Elliptic PDE's

In [6]:
```
import matplotlib.pyplot as plt
import numpy as np
from mpl_toolkits import mplot3d
```

In [44]:
```
N = 100
V = np.zeros((2,N,N))
V
```

Out[44]:
```
array([[[0., 0., 0., ..., 0., 0., 0.],
        [0., 0., 0., ..., 0., 0., 0.],
        [0., 0., 0., ..., 0., 0., 0.],
        ...,
        [0., 0., 0., ..., 0., 0., 0.],
        [0., 0., 0., ..., 0., 0., 0.],
        [0., 0., 0., ..., 0., 0., 0.]],

       [[0., 0., 0., ..., 0., 0., 0.],
        [0., 0., 0., ..., 0., 0., 0.],
        [0., 0., 0., ..., 0., 0., 0.],
        ...,
        [0., 0., 0., ..., 0., 0., 0.],
        [0., 0., 0., ..., 0., 0., 0.],
        [0., 0., 0., ..., 0., 0., 0.]]])
```

In [45]:
```
V[0,0] = np.ones(N)
V[1,0] = np.ones(N)
V
```

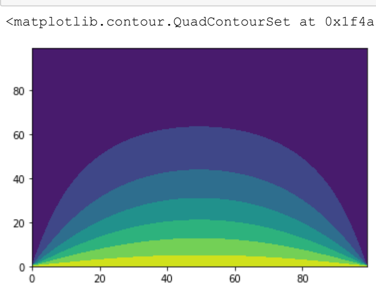
Out[45]:
```
array([[[1., 1., 1., ..., 1., 1., 1.],
        [0., 0., 0., ..., 0., 0., 0.],
        [0., 0., 0., ..., 0., 0., 0.],
        ...,
        [0., 0., 0., ..., 0., 0., 0.],
        [0., 0., 0., ..., 0., 0., 0.],
        [0., 0., 0., ..., 0., 0., 0.]],

       [[1., 1., 1., ..., 1., 1., 1.],
        [0., 0., 0., ..., 0., 0., 0.],
        [0., 0., 0., ..., 0., 0., 0.],
        ...,
        [0., 0., 0., ..., 0., 0., 0.],
        [0., 0., 0., ..., 0., 0., 0.],
        [0., 0., 0., ..., 0., 0., 0.]]])
```

In [46]:
```
V[1,1,1] = (V[0,0,1] + V[0,2,1] + V[0,1,0] + V[0,1,2])/4
V[1,1,1] = (V[0,0,1] + V[0,2,1] + V[0,1,0] + V[0,1,2])/4
V[1,1,2] = (V[0,0,2] + V[0,2,2] + V[0,1,1] + V[0,1,3])/4
```

In [34]:
```
for k in range(10000):
    for i in range(1,N-1):
        for j in range(1,N-1):
            V[k%2,i,j] = (V[(k+1)%2, i-1, j] + V[(k+1)%2, i+1, j] + V[(k+1)%2, i, j-1] + V[(k+1)%2, i, j+1])/4
```

In [35]:
```
plt.contourf(V[1])
```

Out[35]: <matplotlib.contour.QuadContourSet at 0x1f4a3c2cb80>

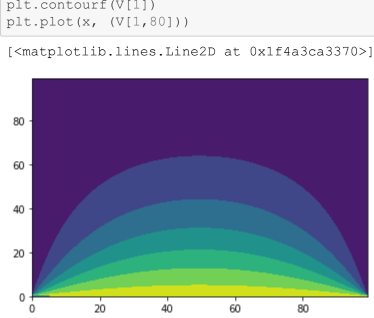


In [38]:
```
for i in range(10000):
    V[0] = V[1]
    V[1,1:-1,1:-1] = (V[0,0:-2,1:-1] + V[0,1:-1,2:] + V[0,1:-1,0:-2] + V[0,2:,1:-1])/4

V
```

Out[38]:
```
array([[[1.00000000e+00, 1.00000000e+00, 1.00000000e+00, ...,
         1.00000000e+00, 1.00000000e+00, 1.00000000e+00],
        [0.00000000e+00, 4.99888342e-01, 6.97429340e-01, ...,
         6.97429340e-01, 4.99888342e-01, 0.00000000e+00],
        [0.00000000e+00, 3.02124028e-01, 4.99553368e-01, ...,
         4.99553368e-01, 3.02124028e-01, 0.00000000e+00],
        ...,
        [0.00000000e+00, 2.23433217e-04, 4.46631262e-04, ...,
         4.46631262e-04, 2.23433217e-04, 0.00000000e+00],
        [0.00000000e+00, 1.11657857e-04, 2.23198210e-04, ...,
         2.23198210e-04, 1.11657857e-04, 0.00000000e+00],
        [0.00000000e+00, 0.00000000e+00, 0.00000000e+00, ...,
         0.00000000e+00, 0.00000000e+00, 0.00000000e+00]],

       [[1.00000000e+00, 1.00000000e+00, 1.00000000e+00, ...,
         1.00000000e+00, 1.00000000e+00, 1.00000000e+00],
        [0.00000000e+00, 4.99888342e-01, 6.97429340e-01, ...,
         6.97429340e-01, 4.99888342e-01, 0.00000000e+00],
        [0.00000000e+00, 3.02124028e-01, 4.99553368e-01, ...,
         4.99553368e-01, 3.02124028e-01, 0.00000000e+00],
        ...,
        [0.00000000e+00, 2.23433217e-04, 4.46631262e-04, ...,
         4.46631262e-04, 2.23433217e-04, 0.00000000e+00],
        [0.00000000e+00, 1.11657857e-04, 2.23198210e-04, ...,
         2.23198210e-04, 1.11657857e-04, 0.00000000e+00],
        [0.00000000e+00, 0.00000000e+00, 0.00000000e+00, ...,
         0.00000000e+00, 0.00000000e+00, 0.00000000e+00]]])
```

In [39]:
```
plt.contourf(V[1])
plt.plot(x, (V[1,80]))
```

Out[39]: [<matplotlib.lines.Line2D at 0x1f4a3ca3370>]



In [47]:
```
def initPotential(N=100):
    V = np.zeros((2,N,N))

    V[0,0] = np.ones(N)
    V[1,0] = np.ones(N)

    dst = 1
    src = 0

    return V, dst, src
```

In [48]:
```
def calcError(V):
    return np.sqrt(((V[1] - V[0])**2).sum()/(N**2))


def doCalculation(V, dst, src, minError = 1e-6):
    error = 1
    count = 0
    while error > minError:
        mat1 = V[src, :-2, 1:-1]  # purple
        mat2 = V[src, 2:, 1:-1]   # yellow
        mat3 = V[src, 1:-1, :-2]  # red
        mat4 = V[src, 1:-1, 2:]   # blue
        V[dst,1:-1,1:-1] = (mat1 + mat2 + mat3 + mat4)/4
        dst = 1 - dst
        src = 1 - src
        error = calcError(V)
        count += 1
        if (count % 1000) == 0:
            print("Counting..", count, "error: ", error)

    return count, error, V
```

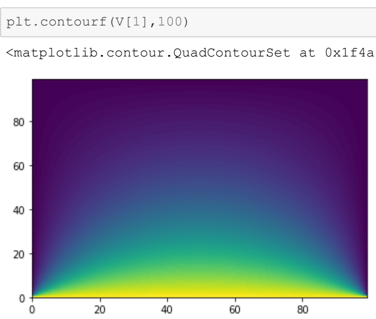
In [49]:
```
#def doBoth(N=100):
#   V, dst, src = initPotential(100)
 #  count, error, V = doCalculation(V, dst, src, minError=1e-4)

#%timeit doBoth()
```

In [50]:
```
count, error, V = doCalculation(V, 1, 0, minError=1e-10)
print(count)
```

```
Counting.. 1000 error:  8.754767552447384e-05
Counting.. 2000 error:  4.038731866263066e-05
Counting.. 3000 error:  2.277786806540147e-05
Counting.. 4000 error:  1.3542807827401835e-05
Counting.. 5000 error:  8.154710674389785e-06
Counting.. 6000 error:  4.924372356551765e-06
Counting.. 7000 error:  2.975560816052434e-06
Counting.. 8000 error:  1.7982400445496382e-06
Counting.. 9000 error:  1.0867757429971941e-06
Counting.. 10000 error:  6.568030687828957e-07
Counting.. 11000 error:  3.969456659680869 4e-07
Counting.. 12000 error:  2.398982611322283e-07
Counting.. 13000 error:  1.4498503157883522e-07
Counting.. 14000 error:  8.762322670753664e-08
Counting.. 15000 error:  5.295601761395 2175e-08
Counting.. 16000 error:  3.200452561664222e-08
Counting.. 17000 error:  1.9342271308599936e-08
Counting.. 18000 error:  1.1689704883238638e-08
Counting.. 19000 error:  7.064795963431981e-09
Counting.. 20000 error:  4.269683666596302 4e-09
Counting.. 21000 error:  2.5804281824433746e-09
Counting.. 22000 error:  1.5595088883027416e-09
Counting.. 23000 error:  9.425055783418317e-10
Counting.. 24000 error:  5.696131485165708e-10
Counting.. 25000 error:  3.442516910737376 7e-10
Counting.. 26000 error:  2.0805212654761062e-10
Counting.. 27000 error:  1.2573848830841773e-10
27455
```

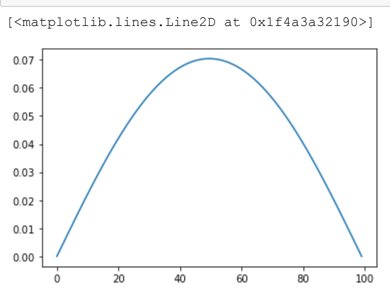
In [51]:
```
plt.contourf(V[1],100)
```

Out[51]: <matplotlib.contour.QuadContourSet at 0x1f4a4cc2820>



In [52]:
```
plt.plot(V[1, 80])
```

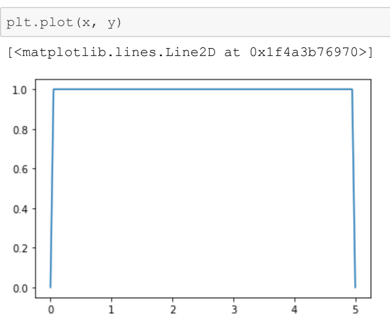
Out[52]: [<matplotlib.lines.Line2D at 0x1f4a3a32190>]



In [53]:
```
L = 5
x = np.linspace(0, L, N)
dx = x[1] - x[0]
y = np.where((x < L) & (x > 0), 1, 0)
```

In [54]:
```
plt.plot(x, y)
```

Out[54]: [<matplotlib.lines.Line2D at 0x1f4a3b76970>]



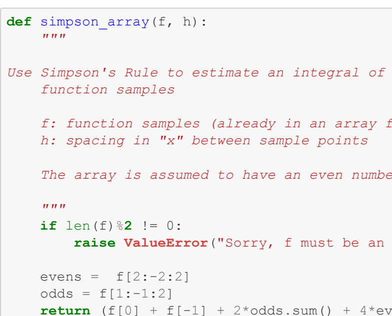
In [55]:
```
def fn_vec(x):
    return np.where((x < L) & (x > 0), 1, 0)

def basis(x, n):
    return np.sqrt(2/L)*np.sin(n*np.pi*x/L)
```

In [56]:
```
def simpson_array(f, h):
    """
    Use Simpson's Rule to estimate an integral of an array of
    function samples

    f: function samples (already in an array format)
    h: spacing in "x" between sample points

    The array is assumed to have an even number of elements.
    """
    if len(f)%2 != 0:
        raise ValueError("Sorry, f must be an array with an even number of elements.")

    evens = f[2:-2:2]
    odds = f[1:-1:2]
    return (f[0] + f[-1] + 2*evens.sum() + 4*odds.sum())*h/3
```

In [57]:
```
def braket(n):
    """
    Evaluate <n|f>
    """
    return simpson_array(basis(x,n)*fn_vec(x),dx)
```

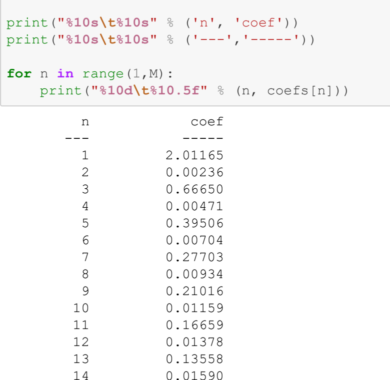
In [58]:
```
plt.plot(x, basis(x,1))
```

Out[58]: [<matplotlib.lines.Line2D at 0x[illegible]>]



In [59]:
```
M=[]
coefs = [0]
coefs_th = [0]
ys = []
sup = np.zeros(N)
for n in range(1,20):
    coefs.append(braket(n))    # do numerical integral

    if n%2==0:
        coefs_th.append(0)

    ys.append(coefs[n]*basis(x,n))
    sup += ys[n]
    plt.plot(x,sup)

print("{:3s}{:10s}" % ("n", "coef"))
print("{:3s}{:10s}" % ("-", "-----"))

for n in range(1,20):
    print("{:3d}{:10.5f}" % (n, coefs[n]))
```

n	coef
1	[illegible]
2	[illegible]
3	[illegible]
4	[illegible]
5	[illegible]
6	[illegible]
7	[illegible]
8	[illegible]
9	[illegible]
10	[illegible]
11	[illegible]
12	[illegible]
13	[illegible]
14	[illegible]
15	[illegible]
16	[illegible]
17	[illegible]
18	[illegible]
19	[illegible]



In [60]:
```
M = []
i = 1
while i < 20:
    A = np.array([[1, 1], [np.exp(-i*np.pi/L*L), np.exp(i*np.pi/L*L)]])
    B = np.array([1, 0])
    X = np.linalg.solve(A, B)
    M.append(X)
    i = i + 1
M
```

Out[60]:
```
[array([ 1.00187094e+00, -1.87093660e-03]),
 array([ 1.00000349e+00, -3.48735452e-06]),
 array([ 1.00000001e+00, -6.51241218e-09]),
 array([ 1.00000000e+00, -1.21615567e-11]),
 array([ 1.00000000e+00, -2.27110107e-14]),
 array([ 1.00000000e+00, -4.24115118e-17]),
 array([ 1.00000000e+00, -7.92010695e-20]),
 array([ 1.00000000e+00, -1.47903462e-22]),
 array([ 1.00000000e+00, -2.76201244e-25]),
 array([ 1.00000000e+00, -5.15790006e-28]),
 array([ 1.00000000e+00, -9.63208298e-31]),
 array([ 1.00000000e+00, -1.79873634e-33]),
 array([ 1.00000000e+00, -3.35903710e-36]),
 array([ 1.00000000e+00, -6.27280941e-39]),
 array([ 1.00000000e+00, -1.17141123e-41]),
 array([ 1.00000000e+00, -2.18754340e-44]),
 array([ 1.00000000e+00, -4.08511201e-47]),
 array([ 1.00000000e+00, -7.62871274e-50]),
 array([ 1.00000000e+00, -1.42461842e-52])]
```

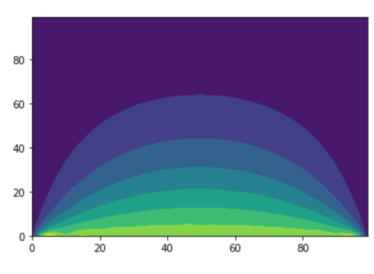
In [61]:
```
M = np.array(M)
M
```

Out[61]:
```
array([[ 1.00187094e+00, -1.87093660e-03],
       [ 1.00000349e+00, -3.48735452e-06],
       [ 1.00000001e+00, -6.51241218e-09],
       [ 1.00000000e+00, -1.21615567e-11],
       [ 1.00000000e+00, -2.27110107e-14],
       [ 1.00000000e+00, -4.24115118e-17],
       [ 1.00000000e+00, -7.92010695e-20],
       [ 1.00000000e+00, -1.47903462e-22],
       [ 1.00000000e+00, -2.76201244e-25],
       [ 1.00000000e+00, -5.15790006e-28],
       [ 1.00000000e+00, -9.63208298e-31],
       [ 1.00000000e+00, -1.79873634e-33],
       [ 1.00000000e+00, -3.35903710e-36],
       [ 1.00000000e+00, -6.27280941e-39],
       [ 1.00000000e+00, -1.17141123e-41],
       [ 1.00000000e+00, -2.18754340e-44],
       [ 1.00000000e+00, -4.08511201e-47],
       [ 1.00000000e+00, -7.62871274e-50],
       [ 1.00000000e+00, -1.42461842e-52]])
```

In [62]:
```
x = np.linspace(0, L, N)
y = np.linspace(0, L, N)

k = np.pi/L

X,Y = np.meshgrid(x, y)
V = np.zeros([N, N])
```

In [63]:
```
for n in range (1, 20):
    V = V + (coefs[n] * np.sqrt(2/L)*np.sin(n*k*X) * (M[n-1, 0] * np.exp(-k * n * Y) + M[n-1, 1] * np.e
xp(k * n * Y)))
V
plt.contourf(V)
```

Out[63]: <matplotlib.contour.QuadContourSet at 0x1f4a4ee3550>



In [64]:
```
plt.plot(x, V[80])
```

Out[64]: [<matplotlib.lines.Line2D at 0x1f4a4f3a7f0>]