrarelativistic particles, $p_F \gg m$.

Exercise. Do that: Use charge neutrality and the Gibbs condition for n-p-e equilibrium to show that the high density limit of n_p/n_n is

$$\frac{n_p}{n_n} = \frac{1}{8}. \tag{XIII.82}$$

(This is a quick calculation. Do *not* use Eq.(XIII.81)≡ (2.5.17) of S&T.)

Doing the easy exercise first is useful to see what's happening; now we embark on the more detailed, less transparent calculation leading to (XIII.81). The Gibbs condition (with the charge-neutrality condition $p_n = p_e$) is

$$\sqrt{p_p^2 + m_e^2} + \sqrt{p_p^2 + m_p^2} = \sqrt{p_n^2 + m_n^2} \quad \text{or} \quad \sqrt{A} + \sqrt{B} = \sqrt{C}. \tag{XIII.83}$$

Our goal is to write this as an equation for n_p/n_n as a function of p_n. Because $n_p/n_n = p_p^3/n_p^3$ and the Gibbs condition involves p_p^2 and p_n^2, we find p_p^2 in terms of p_n^2 and then divide by p_n^2. Now

$$\sqrt{A} + \sqrt{B} = \sqrt{C} \Rightarrow A + B + 2\sqrt{AB} = C \Rightarrow A^2 + B^2 + C^2 - 2(AB + AC + BC) = 0,$$

$$\text{or} \qquad 2(A + B)C = (A - B)^2 + C^2. \tag{XIII.84}$$

This last form is helpful because only the left side involves p_p. We have

$$\begin{aligned}
2(2p_p^2 + m_e^2 + m_p^2)(p_n^2 + m_n^2) &= (m_p^2 - m_e^2)^2 + (p_n^2 + m_n^2)^2 \\
4p_p^2(p_n^2 + m_n^2) &= (m_p^2 - m_e^2)^2 - 2(m_e^2 + m_p^2)(p_n^2 + m_n^2) + (p_n^2 + m_n^2)^2 \\
&= p_n^4 + 2p_n^2(m_n^2 - m_p^2 - m_e^2) + m_e^4 + m_p^4 + m_n^4 - 2(m_e^2 m_p^2 + m_e^2 m_n^2 + m_p^2 m_n^2) \\
&= p_n^4 + 2p_n^2(m_n^2 - m_p^2 - m_e^2) + (Q^2 - m_e^2)[(m_n + m_p)^2 - m_e^2]. \qquad \text{(XIII.85)}
\end{aligned}$$

Now $Q \sim m_e^2 \ll m_p^2 \sim m_n^2$. To an accuracy of one part in 2000, we can replace $m_n^2 - m_p^2 - m_e^2 = Q(m_n + m_p) - m_e^2$ by $2Qm_n$ and we can replace $(m_n + m_p)^2 - m_e^2$ by $4m_n^2$:

$$4p_p^2(p_n^2 + m_n^2) = p_n^4 + 4p_n^2 Q m_n + 4(Q^2 - m_e^2)m_n^2.$$

Dividing by $4p_n^2(p_n^2 + m_n^2)$ gives p_p^2/p_n^2, and we have

$$\frac{n_p}{n_n} = p_p^3/p_n^3 = \frac{1}{8}\left\{\frac{p_n^4 + +4p_n^2 Q m_n + 4(Q^2 - m_e^2)m_n^2}{p_n^2(p_n^2 + m_n^2)}\right\}^{3/2}, \qquad \text{(XIII.86)}$$

equivalent to Eq. (XIII.81) after the replacement $p_n \equiv m_n x_n$. Note that we are not entitled to discard the terms involving Q and m_e in the numerator here, because p_n can have any value: Qm_n will be larger or of order p_n^2 when $p_n \ll m_n$ - when the density is near the critical density for neutron production.

XIII.6 Beta-Equilibrium Between Relativistic Electrons and Nuclei: The Harrison-Wheeler EOS

In white dwarfs, as in the progenitor stars, Coulomb repulsion between nuclei is high enough to prevent the star from turning into iron ($^{56}_{26}$Fe), the lowest energy state at zero pressure. As a result, the composition of a dwarf depends on the composition of its progenitor star. When neutron stars form, photons are energetic enough to destroy the iron core of the star (photodisintegration of iron nuclei) and the energy and density is high enough that the matter is baked to an equilibrium state. As one proceeds inwards from the outer crust of the star, the density rises from the density of iron to 10^{15} g/cm^3, a few to several times nuclear density of $2.7 \times 1014\mathrm{g/cm}^3$. For ordinary nuclei, increasing the size increases the average nuclear binding energy, because a larger fraction of nucleons are in the interior, bonded to all nearest neighbors. For larger nuclei, however, the Coulomb repulsive energy grows as Z^2, overcoming the total nuclear binding energy, which is proportional to the number of nucleons (because only adjacent nucleons bond). This is what makes the binding energy per nucleon decrease after iron.

As the density increases, however, the electron Fermi energy rises, and the stablest nucleus shifts to one with more neutrons. Finally, the Fermi energy of neutrons in neutron-rich nuclei becomes high enough that it is energetically favorable to just add free neutrons: This is neutron drip, the density above which electrons combine with protons in nuclei to form free neutrons, leaving a smaller nucleus. At this density, the free neutrons do not decay because of the high Fermi energy of free electrons.

The equilibrium configuration of protons, neutrons and electrons is specified by finding the number of neutrons and protons in the stablest nucleus together with the number density of free electrons and, once the density is high enough, the number density of free neutrons.

Terminology:

$$\begin{aligned}
A &= \text{number of baryons in a nucleus} \\
Z &= \text{number of protons in a nucleus} \\
n &= \text{number density of baryons} \\
n_N &= \text{number density of nuclei} \qquad && Y_N = n_N/n \\
n_e &= \text{number density of electrons} && Y_e = n_e/n && \text{(XIII.87)} \\
n_n &= \text{number density of neutrons} && Y_n = n_n/n && \text{(XIII.88)} \\
& && && \text{(XIII.89)}
\end{aligned}$$

The configuration is determined by A, Z, Y_n and n: That is, the fraction of baryons in nuclei is $1 - Y_n$, so the number densities of nuclei and electrons are

$$n_N = (1 - Y_n)/A \tag{XIII.90}$$

$$n_e = (1 - Y_n)\frac{Z}{A}. \tag{XIII.91}$$

Because $T = 0$, for a fixed number density n, minimizing the Gibbs free energy is equivalent to minimizing ϵ:

$$d\epsilon = Tds + (\epsilon + P)\frac{dn}{n} + n\sum \mu_i Y_i = n\sum \mu_i Y_i.$$

Our goal then is to write ϵ as a function of A, Z, Y_n and to write

$$\nabla\epsilon = 0, \quad \text{i.e., } \frac{\partial\epsilon}{\partial Z} = 0, \ \frac{\partial\epsilon}{\partial A} = 0, \ \frac{\partial\epsilon}{\partial Y_n} = 0, \tag{XIII.92}$$

as three equations that determine the three quantities A, Z, Y_n as functions of n: We want the point in configuration space at which ϵ is a minimum.

Let $M(A, Z)$ be the mass of nucleus of A baryons and Z neutrons plus the rest mass of Z electrons (including the electrons follows historical and S&T convention). Then

$$\begin{aligned}
\epsilon &= n_N M(A, Z) + (\epsilon_e - m_e) + \epsilon_n \\
&= n(1 - Y_n)\frac{M}{A} + \varepsilon_e(n_e)\big|_{n_e=(1-Y_n)Z/A} + \epsilon_n(n_n)\big|_{n_n=nY_n}.
\end{aligned} \tag{XIII.93}$$

The heart of the problem is to find $M(A, Z)$. The approximation we use is essentially the *liquid drop model* in which A and Z are assumed large enough that the nucleus can be regarded as a sphere whose radius is proportional to $A^{1/3}$ and with the number of surface nucleons proportional to the surface area $\propto A^{2/3}$. Then, with E_b the average nuclear binding energy of a bulk nucleon (a nucleon bonded to all its nearest neighbors), the binding energy is smaller than AE_b by an amount proportional to the surface area,

$$-\text{binding energy} = -AE_b + \beta_2 A^{2/3},$$

with β_2 soon to be related to S&T b_2. For a given number of baryons, the sum of the proton and neutron Fermi energies is minimized by an equal division of protons and neutrons; this preference for an equal division is enhanced by the fact that the $n-p$ bond is stronger than the $p-p$ or $n-p$ nuclear bond. Because the nuclear-interaction binding energy is a minimum for $Z/A = 12$, departures from the minimum increase that binding energy by an amount quadratic in the departure, by a term $\beta_4(1/2 - Z/A)^2$. Finally, The Coulomb repulsion of protons adds an energy $\beta_5 \dfrac{Z^2}{A^{1/3}}$

$$M(A,Z) = (A-Z)m_n + Z(m_e + m_p) - E_b + \beta_2 A^{2/3} + \beta_4 A\left(\frac{1}{2} - \frac{Z}{A}\right) + \beta_5 \frac{Z^2}{A^{1/3}}.$$

S&T follow convention in factoring out the mass per baryon m_u in ^{12}C and almost grouping terms by powers of A

$$M(A,Z) = m_u\left[b_1 A + b_2 A^{2/3} - b_3 Z + b_4 A\left(\frac{1}{2} - \frac{Z}{A}\right)^2 + b_5 \frac{Z^2}{A^{1/3}}\right] \tag{XIII.94}$$
$$b_1 = \frac{m_n - E_b}{m_u}, \qquad b_3 = \frac{m_n - m_p - m_e}{m_u}.$$

The contribution of the binding energy to b_1 then measures a difference between bulk binding energy and nuclear binding energy per nucleon in ^{12}C; similarly b_2 and b_5 measure the difference of surface energy per baryon and Coulomb energy from their ^{12}C values.

The energy density of a neutral collections of (A, Z) nuclei, is

$$\epsilon(Z, A, Y_n) = n_N M(A,Z) + \varepsilon_e + \epsilon_n. \tag{XIII.95}$$

Here $\epsilon_n = \epsilon_n(n_n)$ is the energy density of free neutrons, $\varepsilon_e = \varepsilon_e(n_e) = \epsilon_e - n_e m_e$ is the internal energy density of the electrons (the electron rest mass density was already counted in M). As usual, adding an extra neutron or electron adds it at the Fermi energy, because all lower levels are filled:

$$\frac{d\epsilon_n}{dn_n} = E_F^n, \qquad \frac{d\varepsilon_e}{dn_e} = E_F^n - m_e.$$

The equilibrium condition that ϵ is a minimum means that nearby configurations with the same number of baryons distributed differently (and with the electron number determined by charge neutrality) have the same ϵ to linear order in ΔA, ΔZ and ΔY_n:

$$\frac{\partial\epsilon}{\partial Z} = \frac{\partial\epsilon}{\partial A} = \frac{\partial\epsilon}{\partial Y_n} = 0, \tag{XIII.96}$$

or, as a function on the three-dimensional configuration space with coordinates Z, A, Y_n, $\nabla\epsilon = 0$. These three conditions determine the values of the three variables Z, A, Y_n – the compostion of the nuclei (the values of A and Z and the relative numbers of nuclei and free neutrons. The energy of the equilibrium configuration as a function of n then determines the EOS.

The two most interesting conditions are (1) the relation determining the ratio Z/A (equivalently, the proton/neutron ratio in nuclei) and (2) the relation giving the critical density at which free neutrons appear – the onset of neutron drip. In S&T, the first relation is obtained from several lines of calculatieon that obscure what turns out to be a simple meaning. Here is a quicker and more physically transparent derivation: ϵ is a minimum among configurations with the same number n of baryons in a fixed volume. In particular ϵ is a minimum for rearrangements of the protons and neutrons from a set of nuclei with A baryons to a set with $A+1$ baryons per nucleus (and fewer total nuclei): In each nucleus, we are fixing Z/A while increasing A. Now a line of constant Z/A in the $Z-A$ plane is a radial line: In the $x-y$ planea line of constant x/y is a line of constant ϕ, changing only r. Demanding that a function $f(x,y)$ be a minimum at a point along this line means $\vec{r}\cdot\nabla f=0$, or $(x\partial_x+y\partial_y)f=0$. So we have, at equilibrium,

$$(A\partial_A+Z\partial_Z)\frac{M}{A}=0. \tag{XIII.97}$$

Now

$$\frac{1}{m_u}\frac{M}{A}=b_1+b_4\left(\frac{1}{2}-\frac{Z}{A}\right)^2-b_3\frac{Z}{A}+b_2A^{-1/3}+b_5\frac{Z^2}{A^{4/3}}$$

The first three terms are functions only of Z/A, so their radial derivative is zero. We are left with

$$\begin{aligned}-\frac{1}{3}b_2A^{-1/3}+\frac{2}{3}b_5\frac{Z^2}{A^{4/3}}&=0,\\ Z&=\sqrt{\frac{b_2}{2b_5}}A^{1/2}.\end{aligned} \tag{XIII.98}$$

The condition for neutron drip is easy to see and derive, and here S&T give the obvious interpretation. At equilibrium, taking one neutron from each nucleus and setting it free keeps the energy of the system unchanged. The number of nuclei and electrons are unchanged: The neutron is added to the existing sea of neutrons at the Fermi energy, so the change in mass of the nucleus from losing one neutron is then is then equal to the Fermi energy of the neutron. Formally, $\Delta A=-1$ for each of the n_N nuclei; and the number of free neutrons in a unit volume is increased by $\Delta n_n=n_N$. We then have

$$\begin{aligned}0=n_N\frac{\partial M}{\partial A}\Delta A+\frac{d\epsilon_n}{dn_n}\Delta n_n=n_N(-\frac{\partial M}{\Delta}A+E_F^n)&\Longrightarrow\\ \frac{\partial M}{\partial A}=E_F^n.&\end{aligned} \tag{XIII.99}$$

Lecture XIV Accretion

Let's begins by considering the gravitation potential energy of a proton.

$$E_G(r) = -\frac{GMm_p}{r} \tag{XIV.100}$$

So falling from $r = \infty$, this implies

$$\Delta E = E_G(r = \infty) - E_r = \frac{GMm_p}{r}$$

How much energy is this for $M = 1\mathrm{M}_\odot$:

- $r = 10^9$ cm (10000 km) $\to 133$ keV
- $r = 10^6$ cm (10km) $\to 133$ MeV
- $r = 2GM/c^2$ (black hole) $\to 450$ MeV
- Rest mass energy of proton 1 GeV

The virial theorem (from classical mechanics/stat mech) is

$$2K = -U, \tag{XIV.101}$$

where K is the kinetic energy and U is the potential. KE for a collection of atoms is their internal energy or temperature $k_B T$, where k_B is Boltzmann's constant. So the typical temperature of a self gravitating system is

$$k_B T = \frac{GMm_p}{r} \tag{XIV.102}$$

For the sun $r = R_\odot \approx 7 \times 10^{10}\,$cm, we have $T \approx 20\,$MK, which is a good estimate for the temperature at the sun's core.

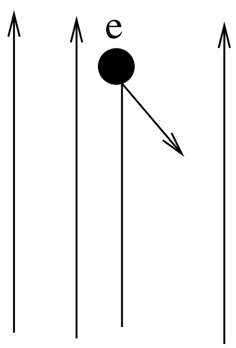
Now suppose material falls toward an object. The energy released per unit time is:

$$L \equiv \dot{E} = \dot{N}\Delta E = \dot{N}\frac{GMm_p}{r} = \frac{GM\dot{M}}{r}, \tag{XIV.103}$$

where L is the luminosity.

Up to this point, L appears that it can be arbitrary large, but this is not the case, as the energy release must escape and the escaping energy can exert a pressure. For most cases of interest, the energy released is light and light exerts a pressure.

Let consider the following picture



An electron scatters light so they can absorb momentum. So the force f that light places on it depends on the number of photons hitting it per unit time and the Δp per photon or

$$f = \dot{n}_\gamma \Delta p = \dot{n}\frac{E_\gamma}{c} = \frac{\dot{E}_\gamma}{c} \tag{XIV.104}$$

So the thing to figure out is the number of photons scattered per unit time. Let us think of the electron as a solid object so the electrostatic potential energy is equal to the rest mass energy or

$$\frac{e^2}{r_e} = m_e c^2, \tag{XIV.105}$$

where r_e is called the classical electron radius. So the cross section is

$$\sigma_T \sim r_e^2 \to \sigma_T = \frac{8\pi}{3} r_e^2 = 6.6 \times 10^{-25}\,\mathrm{cm}^2, \tag{XIV.106}$$

where σ_T is known as the Thomson cross section. The $8\pi/3$ involves a bunch of math, but this OOM estimate got pretty close.

Anyhow, the energy flux in photons is then

$$\dot{E}_\gamma = n_\gamma E_\gamma c \sigma_T = \sigma_T F = \sigma_T \frac{L}{4\pi r^2}, \tag{XIV.107}$$

where in the above, I have assumed a spherical radiation field. The force per electron is then:

$$f = \frac{\sigma_T L}{4\pi r^2 c} \tag{XIV.108}$$

Let's set this equal to gravity on a *proton*.

$$\frac{\sigma_T L}{4\pi r^2 c} = \frac{GMm_p}{r^2} \to L_{\mathrm{Edd}} = \frac{4\pi G M m_p c}{\sigma_T} = 1.4 \times 10^{38} \left(\frac{M}{\mathrm{M}_\odot}\right) \mathrm{ergs\,s^{-1}} = 3.6 \times 10^4\, L_\odot \left(\frac{M}{\mathrm{M}_\odot}\right) \tag{XIV.109}$$

This is known as the Eddington luminosity. Note that is is a function of M and σ_T only.

When the luminosity approaches the Eddington luminosity, radiation pressure starts to interfere with additional accretion and can actually stop it. The rate of accretion necessary to hit the Eddington luminosity it the Eddington mass accretion rate and is given by

$$L = \frac{GM\dot{M}}{r} = L_{\mathrm{Edd}} \to \dot{M}_{\mathrm{Edd}} = \frac{4\pi r c m_p}{\sigma_T} = 1.4 \times 10^{-8} \left(\frac{r}{10\,\mathrm{km}}\right) \mathrm{M}_\odot\,\mathrm{yr}^{-1}. \tag{XIV.110}$$

Note that $M_{\rm Edd}$ depends on r and σ_T, but not mass.

So above this accretion rate, additional accretion will have to fight through the radiation pressure. This can lead to a stagnation of the accretion rate, thought this does not have to happen. See the homework for a counterexample.

In any case, it is natural to think about the luminosity of falling material into a object in terms of the materials rest mass. This sets an efficiency of accretion

$$L = \frac{GM\dot{M}}{r} = \eta\dot{M}c^2 \rightarrow \eta = \frac{GM}{rc^2} \tag{XIV.111}$$

For $r = 2GM/c^2$ for BH's, we have $\eta = 1/2$, which is true for maximally spinning Kerr holes.

For more typically systems like white dwarfs $\eta \sim 10^{-4}$ or NS $\eta \sim 0.1$

XIV.2 Spherical Accretion

We begin with the Euler Equations. This include the continuity equation:

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot\rho\boldsymbol{v} = 0 \tag{XIV.112}$$

and the momentum equation

$$\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla}\boldsymbol{v} = -\frac{1}{\rho}\boldsymbol{\nabla}P - \boldsymbol{\nabla}\Phi. \tag{XIV.113}$$

Here the symbols has the usual meaning. The pressure P and potential Φ requires additional equations to describe them, i.e., to complete the set of equations. For pressure this is accomplished through an equation of state $P = P(\rho, T)$, where T is the temperature. This just replaces P with T, not a great simplification as you need a description for T. But in many cases, one can argue the the thermodynamics are adiabatic and so $P = \kappa\rho^\gamma$, where κ is the constant and γ is the adiabatic exponent.

Now suppose the (gravitational) force we want to consider is due to a point mass:

$$\boldsymbol{f} = -\frac{GM\rho}{r^2} \rightarrow -\boldsymbol{\nabla}\Phi = -\frac{GM}{r^2}. \tag{XIV.114}$$

At this point, we will adopt the first simplification. Let us consider spherically symmetric accretion:

$$\boldsymbol{\nabla}\cdot\rho\boldsymbol{v} \rightarrow \frac{1}{r^2}\frac{\partial}{\partial r}r^2\rho v_r \quad \text{and} \quad \boldsymbol{\nabla} \rightarrow \frac{\partial}{\partial r} \quad \text{and} \quad \boldsymbol{v}\cdot\boldsymbol{\nabla} \rightarrow v_r\frac{\partial}{\partial r} \tag{XIV.115}$$

So in this case, we find

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}r^2\rho v_r = 0 \tag{XIV.116}$$

$$\frac{\partial v_r}{\partial t} + v_r\frac{\partial}{\partial r}v_r = -\frac{1}{\rho}\frac{\partial}{\partial r}P - \frac{\partial}{\partial r}\Phi. \tag{XIV.117}$$

The second simplification that we will apply is steady state where we set the time derivatives $\partial/\partial t \to 0$. This gives:

$$\frac{1}{r^2}\frac{\partial}{\partial r}r^2\rho v_r = 0 \quad \text{and} \quad v_r\frac{\partial}{\partial r}v_r = -\frac{1}{\rho}\frac{\partial}{\partial r}P - \frac{\partial}{\partial r}\Phi. \tag{XIV.118}$$

The first equation gives $r^2\rho v_r =$ constant, which is basically the mass flux, i.e.,

$$4\pi r^2\rho v_r = \dot{M} = \text{constant} \tag{XIV.119}$$

To finish this off, let's calculate the pressure forces in this situation, using the adopted equation of state $P = \kappa\rho^\gamma$:

$$\frac{\partial}{\partial r}P = \frac{\partial P}{\partial \rho}\frac{\partial}{\partial r}\rho = \frac{\gamma P}{\rho}\frac{\partial}{\partial r}\rho = c_s^2\frac{\partial}{\partial r}\rho, \tag{XIV.120}$$

where $c_s = \sqrt{\gamma P/\rho}$ is the sound speed.

Putting this into the spherical momentum equation:

$$v_r\frac{\partial}{\partial r}v_r = -\frac{c_s^2}{\rho}\frac{\partial}{\partial r}\rho - \frac{GM}{r^2} \tag{XIV.121}$$

Continuity is

$$\frac{\partial}{\partial r}\rho = -\frac{2}{r}\rho - \frac{\rho}{v_r}\frac{\partial}{\partial r}v_r \tag{XIV.122}$$

Using this the eliminate $\frac{\partial}{\partial r}\rho$, we find

$$\frac{1}{2}\frac{\partial}{\partial r}v_r^2 = c_s^2\left[\frac{2}{r} + \frac{1}{v_r}\frac{\partial}{\partial r}v_r\right] - \frac{GM}{r^2} \tag{XIV.123}$$

The term $\frac{1}{v_r}\frac{\partial}{\partial r}v_r$ can be written as $\frac{1}{2v_r^2}\frac{\partial}{\partial r}v_r^2$ and so we get

$$\frac{1}{2}\left(1 - \frac{c_s^2}{v_r^2}\right)\frac{\partial}{\partial r}v_r^2 = -\frac{GM}{r^2} + \frac{2c_s^2}{r} \tag{XIV.124}$$

Before solving this equation, let consider the case $P = 0 \to c_s = 0$, i.e., the pressureless case.

$$\frac{1}{2}\frac{\partial}{\partial r}v_r^2 = -\frac{GM}{r^2} \to v_r^2 = \frac{2GM}{r}, \tag{XIV.125}$$

which is just free-fall. From the continuity equation, we recall

$$\dot{M} = 4\pi r^2\rho v_r = 4\pi r^2\rho\sqrt{\frac{2GM}{r}} \to \rho = \frac{\dot{M}}{4\pi\sqrt{2GM}\; r^{3/2}} \tag{XIV.126}$$

Simple enough. Now let suppose you solve the full equations without the restriction of $c_s = 0$. In this case you will notice that when $c_s^2/v_r^2 = 1$, the LHS is zero. Thus the RHS is also zero! This means

$$\frac{GM}{r_s^2} = \frac{2c_s^2}{r_s} \to r_s = \frac{GM}{2c_s^2}, \tag{XIV.127}$$

where r_s is called the sonic radius.

You don't have to worry about it if you can avoid it. But this is generally not the case. The general class of solutions is portrayed in Figure 11. This breaks up to a number of cases

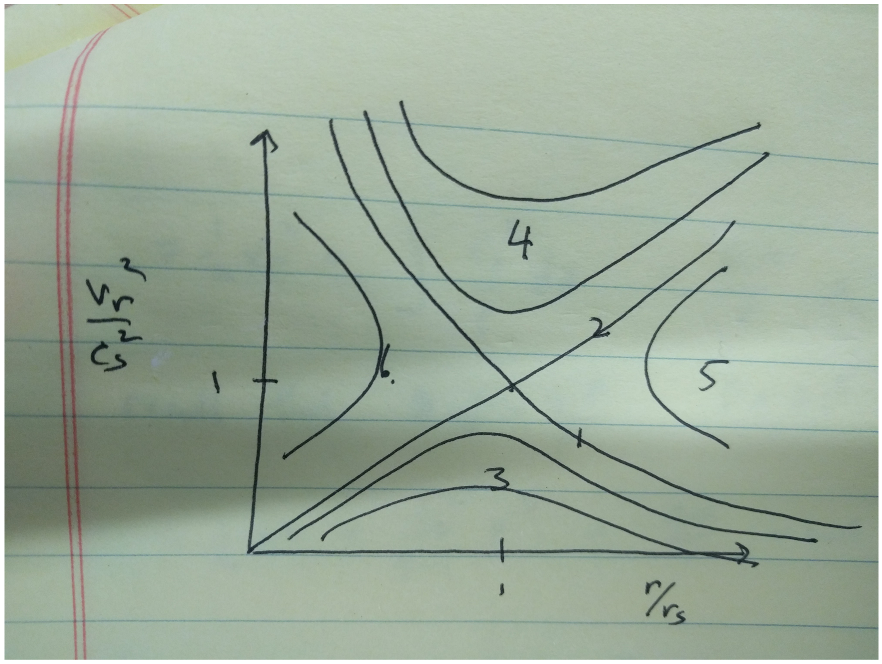


Figure 11: Solutions for spherical accretion.

1. Solution 5,6 have $r < r_s$ or $r > r_s$ so no accretion – killed off immediately
2. Solution 4 has $v_r \gg c_s$ as $r \to \infty$
3. Solution 3 has $v < c_s$ as $r \to 0$ – Settling flow
4. Solution 2 has $v \to 0$ as $r \to 0$ and $v \to \infty$ as $r \to \infty$
5. Solution 1 has $r \to \infty$ as $r \to 0$ and $v \to 0$ as $r \to \infty$

So which solution is the right solution? Well, as $r \to \infty$, we want $v_r \to 0$. This kills off 2 and 4. As $r \to 0$, we want the stuff to accelerate toward the central point, which kills off 3.

Now to make further progress, let's integrate (We're reproducing our conservation of energy equation (III.34), after the time-derivative term is set to zero and conservation of mass, $\nabla \cdot (\rho \mathbf{v}) = \mathbf{0}$, is used):

$$\int v_r \frac{\partial}{\partial r} v_r dr = \int \left(-\frac{c_s^2}{\rho} \frac{\partial}{\partial r} \rho - \frac{GM}{r^2} \right) dr \tag{XIV.128}$$

$$\implies \frac{1}{2} v_r^2 + \int \frac{c_s^2}{\rho} d\rho - \frac{GM}{r} = \text{constant} \tag{XIV.129}$$

Recall $c_s^2 = \gamma P/\rho$ and $P = \kappa \rho^\gamma$, so this gives

$$\frac{1}{2} v_r^2 + \frac{c_s^2}{\gamma - 1} - \frac{GM}{r} = \text{constant} \tag{XIV.130}$$

As $r \to \infty$, $v_r \to 0$ and $\rho \to \rho_0$, so we have the constant $c_{s,0}^2/(\gamma - 1) = c_s^2|_{r=\infty}/(\gamma - 1) = \gamma \kappa \rho_0^{\gamma-1}/(\gamma - 1)$.

Now let's look at what happens at the sonic radius. So at $r = r_s$,

$$\frac{1}{2}c_s^2 + \frac{c_s^2}{\gamma - 1} - \frac{GM}{r_s} = \left(\frac{1}{2} + \frac{1}{\gamma - 1} - 2\right) c_s^2 = c_{s,0}^2/(\gamma - 1) \tag{XIV.131}$$

Let's do some algebra to find:

$$c_s^2(r_s) = c_{s,0}^2 \frac{2}{5 - 3\gamma} \tag{XIV.132}$$

You may be slightly disturbed by this result. Why might you not be?

The density scales like

$$\rho = \rho_0 \left(\frac{c_s^2}{c_{s,0}^2}\right)^{1/(\gamma-1)}$$

So the mass accretion rate is

$$\begin{aligned}
\dot{M} &= 4\pi\rho r^2 v_r = 4\pi\rho_s r_s^2 c_s && \text{(XIV.133)}\\
&= 4\pi\rho_0 \left(\frac{c_s^2}{c_{s,0}^2}\right)^{1/(\gamma-1)} \left(\frac{GM}{c_s^2}\right)^2 c_s && \text{(XIV.134)}\\
&= 4\pi\rho_0 \left(\frac{c_s^2}{c_{s,0}^2}\right)^{1/(\gamma-1)-3/2} \frac{G^2M^2}{c_{s,0}^3} && \text{(XIV.135)}\\
&= 4\pi\rho_0 \left(\frac{2}{5 - 3\gamma}\right)^{(5-3\gamma)/(2(\gamma-1))} \frac{G^2M^2}{c_{s,0}^3}. && \text{(XIV.136)}
\end{aligned}$$

The expression has finite limits for $\gamma = 5/3$ and $\gamma = 1$: In particular, the factor involving γ has the limits[7]

$$\left(\frac{2}{5 - 3\gamma}\right)^{(5-3\gamma)/(2(\gamma-1))} = \begin{cases} 1 & \text{if } \gamma = 5/3 \\ 4.5 & \text{if } \gamma = 1 \end{cases}.$$

You may wonder about when something is $\gamma = 1$. This is an isothermal gas and is frequently used to model the interstellar medium, where the cooling times are fast and the temperature is set by heating from the radiation field of stars.

In any case, for $\gamma = 5/3$, we have

$$\dot{M} = 4\pi\rho_0 \frac{G^2M^2}{c_{s,0}^3}, \tag{XIV.137}$$

where the mass accretion rate is set by the density and temperature of the ambient medium through ρ_0 and $c_{s,0}$ respectively. If it was $\gamma = 1$, the result will just change by a factor of 4.

[7]For $\gamma \to 5/3$, write $\epsilon = \gamma - 5/3$. Then $\epsilon^{-K\epsilon} = e^{-K\epsilon \ln \epsilon}$ and, using $\lim_{\epsilon\to 0} \epsilon \ln \epsilon = 0$, we have $\lim_{\epsilon\to 0} \epsilon^{-K\epsilon} = 1$. Similarly, for $\gamma \to 1$, use $\epsilon = \gamma - 1$. Then $\frac{5 - 3\gamma}{2} = 1 - \frac{3}{2}\epsilon$. We have $(1 - 3\epsilon/2)^{1/\epsilon} = \exp[\epsilon^{-1} \ln(1 - 3\epsilon/2)]$ and $\lim_{\epsilon\to 0} \epsilon^{-1} \ln(1 - 3\epsilon/2)] = -\frac{3}{2}$, implying $\lim_{\gamma\to 1} \left(\frac{2}{5 - 3\gamma}\right)^{(5-3\gamma)/[2(\gamma-1)]} = e^{3/2} \approx 4.5$.

This result can be derived from OOM estimates exactly for $\gamma = 5/3$. For instance, consider a mass embedded in an external medium with the above density and temperature. The distance at which the material feels the gravitational force of the object would be $c_s^2 = GM/r$. So let's say at this radius, the material suddenly falls into the mass at some speed. What would that speed be? Well the only speed scale in this problem $c_{s,0}$. If it falls onto the object spherically then we have

$$\dot{M} \sim 4\pi\rho_0 r^2 c_{s,0} = 4\pi\rho_0 \frac{G^2M^2}{c_{s,0}^3} \tag{XIV.138}$$

which is the exactly the result derived above. Note this result is independent of γ, which is not the case in the exact result. But hey, this is what it means for an OOM estimate.

Incidentally, this result is known as the Bondi-Hoyle accretion rate and in most of astrophysics, the OOM estimate is used. Just as an exercise let us consider the accretion rate of a solar mass point mass in the ambient ISM, where $n = 1\,\mathrm{cm}^{-3}$ and $T = 10^4$ K $\to c_s = 10\,\mathrm{km\,s^{-1}}$. This gives:

$$\dot{M} = 4\pi n m_p \frac{(GM)^2}{c_s^3} \approx 6\times10^{-15}\left(\frac{M}{1\mathrm{M}_\odot}\right)^2\left(\frac{n}{1\,\mathrm{cm}^{-3}}\right)\left(\frac{c_s}{10\,\mathrm{km\,s^{-1}}}\right)^{-3}\mathrm{M}_\odot\,\mathrm{yr}^{-1}, \tag{XIV.139}$$

which corresponds to a luminosity of $L = \eta\dot{M}c^2 \approx 8\times10^{31}\,\mathrm{ergs\,s^{-1}}$ for $\eta = 0.1$.

We should note that the speed scale in the problem can be reset – for instance, one can consider a particle moving through the ambient medium with the replacement $c_s \to \sqrt{c_s^2 + v^2}$. But we will leave the exploration of this as a HW problem.

XIV.3 Accretion Disks

Thus far, we have only considered spherical accretion and this got quite a bit of the way. Now we will consider a disk. Why a disk you ask? Well if you have some material falling in from ∞, it will have some angular momentum. Now rotating stars are nearly spherical when the pressure support is large compared to the rotational support: In other words, when rotational energy $\ll$ internal energy (thermal energy or total Fermi energy in the degenerate case). As one spins up a star, it becomes more oblate, and the opposite limit, when pressure is negligible compared to rotation, is a disk, with each fluid element moving in a Keplerian orbit.

We'll look at that limit and at the correction to it when the pressure is small but nonzero. As one heats up a disk, the ratio of thermal energy to rotational energy increases and the disk becomes thicker. We will soon be working to first order in the ratio H/R of the thickness of the disk to its radius (or to the characteristic length R for which $dP/dr \sim P/R$).

Let's first consider a particle moving toward a central potential with impact parameter b as shown in Figure 12. The energy of the particle with mass m is

$$E = \frac{1}{2}mv^2 - \frac{GMm}{r} = \frac{1}{2}m\left(v_r^2 + r^2\dot{\phi}^2\right) - \frac{GMm}{r} \tag{XIV.140}$$

The key difference is the inclusion of the second term in the kinetic energy. To use this conservation law, we must include the additional angular momentum conservation law (for now):

$$l = \mathrm{constant} = r^2\dot{\phi} \tag{XIV.141}$$

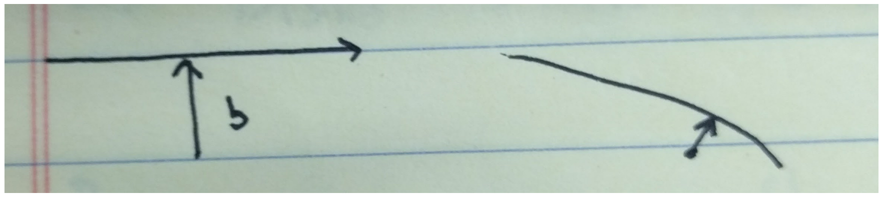


Figure 12

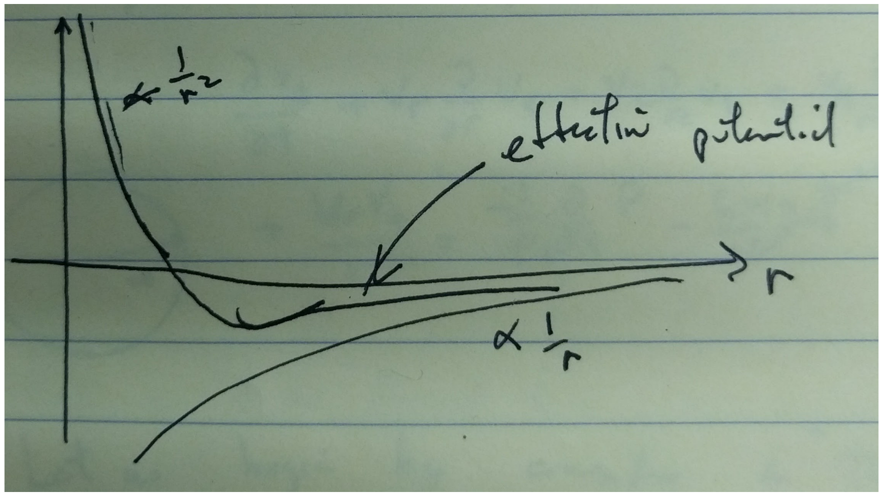


Figure 13

Putting this together, we have

$$E = \frac{1}{2}mv_r^2 + \frac{1}{2}\frac{ml^2}{r^2} - \frac{GMm}{r} = \frac{1}{2}mv_r^2 + m\left(\frac{l^2}{2r^2} - \frac{GM}{r}\right) = \frac{1}{2}mv_r^2 + mU_{\text{eff}}(r) \quad \text{(XIV.142)}$$

The second and third term only depends on r and thus can be thought of as a new type of potential, i.e., the effective potential, U_{eft}. Let's draw this potential as in Figure 13. What we should notice here is that there is a minimum in the effective potential

Let's find the minimum; we set $dU_{\text{eft}}/dr = 0$ to find $r_{\text{min}} = l^2/GM$ or $l^2 = GMr_{\text{min}}$, which is the condition for circular orbits. Thus for a fixed l, there is a minimum $E = U_{\text{eft}}(r_{\text{min}})$ that is allowed. Thus, if you can lose energy via shocks, radiation, etc, then you will come to a circular orbit. So if I throw some material with some net angular momentum toward an object and this stuff loses energy, it is inevitable that a disk (material in circular orbits) will form at some radius.

To make further progress, we must "lose" angular momentum. Toward that end, let us consider the Euler equations in cylindrical coordinate. Continuity becomes

$$\frac{\partial}{\partial t}\rho + \boldsymbol{\nabla} \cdot \rho \boldsymbol{v} = 0 = \frac{\partial}{\partial t}\rho + \frac{1}{r}\frac{\partial}{\partial r} r\rho v_r + \frac{\partial}{\partial z}\rho v_z \quad \text{(XIV.143)}$$

and the momentum equation:

$$\frac{\partial}{\partial t}\boldsymbol{v} + \boldsymbol{v} \cdot \boldsymbol{\nabla}\boldsymbol{v} = -\frac{1}{\rho}\boldsymbol{\nabla}P - \boldsymbol{\nabla}\Phi$$

becomes

$$\begin{aligned}
\frac{\partial}{\partial t}v_r + \left(v_r\frac{\partial}{\partial r} + \frac{v_\phi}{r}\frac{\partial}{\partial \phi} + v_z\frac{\partial}{\partial z}\right)v_r - \frac{v_\phi^2}{r} &= -\frac{1}{\rho}\frac{\partial}{\partial r}P - \frac{\partial}{\partial r}\Phi && \text{(XIV.144)}\\
\frac{\partial}{\partial t}v_\phi + \left(v_r\frac{\partial}{\partial r} + \frac{v_\phi}{r}\frac{\partial}{\partial \phi} + v_z\frac{\partial}{\partial z}\right)v_\phi + \frac{v_\phi v_r}{r} &= -\frac{1}{\rho r}\frac{\partial}{\partial \phi}P - \frac{1}{r}\frac{\partial}{\partial \phi}\Phi && \text{(XIV.145)}\\
\frac{\partial}{\partial t}v_z + \left(v_r\frac{\partial}{\partial r} + \frac{v_\phi}{r}\frac{\partial}{\partial \phi} + v_z\frac{\partial}{\partial z}\right)v_z &= -\frac{1}{\rho}\frac{\partial}{\partial z}P - \frac{\partial}{\partial z}\Phi && \text{(XIV.146)}
\end{aligned}$$

Note here that r is the cylindrical r not the spherical r and the v_ϕ^2/r and $v_r v_\phi/r$ comes from the derivative of $\frac{\partial}{\partial\phi}\hat{\phi} = -\hat{r}$ and $\frac{\partial}{\partial\phi}\hat{r} = \hat{\phi}$ in the ϕ and r momentum equation. This is a rather complicated set of equations to deal with, but becomes much simpler when we look for solutions (1) with $v_z = 0$ as it is in a rotating star and (2) with $H/R \ll 1$. With $v_z = 0$, the z-momentum equation (z-component of the Euler equation) simplifies to

$$0 = -\frac{1}{\rho}\frac{\partial}{\partial z}P + \frac{\partial}{\partial z}\Phi = -\frac{1}{\rho}\frac{\partial}{\partial z}P - \frac{GM}{\mathtt{r}^2}\hat{\mathtt{r}}\cdot\hat{z}, \tag{XIV.147}$$

where $\mathtt{r}$ is the *spherical* radius. Note that $\hat{r}\cdot\hat{z} = \sin\theta \approx z/r$, where that last r is cylindrical. For "thin" disk the difference between the spherical and cylindrical r are similar. This gives

$$0 = -\frac{1}{\rho}\frac{\partial}{\partial z}P - \frac{GM}{r^2}\frac{z}{r} \tag{XIV.148}$$

Now we will use cylindrical symmetry, which implies that $\frac{\partial}{\partial\phi} = 0$. So we have for our sets of equations

$$\begin{aligned}
\frac{\partial}{\partial t}\rho + \frac{1}{r}\frac{\partial}{\partial r}r\rho v_r &= 0 && \text{(XIV.149)}\\
\frac{\partial}{\partial t}v_r + v_r\frac{\partial}{\partial r}v_r - \frac{v_\phi^2}{r} &= -\frac{1}{\rho}\frac{\partial}{\partial r}P - \frac{\partial}{\partial r}\Phi && \text{(XIV.150)}\\
\frac{\partial}{\partial t}v_\phi + v_r\frac{\partial}{\partial r}v_\phi + \frac{v_\phi v_r}{r} &= 0 && \text{(XIV.151)}\\
-\frac{1}{\rho}\frac{\partial}{\partial z}P - \frac{GM}{r^2}\frac{z}{r} &= 0 && \text{(XIV.152)}
\end{aligned}$$

The equations above are for a thin, inviscid disk. How thin is thin? Well consider the last equation in the limit where $z \ll r$. Let $z = H$, where H is the scale height of the disk (typical height of the disk). So we have

$$-\frac{1}{\rho}\frac{\partial}{\partial z}P - \frac{GM}{r^3}\frac{z}{r} \rightarrow \frac{P}{\rho H} = \frac{GMH}{r^3} = \frac{c_s^2}{H} = v_{\rm orb}^2\frac{H}{r^2} \rightarrow c_s^2 = v_{\rm orb}^2\left(\frac{H}{r}\right)^2, \tag{XIV.153}$$

where $v_{\rm orb} = GM/r$ is the orbital speed. So the condition that $z \ll r \rightarrow H \ll r \rightarrow c_s \ll v_{\rm orb}$. Another way of thinking about this is $c_s^2 = k_BT \ll v_{\rm orb}^2 = k_BT_{\rm vir}$, i.e., $T \ll T_{\rm vir}$, i.e., the gas temperature is much smaller than the virial temperature.

With this in mind, lets consider steady state as we did last time. So we have

$$\frac{1}{r}\frac{\partial}{\partial r}r\rho v_r = 0 \tag{XIV.154}$$

$$v_r\frac{\partial}{\partial r}v_r = -\frac{GM}{r^2}+\frac{v_\phi^2}{r} \tag{XIV.155}$$

$$v_r\frac{\partial}{\partial r}v_\phi+\frac{v_\phi v_r}{r} = 0 \tag{XIV.156}$$

$$0 = -\frac{1}{\rho}\frac{\partial}{\partial z}P-\frac{GM}{r^2}\frac{z}{r} \tag{XIV.157}$$

An alert reader will notice that $\rho^{-1}dP/dr$ disappeared. This is no accident as $\rho^{-1}dP/dr \sim P/\rho r = c_s^2/r \ll v_{\rm orb}^2/r = GM/r^2$, much smaller that the competing terms.

One solution to the above set is $v_r = 0$. In this case, we have a complete solution: $v_\phi^2 = GM/r$, i.e., the Keplerian disk, with the fluid moving in circular orbits. On the other hand, if $v_r \neq 0$, then no steady state solution is possible as one of the equations explicitly prevents a solultion close to the Keplerian disk for small v^r (can you see which one?). This just means you have to consider time variations or extra physics to describe an accreting disk. (Answer to the question is in the footnote below[8]). In a Keplerian disk, orbits at different radii have different angular velocity. When one includes viscosity, two things happen: The viscous dissipation heats the fluid and, as a ring of fluid loses energy, it loses angular momentum, moving inward and thus acquiring a nonzero radial velocity. Because the thermal energy of a thin disk is of order e^2, it is natural to look for a solution in which the viscous force of order order e^2 times the gravitational force. We begin by formally introducing shear viscosity and then giving a separate argument that thin disk viscosity is of that order.

We return to the Euler equations, adding an extra term

$$\frac{\partial \boldsymbol{v}}{\partial t}+\boldsymbol{v}\cdot\boldsymbol{\nabla}\boldsymbol{v} = -\frac{1}{\rho}\boldsymbol{\nabla}P-\boldsymbol{\nabla}\Phi+\frac{1}{\rho}\boldsymbol{\nabla}\cdot\boldsymbol{T}, \tag{XIV.158}$$

where $\boldsymbol{T}$ is the viscous stress tensor. The off-diagonal terms in the stress tensor will allow flows to exchange momentum with each other via contact and will not require them to physically collide with one anther.

In particular, we will look at shearing flows with nonzero viscosity. A good introduction to viscosity is Feynman's second chapter on fluids, Chap. 41 of the second volume (feynmanlectures.caltech.edu/II_41.html).[9] The stress tensor associated with viscosity has the form $T_{ij} = \eta(\nabla_i v_j + \nabla_j v_i)$, when $\nabla\cdot v = 0$, as is the case for a rotating disk or for a star in which rings at different cylindrical radii rotate with different angular velocities. When $\nabla\cdot v \neq$, one subtracts off the divergence:

$$T_{ij} = \eta(\nabla_i v_j + \nabla_j v_i - \frac{2}{3}\delta_{ij}\nabla_k v_k),$$

[8] When $v_r \neq 0$, the φ component (XIV.156) of the Euler equation has the form $\frac{v_r}{r}\partial_r(rv_\phi) = 0$, implying $v_\phi = C/r$, and contradicting the $v_\phi = \sqrt{GM/r}+O(e)$ behavior of a thin disk.

[9] See also Section 13.7 of Blandford-Thorne.

making T_{ij} tracefree. In a gas, shear viscosity is dominated not by the chemical bonds that determine the viscosity of ordinary liquids but by the momentum carried by the particles. Viscosity *increases* with an increase in the mean free path of particles, because a larger mean-free path allows particles to transfer momentum between fluid elements having larger differences in their fluid velocities. The viscosity of each particle species is then limited by the dominant interaction that limits its mean free path. (The $\nabla \cdot v$ term that is removed from the shear tensor measures the change in volume, not the shear; dissipation due to a change in volume is called bulk viscosity. It comes, for example, from induced nuclear reactions as a fluid element expands or contracts, and it has a coefficient that is ordinarily unrelated to shear viscosity.)

Here, for our thin disk, the important term is $T_{xy} = \eta \partial v_x / \partial y$ in cartesian coordinates, where η is the dynamical viscosity. Shear flows are particularly important because the orbital velocity scales like $v_{\rm orb} \propto r^{-1/2}$, so at any radii the local fluid flow in disk experiences some shear. Noting the form of the stress tensor is of the form $T_{ij} \propto \partial v_i / \partial x_j$ and for a thin disk the dominant velocity is $v_\phi = v_{\rm orb}(r)$, the only component of the stress tensor that we need to concern ourselves with is the

$$T_{\phi r} = \eta r \frac{\partial}{\partial r} \Omega$$

component.

With this in mind, the ϕ momentum equation becomes

$$\frac{\partial}{\partial t} v_\phi + v_r \frac{\partial}{\partial r} v_\phi + \frac{v_\phi v_r}{r} = \frac{1}{r\rho}\left(\frac{\partial}{\partial r} r T_{\phi r} + T_{\phi r}\right) \tag{XIV.159}$$

where the last term in the middle equation comes from an r-derivative of $\hat{\phi}$. We group terms together to find

$$\frac{1}{r}\left(\frac{\partial}{\partial t} + v_r \frac{\partial}{\partial r}\right) r v_\phi = \frac{1}{r}\frac{d}{dt} l = \frac{1}{r^2 \rho}\frac{\partial}{\partial r} r^2 T_{\phi r} = \frac{1}{r^2\rho}\frac{\partial}{\partial r} r^3 \eta \frac{\partial}{\partial r}\Omega, \tag{XIV.160}$$

where $l = r v_\phi$ is the specific angular momentum and $\frac{d}{dt} \equiv \frac{\partial}{\partial t} + v_r \frac{\partial}{\partial r}$ is the complete time derivative. Written another way the above equation becomes:

$$\frac{d}{dt} l = \frac{1}{r\rho}\frac{\partial}{\partial r} r^3 \eta \frac{\partial}{\partial r}\Omega. \tag{XIV.161}$$

This equation is simply an evolution equation for the angular momentum. If the viscosity is zero, i.e., $\eta = 0$, then we have a conservation law for angular momentum. But the fact that there is a source term on the RHS suggest that angular momentum is in fact not *locally* conserved, but can evolve. We note that the above looks like a diffusion equation, something that will be important in a second. Let us OOM the equation above

$$\frac{l}{t} \sim \frac{1}{\rho}\frac{1}{r^3}\eta r^3 \Omega = \frac{\eta l}{\rho r^2} \to t_{\rm visc} \sim \frac{\rho}{\eta} r^2 = \frac{r^2}{\nu} \tag{XIV.162}$$

where $\nu = \eta/\rho$ is the kinetimatic viscosity and has unit of cm^2/s, aka a diffusion coefficient.

The determination of this ν for astrophysical systems was a key problem and remains one today, though we have a better idea of how this might work. Before we go on let's discuss briefly how big viscosity might be. For molecular viscosity, the fact that it is a diffusion coefficent allows us to estimate it in the standard way:

$$\nu \sim \lambda v_{\rm th}$$

where $v_{\rm th} = c_s$ is the thermal velocity and λ is the mean free path. The combination of λ, which is short and $v_{\rm th}$, which is slow tends to give very slow diffusive processes. This gives very weak viscous forces.

How weak is weak? Well one must compare this the other forces in the systems, the so-called inertial forces, which come into play in the hydrodynamic equations as the $\boldsymbol{v} \cdot \boldsymbol{v}$ terms. Because the size of the system is r and $v_{\rm orb}$ is the largest velocity, the inertial forces have an associate acceleration that is $v_{\rm orb}^2/r$. Now the viscous force is $\boldsymbol{\nabla} \cdot \boldsymbol{T} \sim \nu r^{-2} v_\phi = \lambda v_{\rm th} v_\phi r^{-2}$. The ratio of these two is known as the Reynold's number

$$\mathrm{Re} = \frac{\text{inertial forces}}{\text{viscous forces}} \sim \frac{v_\phi^2/r}{\lambda v_{\rm th} v_\phi / r^2} \sim \frac{r}{\lambda}\frac{v_{\rm orb}}{v_{\rm th}} \tag{XIV.163}$$

This ratio is typically really large $\sim 10^{14}$ though I'll leave it to you to convince yourself of that. Prior to 1990, the usual way out was to note that such large Reynolds numbers inevitable lead to turbulent flow in terrestial environments. This is not clear if it is the case in Keplerian disks as the angular momentum tends to keep everything in check. Nevertheless one can try to make this assumption. If this leads to turbulence, then one can think of a turbulent viscosity in the same manner as molecular viscosity, i.e.,

$$\nu \sim \text{length scale} \times \text{velocity scale} \tag{XIV.164}$$

In a disk, there are two velocity scales c_s and $v_{\rm orb}$. If one thinks about local turbulence, supersonic flow quickly damps out, so the typical velocity scale for turbulence is $< c_s$. Now the length scale is set by the size of the largest eddies, which is limited by the size of the systems and here again there are two scales H and r. Again if one thinks about isotropic turbulence, then scales above H are no longer isotropic. So we conclude:

$$\nu = \alpha H c_s = \alpha \frac{H^2}{r^2} v_{\rm orb}, \tag{XIV.165}$$

where $\alpha \sim 0.1$ is a numerical constant than encapsulates our ignorance and we recalled that $H/r = c_s/v_{\rm orb}$. This gives an all important viscous time of

$$t_{\rm visc} = \frac{r^2}{\alpha v_{\rm orb} H^2/r^2} = \alpha^{-1}\left(\frac{r}{H}\right)^2 t_{\rm orb} \tag{XIV.166}$$

XIV.4 Steady Disks

Armed with this α prescription, we now turn to developing a steady disk model

$$\begin{aligned}
\frac{\partial}{\partial t}\rho + \frac{1}{r}\frac{\partial}{\partial r}r\rho v_r &= \frac{d}{dt}\rho + \frac{\rho}{r}\frac{\partial}{\partial r}rv_r = 0 && \text{(XIV.167)}\\
\frac{\partial}{\partial t}v_r + v_r\frac{\partial}{\partial r}v_r - \frac{v_\phi^2}{r} &= \frac{d}{dt}v_r - \frac{v_\phi^2}{r} = -\frac{1}{\rho}\frac{\partial}{\partial r}P - \frac{GM}{r^2} && \text{(XIV.168)}\\
\frac{\partial}{\partial t}v_\phi + v_r\frac{\partial}{\partial r}v_\phi + \frac{v_\phi v_r}{r} &= \frac{d}{dt}v_\phi + \frac{v_\phi v_r}{r} = (\text{sources}) && \text{(XIV.169)}\\
\frac{\partial}{\partial t}v_z + v_r\frac{\partial}{\partial r}v_z &= \frac{d}{dt}v_z = -\frac{1}{\rho}\frac{\partial}{\partial z}P - \frac{GM}{r^2}\frac{z}{r}, && \text{(XIV.170)}
\end{aligned}$$

This is different from the previous set that we used because we found that the previous set didn't work. Namely for $v_r \neq 0$, we cannot have a steady state equation for v_ϕ if the source for the third equation is 0 (and if the solution is close to a Keplerian disk for small v^r). Now we have computed the source, which we found compactly written as:

$$\frac{d}{dt}rv_\phi = \frac{1}{r\rho}\frac{\partial}{\partial r}r^3\eta\frac{\partial}{\partial r}\Omega. \tag{XIV.171}$$

It turns out that steady-state can be simplified further in the limit where v_r is small. So we will make the approximation $v_r \approx 0$ and $\partial/\partial t \approx 0$. This gives $d/dt \approx 0$, which is the reason why we wrote the set of equation using the total time derivative. So the equations are now:

$$\begin{aligned}
\frac{d}{dt}\rho + \frac{\rho}{r}\frac{\partial}{\partial r}rv_r &= 0 && \text{(XIV.172)}\\
-\frac{v_\phi^2}{r} &= -\frac{1}{\rho}\frac{\partial}{\partial r}P - \frac{GM}{r^2} && \text{(XIV.173)}\\
\frac{d}{dt}rv_\phi &= \frac{1}{r\rho}\frac{\partial}{\partial r}r^3\rho\nu\frac{\partial}{\partial r}\Omega && \text{(XIV.174)}\\
0 &= -\frac{1}{\rho}\frac{\partial}{\partial z}P - \frac{GM}{r^2}\frac{z}{r}, && \text{(XIV.175)}
\end{aligned}$$

where $\nu = \alpha H c_s$. An alert reader will recognize that first equation is not completely consistent with this set. We will ignore this inconsistency for now because we need this result in a second.

Now if the pressure gradient (in radius) is small compared to gravity, then

$$v_r \text{is small} \rightarrow \frac{v_\phi^2}{r} = \frac{GM}{r^2}, \tag{XIV.176}$$

i.e., Keplerian rotation. Now the z equation decouples so we can ignore this equation for now. So we only need to solve everything using the first and third equations. Expand out the third equation:

$$\frac{\partial}{\partial t}r^2\Omega + v_r\frac{\partial}{\partial r}r^2\Omega = \frac{1}{r\rho}\frac{\partial}{\partial r}r^3\rho\nu\frac{\partial}{\partial r}\Omega, \tag{XIV.177}$$

Multiply by ρ to get:

$$\frac{\partial}{\partial t}\rho r^2\Omega + \frac{1}{r}\frac{\partial}{\partial r} r v_r \rho r^2\Omega = \frac{1}{r}\frac{\partial}{\partial r} r^3 \rho\nu \frac{\partial}{\partial r}\Omega, \tag{XIV.178}$$

where we have used the first equation in this substitution.

Now ρ is a function of z, so I will integrate this out. Technically, v_r and Ω are also functions of z, but they are weak functions, so we will ignore this dependence. Define

$$\Sigma = \int \rho dz, \tag{XIV.179}$$

and integrate the continuity and angular momentum equation in dz

$$\frac{\partial}{\partial t}\Sigma + \frac{1}{r}\frac{\partial}{\partial r} r\Sigma v_r = 0 \tag{XIV.180}$$

$$\frac{\partial}{\partial t}\Sigma r^2\Omega + \frac{1}{r}\frac{\partial}{\partial r} r^3 v_r \Sigma\Omega = \frac{1}{r}\frac{\partial}{\partial r} r^3 \Sigma\nu \frac{\partial}{\partial r}\Omega, \tag{XIV.181}$$

Using steady state now, we have $\frac{\partial}{\partial t} = 0$

$$\frac{1}{r}\frac{\partial}{\partial r} r\Sigma v_r = 0 \tag{XIV.182}$$

$$\frac{1}{r}\frac{\partial}{\partial r} r^3 v_r \Sigma\Omega = \frac{1}{r}\frac{\partial}{\partial r} r^3 \Sigma\nu \frac{\partial}{\partial r}\Omega. \tag{XIV.183}$$

The first equation implies that $r\Sigma v_r$ = constant, which we recognize as the mass accretion rate $\dot{M} = 2\pi r\Sigma v_r$. For the second equation, we multiply by r on both sides and integrate in dr to eliminate the leading $\frac{\partial}{\partial r}$, which gives

$$r^3 v_r \Sigma\Omega = r^3\Sigma\nu\frac{\partial}{\partial r}\Omega + \frac{C}{2\pi}, \tag{XIV.184}$$

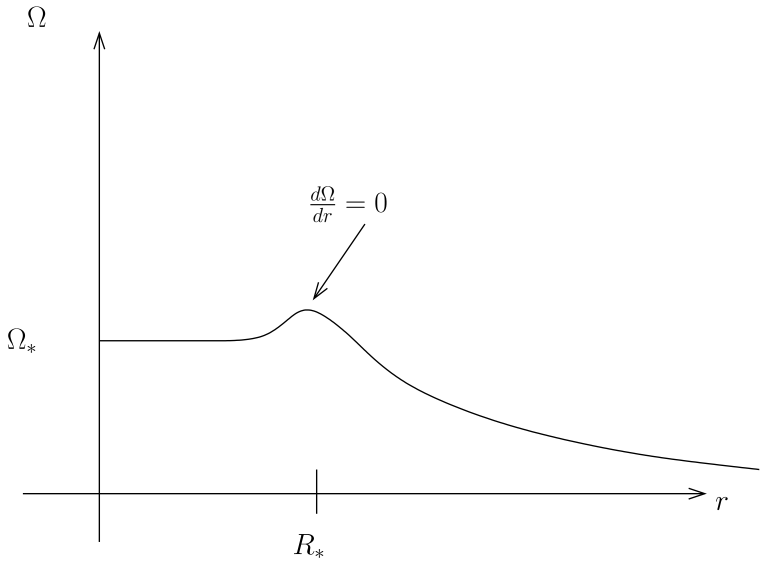
where C is some integration constant that we will need to set. Staring at the first term on the right had side, we define

$$\mathcal{G} \equiv 2\pi r\nu\Sigma r^2 \frac{\partial}{\partial r}\Omega, \tag{XIV.185}$$

as the local viscous torque. In any case, the steady state angular momentum equation can be simplified to

$$v_r\Sigma\Omega = \Sigma\nu\frac{\partial}{\partial r}\Omega + \frac{C}{2\pi r^3}. \tag{XIV.186}$$

We have to set C now via boundary conditions. To do so, let's think about what the right BC might look like. Suppose you have an accretion disk that comes down to the surface of a star at R_* and the star's intrinsic rotation rate is Ω_*



Far away from the star, the material is in Keplerian orbit so $\Omega \propto r^{-3/2}$. Deep in the star, the material is rotating at Ω_*. So somewhere in between there must be a peak in Ω, which gives $d\Omega/dr = 0$. Let say that this point is close to the surface of the star R_*. This allows us derive the BC. From the preceding equation as the first term on the RHS is zero:

$$v_r(R_*)\Sigma(R_*)\Omega(R_*) = \frac{C}{2\pi R_*^3} \rightarrow C = -\dot{M}\sqrt{GM_* R_*}. \tag{XIV.187}$$

This boundary condition is know as the torque free boundary condition because there is no torque on the inner edge of the accretion disk, i.e., $\mathcal{G} = 0$ near the surface of the star since $d\Omega/dr = 0$. Plugging this result back in, we find the following

$$2\pi r^3 v_r \Sigma\Omega = 2\pi r^3 \Sigma\nu \frac{\partial}{\partial r}\Omega - \dot{M}\sqrt{GM_* R_*} \tag{XIV.188}$$

$$\dot{M}\sqrt{GM_* r} = 3\pi\sqrt{GM_* r}\nu\Sigma + \dot{M}\sqrt{GM_* R_*} \rightarrow \nu\Sigma = \frac{\dot{M}}{3\pi}\left(1 - \sqrt{\frac{R_*}{r}}\right) \tag{XIV.189}$$

Under viscous dissipation the energy dissipation rate is

$$\frac{d\epsilon}{dt} = \frac{1}{2}\rho\nu\sum_{i,j}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)^2 = \frac{1}{2}\rho\nu r^2\left(\frac{\partial}{\partial r}\Omega\right)^2, \tag{XIV.190}$$

where ϵ is the energy dissipation rate per unit volume. As we argue previously, the only term that matters is the $i, j = \phi, r$ term. Additionally if we integrate over z, we get the flux, i.e., dissipation per unit area

$$F = \int \frac{d\epsilon}{dt} dz = \frac{1}{2}\Sigma\nu r^2\left(\frac{\partial}{\partial r}\Omega\right)^2. \tag{XIV.191}$$

Plugging in our result for $\nu\Sigma$, we find

$$F = \frac{1}{2}\frac{\dot{M}}{3\pi}\left(1 - \sqrt{\frac{R_*}{r}}\right)^{1/2} r^2\left(\frac{3}{2}\frac{\Omega}{r}\right)^2 = \frac{3}{4\pi}\frac{GM\dot{M}}{r^3}\left(1 - \sqrt{\frac{R_*}{r}}\right)^{1/2} \rightarrow T_e \propto r^{-3/4} \tag{XIV.192}$$

This flux needs to be integrated over the area of the disk to get the total luminosity:

$$L = \int_{r_*}^{r} F 2\pi r dr \approx \frac{GM\dot{M}}{2R_*}, \tag{XIV.193}$$

assuming $r \gg R_*$. This is in line with the expections of the virial theorem that $T = -2U$ so $\Delta e = GM/2R_*$.

Armed with this, we can figure out how a disk will look like if it emitted like a blackbody – which depending on the system is a good or poor assumption. So we set

$$2\sigma T_e^4 = F = \frac{3}{4\pi}\frac{GM\dot{M}}{r^3}\left(1 - \sqrt{\frac{R_*}{r}}\right)^{1/2}, \tag{XIV.194}$$

where the leading 2 comes from the fact that the disk emits through the top and bottom. Now the spectrum of a blackbody with effective temperature T_e is

$$I_\nu = \frac{2h\nu^3}{c^2(\exp(h\nu/k_B T_e) - 1)}. \tag{XIV.195}$$

Noting that T_e is a function of r in a disk, the observed flux that someone sees a distance d away is

$$F_\nu = \frac{\cos i}{d^2}\int 2\pi r dr I_\nu = \frac{4\pi \cos i h\nu^3}{c^2 d^2}\int \frac{r dr}{\exp(h\nu/k_B T_e) - 1} \tag{XIV.196}$$

where i is the inclination angle $i = 90$ is face on. To do this integral, let make note that $T_e \propto r^{-3/4} \to r \propto T^{-4/3}$ and thus $dr \sim - = - = T^{-1/3} d(1/T)$. This gives

$$F_\nu \sim \int \frac{\nu^3 T^{-4/3} T^{-1/3} d(1/T)}{\exp(h\nu/k_B T_e) - 1}, \tag{XIV.197}$$

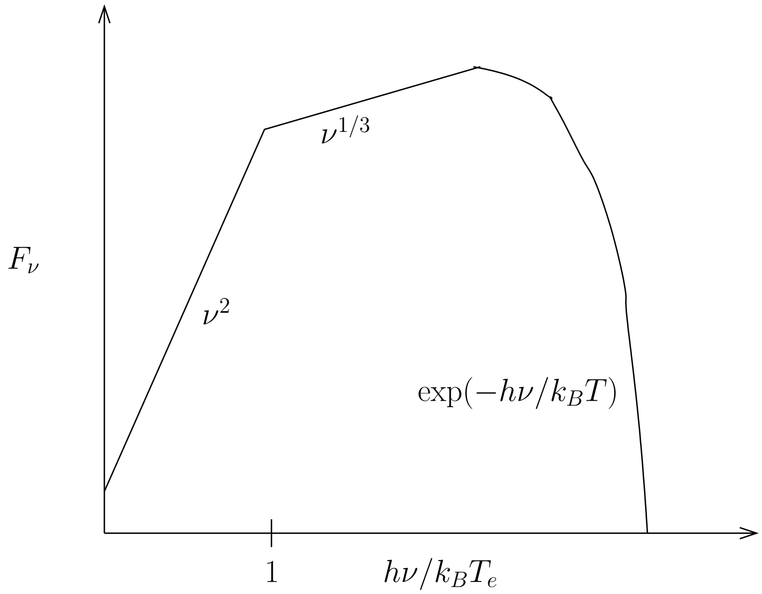
Now let $x = h\nu/k_B T \to 1/T = k_B x/h\nu, d(1/T) = k_B dx/h\nu$. This gives

$$F_\nu \sim \int \frac{\nu^3 \nu^{-5/3} x^{5/3} \nu^{-1} dx}{\exp(x) - 1} = \nu^{1/3}\int \frac{x^{5/3} dx}{\exp(x) - 1}. \tag{XIV.198}$$

At this point three possibilities are available

$$F_\nu \propto \begin{cases} \nu^2 & h\nu \ll k_B T_e \to x \ll 1 \\ \nu^{1/3} & h\nu \sim k_B T_e \to x \sim 1 \\ \exp(-h\nu/k_B T_e) & h\nu \gg k_B T_e \to x \gg 1 \end{cases} \tag{XIV.199}$$

We can draw that this looks like as follows:



This spectrum is known as the multicolor blackbody. Note that this spectra is just assumes that

1. $\dot{M}$ is constant throughout the disk – it is a steady state disk.
2. dissipation is entirely local, i.e., at every point the disk is in roughly Keplerian rotation.

It is important that this spectra does not depend on the details on how dissipation occurs, just that it does and is roughly partitioned by the mass of the disk – equal dissipation per gram. So this profile is fairly free of the assumptions and microphysics of dissipation, an attractive situation to be in. For this reason this is one of the key predictions of accretion disk theory and has been more or less confirmed from (some) astrophysical observations.

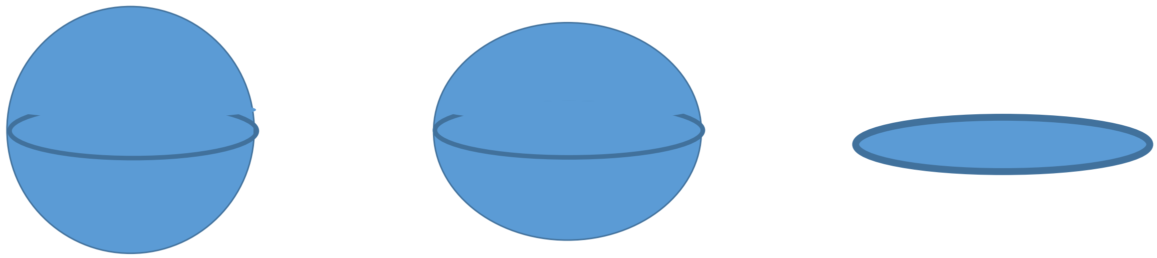
XIV.4.1 Derivation from Perturbation Theory

This section derives the thin disk equations as perturbation of a Keplerian disk at first order in the viscosity. It is helpful to think first of a spherical star and a Keplerian disk as the two extremes of a family of rotating stars. When $\Omega = 0$, the star is supported against gravity entirely by pressure, and it is spherical. As Ω increases, the star grows increasing oblate, supported by both rotation and pressure:

$$\rho\nabla\Phi = -\nabla P + \rho\Omega^2 r. \tag{XIV.200}$$

To reach a disk as the final configuration, the pressure must decrease to zero along the sequence. Think of gas whose temperature drops to zero, with the gas sparse enough that the particles are not degenerate. In the thin disk we are considering, a small viscosity dissipates energy, slightly heating the disk and so leading to a small nonzero pressure. We will look at perturbations of the Keplerian disk arising from adding a small viscosity.

One can describe perturbation theory as follows. Consider a family of solutions $P(e), \rho(e), \mathbf{v}(e)$ to the mass conservation equation and the Euler equation with viscosity (XIV.158) (the Navier-Stokes



equation). For $e = 0$, the solution is the Kepleran disk:

$$P(0,t,\mathbf{r}) = 0, \quad \rho(0,t,\mathbf{r}) = \Sigma(r)\delta(z), \quad \mathbf{v}(0,t,\mathbf{r}) = r\Omega(r), \text{ with } \Omega(r) = \sqrt{\frac{GM}{r^3}}. \tag{XIV.201}$$

To make precise the statement that we work to first order in viscosity, one takes the the coefficent of viscosity ν to be proportional to e, vanishing at $e = 0$:

$$\nu = e\overset{1}{\nu}. \tag{XIV.202}$$

We can expand the solution in powers of e, writing for each quantity Q

$$Q(e) = Q(0) + e\overset{1}{Q} + O(e^2), \text{ where } \overset{1}{Q} := \left.\frac{\partial Q}{\partial e}\right|_{e=0}. \tag{XIV.203}$$

The usual physicist's notation is

$$Q = Q_0 + \delta Q; \tag{XIV.204}$$

here $\delta Q = e\overset{1}{Q} = e\left.\frac{\partial Q}{\partial e}\right|_{e=0}$. If Q represents all of the fluid variables, then for each e, the exact solution $Q(e)$ satisfies a set of equations of the form

$$E(e, Q(e)) = 0, \tag{XIV.205}$$

and the first-order perturbation $\overset{1}{Q}$ satisfies the first-order equation

$$\left.\frac{\partial}{\partial e}E(e, Q(e))\right|_{e=0} = 0, \text{ or, equivalently } \delta E = 0. \tag{XIV.206}$$

A finite disk is a stationary axisymmetric exact solution $P(e,t,\mathbf{r}),\ \rho(e,t,\mathbf{r}),\ \mathbf{v}(e,t,\mathbf{r})$ to the set of equations (XIV.167)-(XIV.170)

$$\begin{aligned}
\frac{\partial}{\partial r}(r\rho v_r) &= 0 && \text{(XIV.207)}\\
\rho\left(v_r\frac{\partial}{\partial r}v_r - \frac{v_\phi^2}{r}\right) &= -\frac{\partial}{\partial r}P - \frac{GM\rho}{\mathtt{r}^2}\frac{r}{\mathtt{r}} && \text{(XIV.208)}\\
\rho v_r\frac{\partial}{\partial r}\left(r v_\phi\right) &= \frac{1}{r}\frac{\partial}{\partial r}\left(r^3\rho\nu\frac{\partial}{\partial r}\Omega\right) && \text{(XIV.209)}\\
\rho v_r\frac{\partial}{\partial r}v_z &= -\frac{\partial}{\partial z}P - \frac{GM\rho}{\mathtt{r}^2}\frac{z}{\mathtt{r}}. && \text{(XIV.210)}
\end{aligned}$$

At $e = 0$, the mass conservation equation and the ϕ and z components of the Euler equation vanish when $v_r = 0$. With $\rho_0 = \Sigma(r)\delta(z)$, the remaining equation implies for Ω the Keplerian angular velocity, and it leaves the surface density profile $\Sigma(r)$ arbitrary.

$$-\rho_0\Omega^2 r = \frac{GM\rho_0}{r^2} \qquad \text{(XIV.211)}$$

The first-order equations (XIV.206) are

$$\begin{aligned}
\frac{\partial}{\partial r}(r\Sigma\overset{1}{v}_r) &= 0 && \text{(XIV.212)}\\
-\overset{1}{\rho}\Omega^2 r - 2\rho_0\Omega\overset{1}{v}_\phi &= -\frac{\partial}{\partial r}\overset{1}{P} - \frac{GM\overset{1}{\rho}}{r^2} && \text{(XIV.213)}\\
\Sigma\overset{1}{v}_r\frac{\partial}{\partial r}(r^2\Omega) &= \frac{1}{r}\frac{\partial}{\partial r}\left(r^3\sigma\overset{1}{\nu}\frac{\partial}{\partial r}\Omega\right) && \text{(XIV.214)}\\
0 &= -\frac{\partial}{\partial z}\overset{1}{P} - \frac{GM\overset{1}{\rho}z}{r^3}. && \text{(XIV.215)}
\end{aligned}$$

As before, using the first of these equations to bring $r\Sigma\overset{1}{v}_r$ inside ∂_r in the third equation gives

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r^3\overset{1}{v}_r\Sigma\Omega\right) = \frac{1}{r\rho}\frac{\partial}{\partial r}\left(r^3\sigma\overset{1}{\nu}\frac{\partial}{\partial r}\Omega\right) \qquad \text{(XIV.216)}$$

The first and third equations, with $\overset{1}{v}_r$ replaced by $v_r = e\overset{1}{v}_r$ are just equations (XIV.182) and (XIV.183). One needs the additional physical assumption that the disk extends to the star's surface, which rotates more slowly that the Keplerian speed, to infer that $\partial_r\Omega = 0$ close to the surface of the star and thereby get the boundary condition (XIV.182).

XIV.5 Local Structure of Thin Disks

In the previous lecture, we saw that the bulk properties of the disks depends only on their mass accretion rate. It turns out that the simple assumptions the determine the bulk structure also determines their local structure as we will see here.

Because these disks are thin, we can solve for their radial and vertical structure independently. Let begin with the vertical momentum equation, which is just vertical hydrostatic balance:

$$\frac{1}{\rho}\frac{\partial}{\partial z}P = -\frac{GMz}{r^3} \tag{XIV.217}$$

Define H as the scale height, we can approximate the above equation via

$$-\frac{P}{\rho H} = -\frac{c_s^2}{H} = \frac{GMH}{r^3}, \tag{XIV.218}$$

where I used $P \sim \rho c_s^2$. This gives:

$$\frac{c_s^2}{v_{\rm orb}^2} = \frac{H^2}{r^2} \quad \text{or} \quad H = \frac{c_s}{\Omega} \tag{XIV.219}$$

We also note that

$$\Sigma = \int \rho dz \approx \rho H \to \rho = \Sigma/H. \tag{XIV.220}$$

This gives ρ in terms of Σ and H. And we know H in terms of c_s. To get c_s, we will assume an ideal gas equation of state $P = \rho k_B T/\mu m_p$, where $\mu = A/Z + 1$ is mean molecular weight. Now the relevant pressure is for the midplane temperature T_m so:

$$c_s^2 = \frac{k_B T_m}{\mu m_p} \tag{XIV.221}$$

To get this midplane temperature we have to work backwards. We will get the flux, which is determined by local dissipation, then work back to get the midplane temperature to support that flux. So recall the local flux

$$F = \sigma T_e^4 = \frac{3}{8\pi}\frac{GM\dot{M}}{r^3}\left(1 - \sqrt{\frac{r_*}{r}}\right), \tag{XIV.222}$$

where we have divided by a factor of 2 to account for the flux leaving through the top and bottom. In radiative equilibrium, the flux equation is

$$F = -\frac{4\sigma}{3\kappa\rho}\frac{\partial}{\partial z}T^4 \sim \frac{4\sigma}{3\kappa\rho}\frac{T^4}{H} = \frac{4\sigma T^4}{3\kappa\Sigma}, \tag{XIV.223}$$

where κ is the opacity and is a function of $\kappa(\rho, T)$ or $\kappa(\Sigma, H, T)$. We approximate $\frac{\partial}{\partial z} = 1/H$. This allows us to connect the midplane temperature to the flux. In particular:

$$F = \frac{4\sigma T^4}{3\kappa\Sigma} = \frac{3}{8\pi}\frac{GM\dot{M}}{r^3}\left(1 - \sqrt{\frac{r_*}{r}}\right) \tag{XIV.224}$$

All we need now is Σ which we get from

$$\nu\Sigma = \frac{\dot{M}}{3\pi}\left(1 - \sqrt{\frac{r_*}{r}}\right), \tag{XIV.225}$$

and the kinetimatic viscosity is

$$\nu = \alpha c_s H \tag{XIV.226}$$

Equations (XIV.219), (XIV.220), (XIV.221), (XIV.224),(XIV.225), and (XIV.226) are six equations with seven unknowns ν, c_s, T_m H, Σ, ρ, and κ. To complete the set, we need a prescription for $\kappa(\rho, T)$ and then we can solve this set of seven algebraic equations. For instance, let me assume a Kramers (free-free) opacity:

$$\kappa = 5 \times 10^{24} \rho T_m^{-3.5}\,\mathrm{cm}^2\,\mathrm{g}^{-1}. \tag{XIV.227}$$

You can then show (in your HW):

$$\Sigma = 5.2\alpha^{-4/5} \dot{M}_{16}^{7/10} M_1^{1/4} r_{10}^{-3/4} f^{14/5}\,\mathrm{g\,cm}^{-2}, \tag{XIV.228}$$

where $M_1 = M/\mathrm{M}_\odot$, $r_{10} = r/10^{10}\,\mathrm{cm}$, $\dot{M}_{16} = \dot{M}/10^{16}\,\mathrm{g\,s}^{-1}$. The complete set is in your HW. This solution is known at the Shakura-Sunyaev alpha disk solution and is one of the great results in astrophysics.

XIV.6 Radiative Inefficient Accretion Flows

Thus far we have studied the radiative efficient accretion flow. We will note that the solutions that we found are power-law in radius, ignoring the $(1 - \sqrt{r_*/r})$ term. So the question is if there are other solutions that are available. Let us return to the Navier-Stokes equations:

$$\begin{aligned}
\frac{\partial}{\partial t}\rho + \frac{1}{r}\frac{\partial}{\partial r} r\rho v_r &= 0 && \text{(XIV.229)}\\
\frac{\partial}{\partial t} v_r + v_r \frac{\partial}{\partial r} v_r - r\Omega^2 &= -\frac{1}{\rho}\frac{\partial}{\partial r} P - \frac{GM}{r^2} && \text{(XIV.230)}\\
\frac{\partial}{\partial t} r v_\phi + v_r \frac{\partial}{\partial r} r^2 \Omega &= \frac{1}{r\rho}\frac{\partial}{\partial r} r^3 \rho\nu \frac{\partial}{\partial r}\Omega && \text{(XIV.231)}\\
0 &= -\frac{1}{\rho}\frac{\partial}{\partial z} P - \frac{GM}{r^2}\frac{z}{r}, && \text{(XIV.232)}
\end{aligned}$$

As usual the last equation gives me $H = c_s/\Omega_K$, where $\Omega_K = \sqrt{GM/r^3}$ is the Keplerian rotation rate. I will also use $\nu = \alpha c_s^2/\Omega_K$. To this set I will add an energy equation

$$\rho\frac{\partial}{\partial t} e + \rho v_r \left(\frac{\partial}{\partial r} e - \frac{P}{\rho^2}\frac{\partial}{\partial r}\rho\right) = \rho\nu r^2 \left(\frac{\partial}{\partial r}\Omega\right)^2 - (\text{cooling}), \tag{XIV.233}$$

where $e \approx c_s^2$ is the internal energy, the second term on the LHS contains pdV work, and the right hand side contains the dissipation term that we discussed last time. Now at this point a few thing should bother you. First there is a difference between Ω and Ω_K. Second, the energy equation has this unspecified cooling term there. We have ignored the cooling term up to this point as we have set it to exactly balance the heating term. So it seems strange to include it now.

However, let us consider the possibility that it doesn't cool effectively or just partially. Let us rewrite the energy equation as

$$\rho\frac{\partial}{\partial t} e + \rho v_r \left(\frac{\partial}{\partial r} e - \frac{P}{\rho^2}\frac{\partial}{\partial r}\rho\right) = f\rho\nu r^2 \left(\frac{\partial}{\partial r}\Omega\right)^2, \tag{XIV.234}$$

where f is some constant between 0 and 1. If it is 1, then all the heat that is produced goes into heating – none is radiate away. If it is 0, the all of it is radiated away. We now use the steady state approximation and drop the vertical equation:

$$\begin{aligned} \frac{1}{r}\frac{\partial}{\partial r} r\rho v_r &= 0 && \text{(XIV.235)} \\ v_r \frac{\partial}{\partial r} v_r - r\Omega^2 &= -\frac{1}{\rho}\frac{\partial}{\partial r} P - \frac{GM}{r^2} && \text{(XIV.236)} \\ v_r \frac{\partial}{\partial r} r^2\Omega &= \frac{1}{r\rho}\frac{\partial}{\partial r} r^3 \rho \frac{\alpha c_s^2}{\Omega_K}\frac{\partial}{\partial r}\Omega && \text{(XIV.237)} \\ v_r \frac{\partial}{\partial r}\rho c_s^2 &= f\rho \frac{\alpha c_s^2}{\Omega_K} r^2 \left(\frac{\partial}{\partial r}\Omega\right)^2. && \text{(XIV.238)} \end{aligned}$$

Let me define $\tilde{r} = r/r_0$, where r_0 is some characteristic radius. Let me also assume power law solutions for each

$$\rho = \rho_0 \tilde{r}^a, \quad v_r = v_{r,0}\tilde{r}^b, \quad \Omega = \Omega_0 \tilde{r}^c, \quad c_s^2 = c_{s,0}^2 \tilde{r}^d. \tag{XIV.239}$$

and $P = \rho c_s^2$. Plugging in these power law solutions, we find

$$\begin{aligned} 2 + a + b + d/2 + 1/2 &= 0 && \text{(XIV.240)} \\ 2b - 1 &= 1 + 2c = d - 1 = -2 && \text{(XIV.241)} \\ b - 1 + 2 + c &= d + c + 3/2 && \text{(XIV.242)} \\ b - 1 + d &= d + 3/2 + 2 - 2 + 2c && \text{(XIV.243)} \end{aligned}$$

A solution to this set is

$$a = -3/2, \quad , b = -1/2, \quad c = -3/2, \quad \text{and} \quad d = -1 \tag{XIV.244}$$

or

$$\rho = \rho_0 \tilde{r}^{-3/2}, \quad v_r = v_{r,0}\tilde{r}^{-1/2}, \quad \Omega = \Omega_0 \tilde{r}^{-3/2}, \quad c_s^2 = c_{s,0}^2 \tilde{r}^{-1}. \tag{XIV.245}$$

Now let's OOM some of the normalizations: looking at the energy equation, we can show

$$v_r \frac{\partial}{\partial r}\rho c_s^2 = f\rho\frac{\alpha c_s^2}{\Omega_K} r^2 \left(\frac{\partial}{\partial r}\Omega\right)^2 \rightarrow \frac{v_{r,0}\rho_0 c_{s,0}^2}{r_0} = f\alpha \frac{\rho_0 c_{s,0}^2 \Omega_0^2}{\Omega_{K,0}} \rightarrow v_{r,0} = \alpha f \frac{\Omega_0}{\Omega_{K,0}} r_0 \Omega_0, \tag{XIV.246}$$

This implies that the radial infall velocity scales likes f. For $f = 0$ (efficient cooling), v_r is small as it is in radiative efficient accretion disks, but if f is near unity, then the radial infall velocity is comparable to the rotational velocity, with a factor of α. Let look at the angular momentum equation:

$$v_{r,0} r_0 \Omega_0 = \frac{\alpha c_{s,0}^2}{\Omega_{K,0}}\Omega_0 \rightarrow c_{s,0}^2 = f r_0^2 \Omega_0^2 \tag{XIV.247}$$

Now this one is interesting. The sound speed, i.e., internal energy is similar to the rotation velocity, i.e., gas is near the virial temperature for $f \neq 0$. Finally let's look at the angular momentum equation:

$$-r_0\Omega_0^2 = -\frac{c_{s,0}^2}{r_0} - \frac{GM}{r_0^2} \rightarrow \Omega_0^2 = \frac{GM}{(1-f)r_0^3} \tag{XIV.248}$$

Now don't take the $1 - f$ too seriously as we have dropped factors of order unity, But the rotation rate is going to be different from the Keplerian rate due to the nontrivial contribution of the gas pressure. The structure of this solution make up what is called a radiative inefficient accretion flow or RIAF. This is increasing important in astrophysics as they have many important properties for high energy systems. In particular

1. Inefficient cooling implies gas near the virial temperature.
2. Larger radial velocities $v_r \sim \alpha c_s$
3. Sub-Keplerian angular velocities
4. The binding energy of the gas is near zero – possibility for outflows.

This last fact is perhap the most important aspect of RIAF that you should be concerned with because these flows mean jets and mechanical power which is a powerful way by which these flows might be observed. In addition just because these things are radiatively inefficient does not mean that they don't radiate. Instead the hot gas can radiate via synchrotron emission or inverse Compton scattering – They radiate nonthermally, which implies hard photons.

Interestingly, x-ray binary systems have been observed to migrate between a radiatively efficient thin disk state and a radiative inefficient state. Because it undergoes the transition in one system, we can learn alot about these systems. Here, there is a definitive link between the type of outflows and the state of the disk. In Figure 14, we show how this works. On the x-axis we show the x-ray hardness which is a proxy for radiative efficient vs inefficient. Blackbody spectrum are the softest so they are associated with thin disks. Hard spectra are associated with thick disks. On the y-axis is the luminosity, which is a proxy for the accretion rate. What is not shown is that jets appear when the disk is on the right side of the plot (hard state, generally lower accretion rate) whereas these jets disappear on the left, replaced by disk winds (a more recent observed phenomenon).

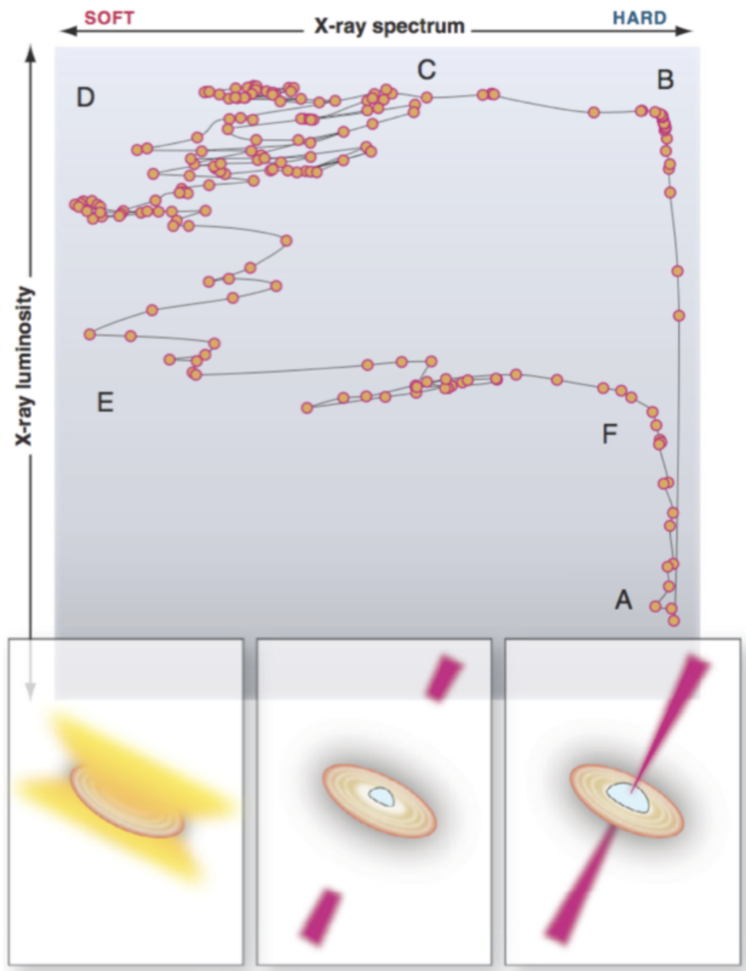


Figure 14: Accretion states and outflows

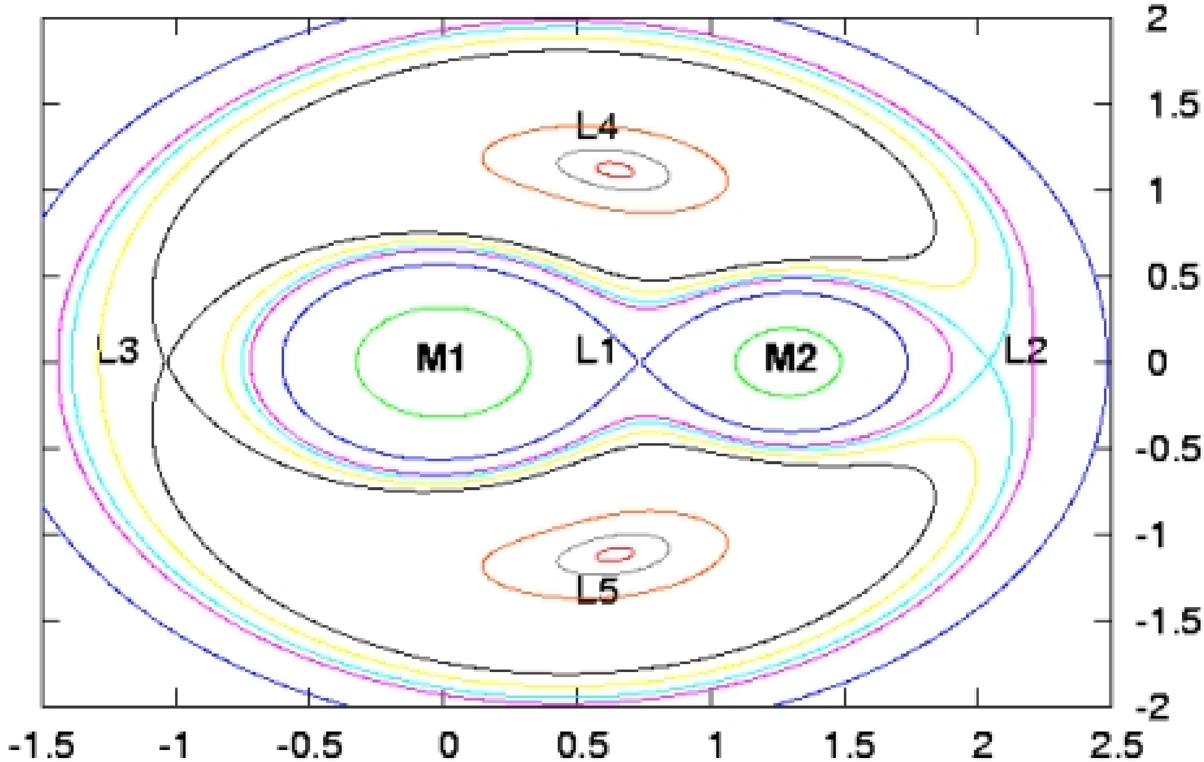
Lecture XV Interacting Binaries

Interacting means accretion (mass transfer).

XV.2 Gravity and the Roche Potential

Consider two stars orbiting their COM. They have angular velocity $\omega = v_1/a_1 = v_2/a_2$. We will work in a coordinate system that follows this rotation, where the gravitational force will be balanced by a centrifugal force $F_c = m\omega^2 r$.

Instead of forces consider the potential energy. From gravity this is $U_g = -GM_1M_2/a$. We can integrate the centrifugal force to find a centrifugal potential, with $U_c = -m\omega^2 r^2/2$ for a mass m a distance r from the origin.



The full potential for a test mass in the plane of the orbit is:

$$U = -G\left(\frac{M_1 m}{s_1} + \frac{M_2 m}{s_2}\right) + \frac{1}{2}m\omega^2 r^2 \tag{XV.249}$$

where s_1 and s_2 are the distances from the test mass to the stars. Or, we can find:

$$\Phi = -G\left(\frac{M_1}{s_1} + \frac{M_2}{s_2}\right) + \frac{1}{2}\omega^2 r^2 \tag{XV.250}$$

We can solve for s_1 and s_2 via geometry (θ is the angle from the COM to the test mass):

$$s_1^2 = a_1^2 + r^2 + 2a_1 r\cos\theta \tag{XV.251}$$

$$s_2^2 = a_2^2 - r^2 + 2a_2 r\cos\theta \tag{XV.252}$$

And we can get ω from Kepler:

$$\omega^2 = \frac{G(M_1+M_2)}{a^3} \tag{XV.253}$$

With this potential, we can evaluate the forces in 2D ($\nabla\Phi$) to find the Lagrange points where there is no force. There are three Lagrange points on the axis between the stars: L_1 between then, and L_2 and L_3 on either side. These are equilibrium points, but they are unstable since they are local maxima of Φ.

The inner point L_1 is the most important for this discussion. When one star expands, the material fills equipotential surfaces. As the radius of the star approaches the L_1 point matter can overfill its potential well and fall onto the other star. This is mass transfer. The potential well is known as the "Roche Lobe". We have

$$r_L = \frac{R_L}{a} = 0.5 - 0.227\log_{10} q \tag{XV.254}$$

with $q = M_2/M_1$ is the mass ratio. The other point has $1/q$, so the sign on the log is flipped. There are various useful approximations for r_L from Eggleton (1983, ApJ, 268, 368) and Paczyński (1967, Acta Astrononica, 17, 287).

If mass is transferred, we say a star is experiencing "Roche Lobe overflow."

XV.2.1 Accretion Disks

Convert bulk motion into heat through viscosity, leading to matter traveling inward in the disk. For a start, assume that all of that energy is radiated as a uniform blackbody disk in steady-state. In a ring at radius r, the matter has $E = -GMm/2r$ (Virial theorem). As it moves inward, E gets more negative, and this energy is radiated:

$$dE = \frac{dE}{dr}dr = \frac{d}{dr}\left(-G\frac{Mm}{2r}\right)dr = G\frac{M\dot{M}t}{2r^2}dr \tag{XV.255}$$

with $m = \dot{M}t$ matter entering (and leaving) the disk. And this will be radiated:

$$dLt = dE = G\frac{M\dot{M}t}{2r^2}dr \tag{XV.256}$$

Assume blackbody, with $dL = 4\pi r\sigma T(r)^4 dr$ (top and bottom, so $A = 2\times 2\pi r dr$). We can solve:

$$T = \left(\frac{GM\dot{M}}{8\pi\sigma R^3}\right)^{1/4}\left(\frac{R}{r}\right)^{3/4} \tag{XV.257}$$

This says that the inner edge will be the hottest, at $T_{\rm disk} = (GM\dot{M}/8\pi\sigma R^3)^{1/4}$, although a better fit to data says that the inner edge is at roughly half this. Beyond the inner edge, the temperature falls as $T(r) \propto r^{-3/4}$. Integrating, we find:

$$L_{\rm disk} = G\frac{M\dot{M}}{2R} \tag{XV.258}$$

which is 50% of the total amount of energy that could be tapped. So 50% can be radiated while the gas accretes, which leaves 50% to be radiated upon accretion.

Looking at the luminosity and temperature of the accretion flow, we can determine whether the object is a WD or a NS. For instance, we might find $T_{\rm max} = 3 \times 10^4\,\rm K$ for a WD, but $T_{\rm max} = 7 \times 10^6\,\rm K$ for a NS.

XV.3 Period-Density relation

Star is losing mass through L_1 point. So it fills the whole Roche lobe. Size of Roche lobe is:

$$\frac{R_2}{a} \approx \frac{2}{3^{4/3}}\left(\frac{M_2}{M_1 + M_2}\right)^{1/3} \tag{XV.259}$$

(approximate relation from Paczynski). Cube both sides:

$$\frac{3^4 R_2^3}{8M_2} = \frac{a^3}{M_1 + M_2} \tag{XV.260}$$

But Kepler's third law is:

$$\frac{a^3}{G(M_1 + M_2)} = \frac{P^2}{4\pi^2} \tag{XV.261}$$

So we have:

$$\frac{3^4 R_2^3}{8M_2} = \frac{GP^2}{4\pi^2} \tag{XV.262}$$

We can relate this to $\bar{\rho} = M_2/(4\pi R_2^3/3)$, with:

$$\bar{\rho} \approx 110\, P_{\rm hr}^{-2}\,\rm g\,cm^{-3} \tag{XV.263}$$

This is the period mean-density relation. We can turn this into a period-mass relation by using a mass-radius relation. e.g., for main-sequence we know that $R \propto M$, which says:

$$M_2 \approx 0.11\, M_\odot P_{\rm hr} \tag{XV.264}$$

XV.4 Effect of Mass Transfer

Does the orbit get wider or narrower?

Assume conservative mass transfer: mass and L are fixed. So $M_1 + M_2 = M$, and $\dot{M}_1 = -\dot{M}_2$ (this need not be the case). And:

$$L = \mu\sqrt{GMa} = M_1 M_2 \sqrt{\frac{Ga}{M}} \tag{XV.265}$$

is also constant. We can take a total derivative of the above to find:

$$\frac{\dot{a}}{a} = 2\frac{\dot{L}}{L} - 2\left(1 - \frac{M_2}{M_1}\right)\frac{\dot{M}_2}{M_2} \tag{XV.266}$$

assuming that star 2 is the donor ($\dot{M}_2 < 0$) and that $\dot{L} = 0$, we find that:

$$\frac{\dot{a}}{a} = -2\dot{M}_2 \frac{M_1 - M_2}{M_1 M_2} \tag{XV.267}$$

So if $M_1 > M_2$ (transfer from less to more massive) then $\dot{a} > 0$ and the orbit gets wider. if $M_1 < M_2$ (transfer from more to less massive) then $\dot{a} < 0$ and the orbit gets smaller. As that happens, ω increases. However, this treatment ignores possible losses of L. What could that be? Simplest is GR:

$$\dot{L}_{GR} = -\frac{32}{5}\frac{G^{7/2}}{c^5}\frac{M_1^2 M_2^2 (M_1 + M_2)^{1/2}}{a^{7/2}} \tag{XV.268}$$

With this as a minimum, it often drives systems to closer orbits even if $M_1 > M_2$.

Is this mass transfer stable? Will the movement of mass from one star to the other proceed slowly (on the nuclear evolutionary timescale of the star, $\sim 10^{10}$ yr for the Sun) or faster (thermal timescale $\sim 10^7$ yr, dynamical timescale $\sim 90\ \text{min}$)?

Consider how the star and the Roche lobe adjust to the change in mass. The star loses a parcel of mass, and it then adjusts its radius. First it restores hydrostatic equilibirum on a dynamical timescale (so fast it's adiabatic: no energy can enter or leave). Then adjusts thermal equilibrium on thermal timescale. If it becomes bigger compared to the change in the Roche lobe, we get unstable transfer and it runs away on the relevant timescale. If it is stable, it will continue slowly and only transfer matter as the star expands on the nuclear timescale.

If transfer is stable, then the radius of the donor will essentially track the Roche lobe radius as they both evolve slowly. Or, $R_2 = R_L = ar_L$. So we can re-write $\dot{a}/a$ in terms of these.

Assume that the donor star (M_2) has $R_2 \propto M_2^{\zeta}$, where $\zeta \equiv d\ln R_2/d\ln M_2$. For a normal star this would be ≈ 1, or $-1/3$ for a WD or NS. Would actually need to be careful to consider changes in radius on the appropriate timescales. Can also find ζ_L for the Roche lobe r_L. Use the slightly better Paczyński formula:

$$r_L \approx 0.46\left(\frac{M_2}{M_1 + M_2}\right)^{1/3} \tag{XV.269}$$

We find (assuming $dM_1 = -dM_2$) $\zeta_L = 1/3$. So the change in a will be:

$$\frac{\dot{a}}{a} = \frac{\dot{M}_2}{M_2}\left(\zeta - \zeta_L\right) \tag{XV.270}$$

Equate this with $\dot{a}/a$ from above, and solve:

$$\frac{\zeta - \zeta_L}{2} = \frac{M_2}{M_1} - 1 \tag{XV.271}$$

If the LHS is < 0, then mass transfer will be dynamically stable: the star's radius will shrink relative to R_L, and we will need to wait for nuclear evolution to make it larger. This requires:

$$\frac{M_2}{M_1} < \frac{5}{6} + \frac{\zeta}{2} \tag{XV.272}$$

So for a WD or degenerate object, $\zeta \approx -1/3$, and we must have $M_2/M_1 < 2/3$ for stability.

Lecture XVI Magnetic Accretion

So far we have ignored how the material gets to the star. For a WD/NS with $B = 0$, it will just fall. But what if $B > 0$?

Consider a dipole magnetic field with dipole moment $\mu \sim B_0 R^3$. So the strength of the field will vary as:

$$B \sim \frac{\mu}{r^3} \tag{XVI.273}$$

Far away it is negligible. But close to the star it will be strong enough to interrupt the accreting gas. The magnetic field has a pressure:

$$P_B = \frac{B^2}{8\pi} \tag{XVI.274}$$

which increases like r^6 as you approach the star. It will channel the material. It can't stop it overall, but the material (which we assume is ionized) is forced to follow the magnetic field lines. It end up moving along them and only hitting the star at the magnetic poles.

The field interrupts the flow where the magnetic pressure is equal to the gas pressure + ram pressure. But since the flow is highly supersonic, we can neglect gas pressure and say that r_M is where:

$$\frac{B^2}{8\pi} = \frac{\rho v^2}{2} \tag{XVI.275}$$

If the flow is spherically symmetric, then we have v^2 close to the free-fall value of $2GM^2/r$ and $4\pi r^2 \rho v = \dot{M}$. So we can solve for r_M and find:

$$r_M = 5.1 \times 10^8\,\mathrm{cm} \dot{M}_{16}^{-2/7} \mu_{30}^{4/7} M^{-1/7} \tag{XVI.276}$$

with μ expressed in $10^{30}\,\mathrm{G\,cm}^3$ (a WD with $B = 10^4\,\mathrm{G}$ and $R = 5 \times 10^8$ cm or a NS with $B = 10^{12}\,\mathrm{G}$ and $R = 10^6$ cm).

This is pretty rough, and in general things will be non-spherical etc. But P_B increases so steeply with radius that gas will not penetrate much beyond r_M before it gets caught. This radius is known as the Alvén radius. Clearly this will happen well outside a NS, and if B is strong outside a WD too.

A common sign of channeled accretion is X-ray hotspots which cause pulsations when the star rotates (this requires misalignment). How big will the spots be? Consider a NS with a disk with an angle α between the magnetic axis and the disk. In polar coordinates (r, θ) the dipole geometry says that a field line will collor $r = C \sin^2 \theta$, with C a constant labeling individual field lines. At the disk we have $r = r_M$ and $\theta = \alpha$, so $C = r_m / \sin^2 \alpha$. At the star we have $r = R$ and $\theta = \beta$. We can solve:

$$\sin^2 \beta = \frac{R}{C} = \frac{R}{r_M} \sin^2 \alpha \tag{XVI.277}$$

Only the part of the star with $\theta < \beta$ will be hit by accreted material. The area of this cap is (as a fraction of the total area of the star):

$$f \sim \frac{\pi R^2 \sin^2 \beta}{4\pi R^2} = \frac{R \sin^2 \alpha}{4 r_M} \tag{XVI.278}$$

which we multiply by two for the opposite pole. Typically this is 0.01%–10%.

What happens to the angular momentum of the star? The material in the disk will have a Keplerian orbit with angular velocity $\Omega_K(r) = \sqrt{GM/r^3}$. It will hit the star's field lines, which rotate with Ω_0. If:

$$\Omega_0 > \Omega_K(r_M) \tag{XVI.279}$$

then the star will be rotating faster than the disk when it encounters the field lines. The material will hit a centrifugal barrier and be flung away (known as "propeller accretion"). We parameterize this with the "fastness parameter":

$$\omega_0 = \frac{\Omega_0}{\Omega_K(r_M)} \tag{XVI.280}$$

and require $\omega_0 < 1$ for actual accretion.

Assuming that it can penetrate this barrier, the material can still drag on the star. P (spin period) is typically $1\,\mathrm{s}$–$10^3\,\mathrm{s}$ for accreting systems (not for rotation-powered pulsars). But P is getting shorter slowly, spinning up. If $\omega_0 \ll 1$ we can treat it simply. Basically the star gains angular momentum at a rate:

$$\dot{L} \sim \dot{M} r_M^2 \Omega_K(r_M) = I\dot{\Omega}_0 \tag{XVI.281}$$

We can then work out a relation between the spin-up rate and the star's properties.

Eventually we get $\omega_0 \sim 1$ and the star is in near-equilibrium. Here we see periods of spin-up and spin-down. We can equivalently express this by asking whether the corotation radius:

$$R_\Omega = \left(\frac{GMP^2}{4\pi^2}\right)^{1/3} \tag{XVI.282}$$

is bigger or smaller than r_M. For $\omega_0 < 1$ that is $R_\Omega > r_M$. We end up with an equilibrium period where these are equal.