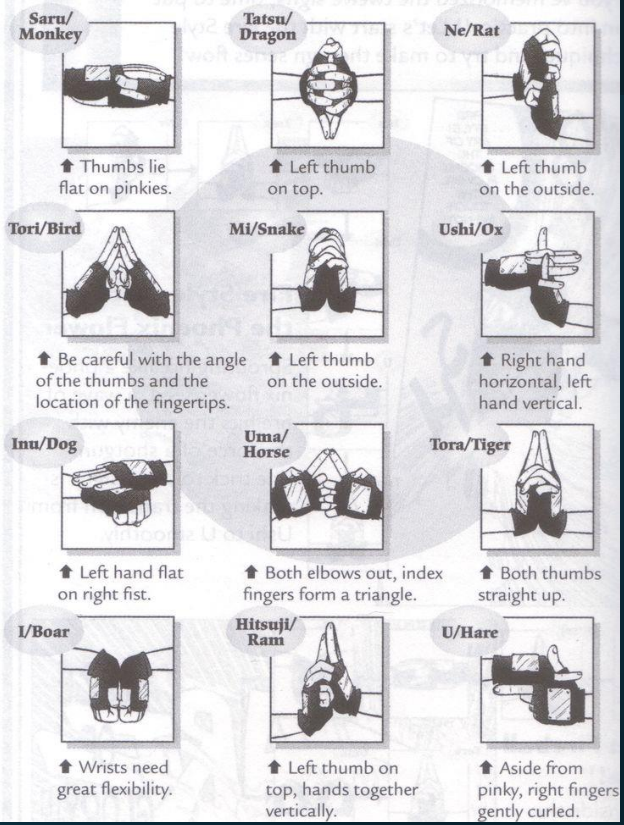
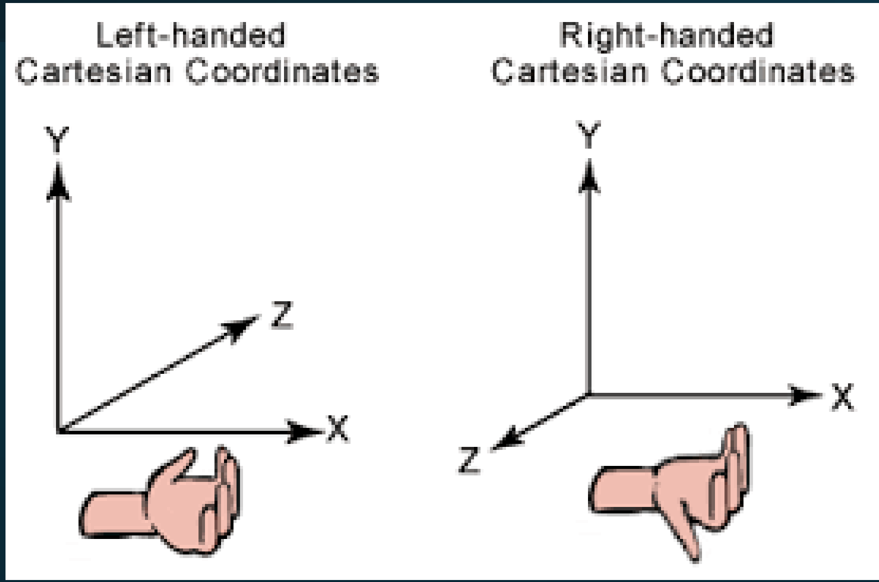
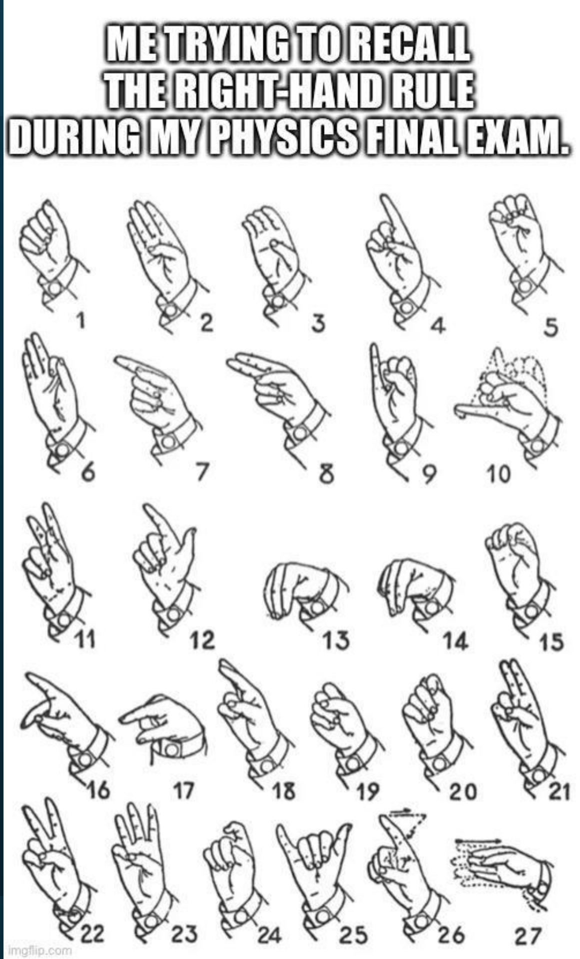
Introduction: Spatial descriptions and transformation



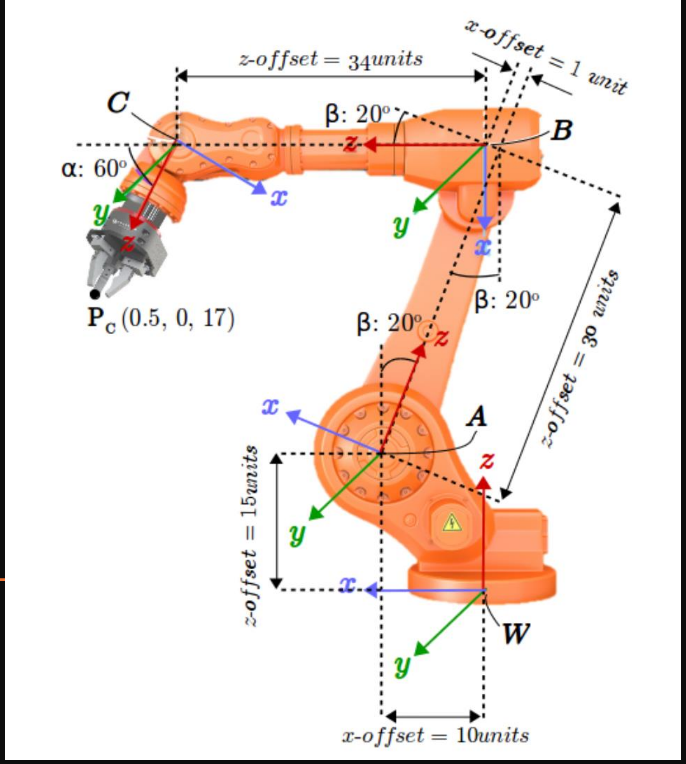




What are spatial descriptions and transformations ?

Represent information of objects in space:

- Positional
- Orientational
- Frames



In form of vectors and matrices

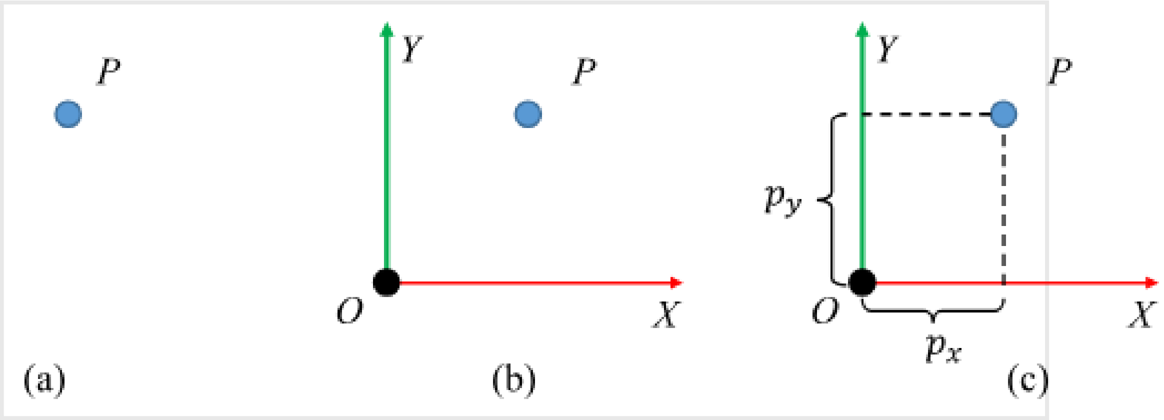
The size of vectors and matrices depends on the space dimension they're used in, typically 2D or 3D. In 2D, vectors have two components, while in 3D, they have three. Similarly, matrix sizes adjust accordingly

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix}_{(T)} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}_{(S)} + \begin{bmatrix} T_X \\ T_Y \\ T_Z \end{bmatrix} + \begin{bmatrix} 0 & -R_Z & R_Y \\ R_Z & 0 & -R_X \\ -R_Y & R_X & 0 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}_{(S)} + D \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}_{(S)}$$

(T)	Target Datum	
(S)	Source Datum	
T_X, T_Y, T_Z	geocentric X/Y/Z translations	[m]
R_X, R_Y, R_Z	rotations around X/Y/Z axis	[radian]
D	correction of scale	[ppm]

Remark: the rotations R_X,R_Y,R_Z must be small

Position – 2D

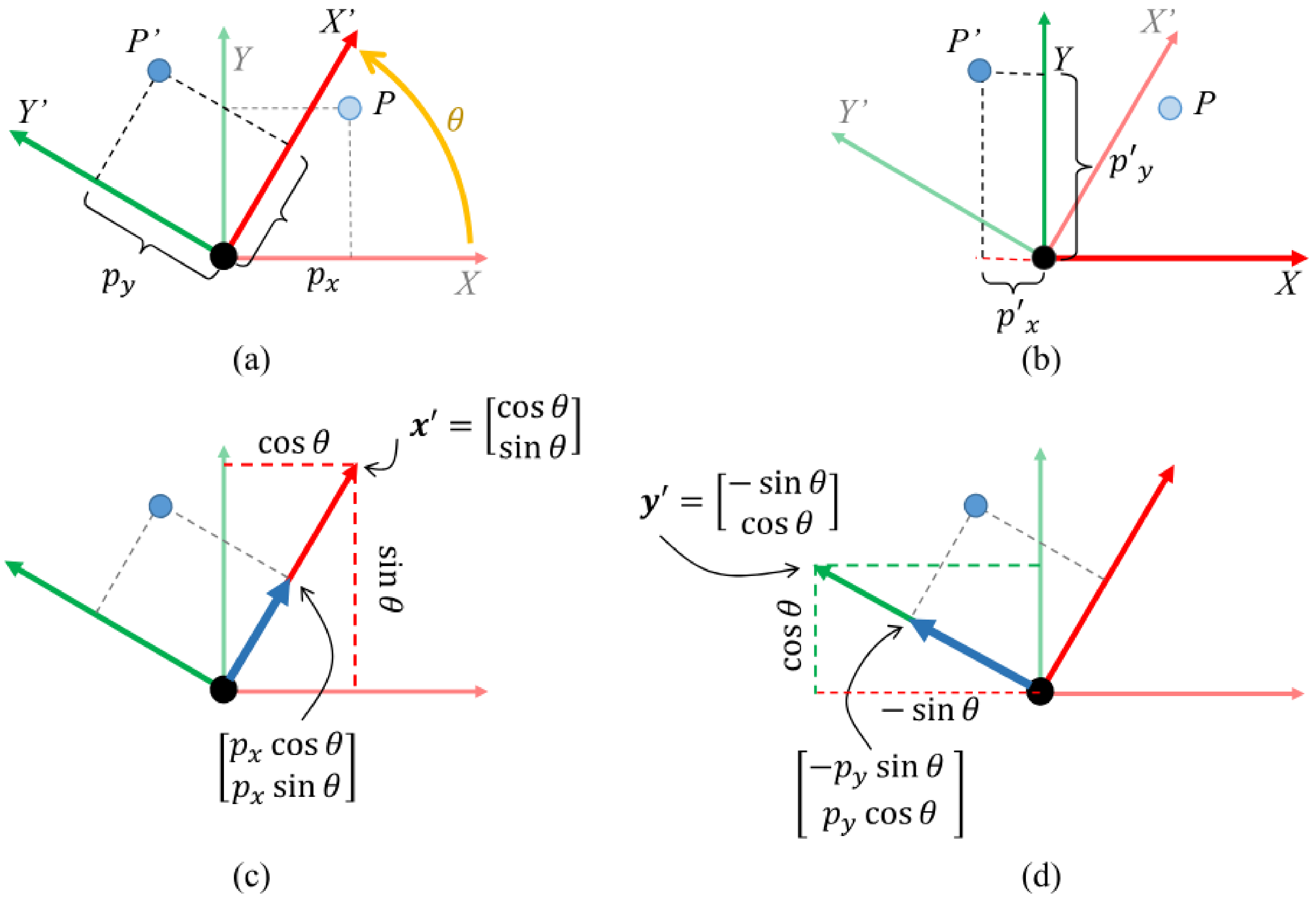


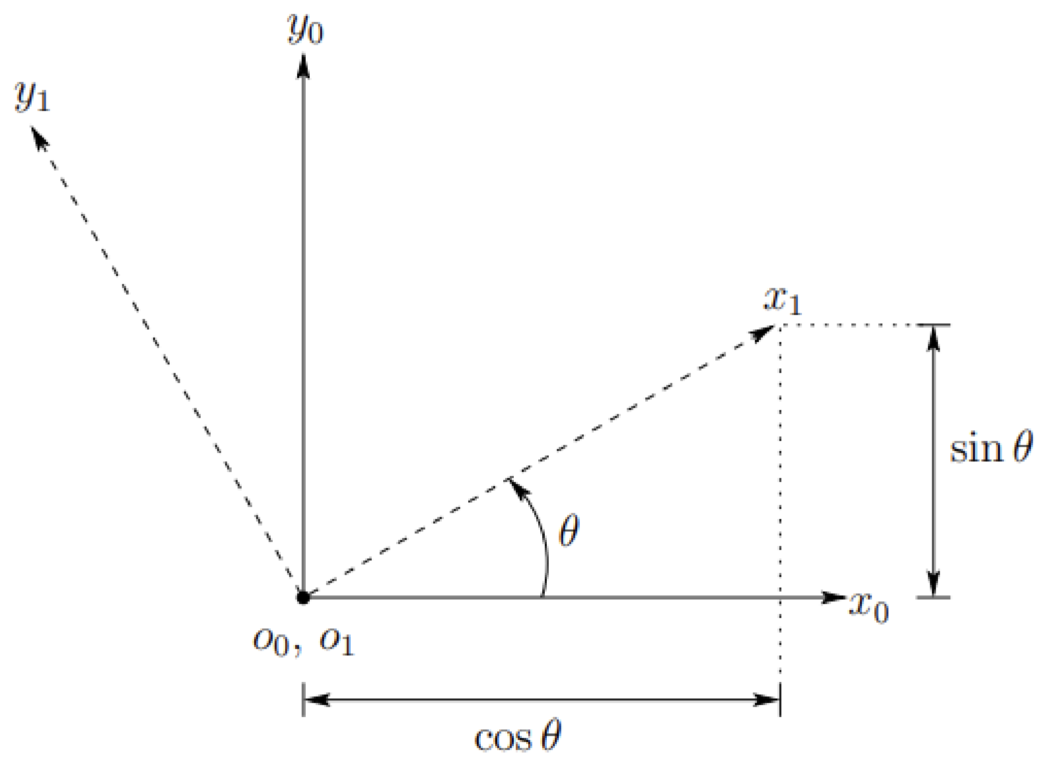
- For a 2D coordinate frame, a position is represented as a vector of two elements

$$\mathbf{p} = \begin{bmatrix} p_x \\ p_y \end{bmatrix}$$

Planar Rotations – 2D

- When describing orientations, we typically use two frames: a world or global frame or any (referenced frame), and a body frame (any other frame) attached to the object of interest.





Rotation Matrix 2D - Basics

$$\boldsymbol{R}_{frame}^{ref\ frame} = R_1^0 = \begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix}$$

$$(R_1^0)^T = (R_1^0)^{-1} = R_0^1$$

A more compact and convenient way of writing this is with a matrix equation

$$\mathbf{p}' = R(\theta)\mathbf{p}$$

with the rotation matrix given by

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

There are several useful properties of such matrices:

1. The matrix composition $R(\theta_1)R(\theta_2) = R(\theta_1 + \theta_2)$ gives the rotation matrix for the sum of the angles.
2. The determinant $\det(R(\theta)) = \cos^2\theta + \sin^2\theta = 1$ for all θ.
3. Due to the identities $\cos(-\theta) = \cos(\theta)$ and $\sin(-\theta) = -\sin(\theta)$, the operation of rotating about $-\theta$ is equivalent to a matrix transpose:

$$R(-\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = R(\theta)^T.$$

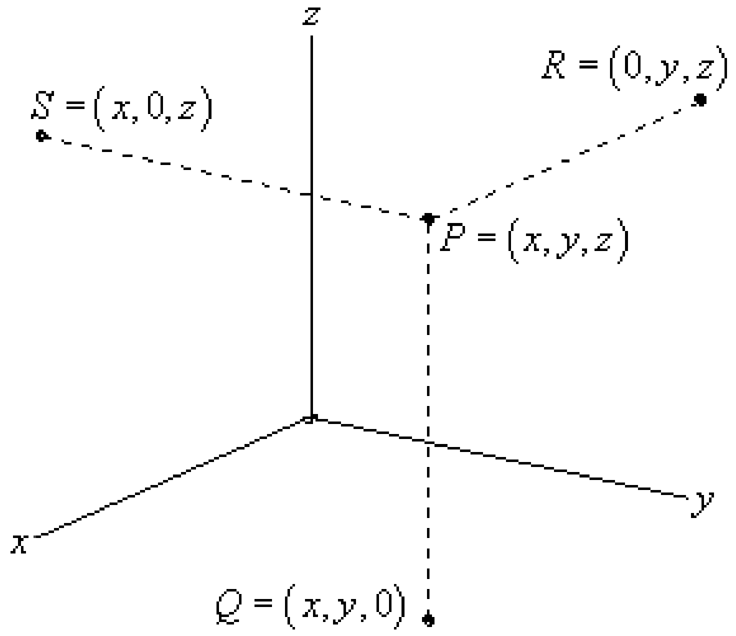
4. Moreover, the transpose is equivalent to the matrix inverse:

$$R(-\theta) = R(\theta)^T = R(\theta)^{-1}.$$

Rotation Matrix 2D – Generalized + Extra information

Position– 3D

- For a 3D coordinate frame, a position is represented as a vector of three elements



$$\boldsymbol{P}^{frame} = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}$$

"frame" – Refers to the reference frame relative to which the point is defined.

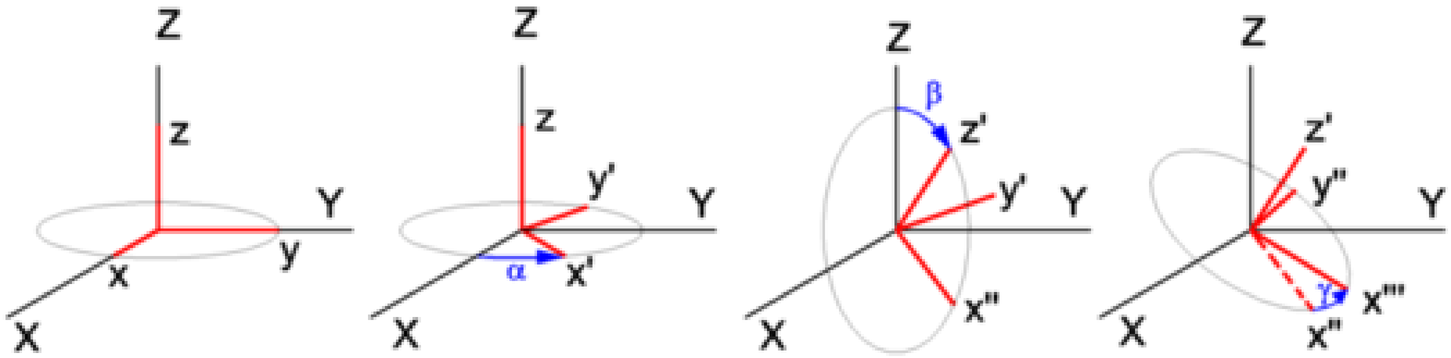
Rotational – 3D

- Rotational matrix in 3D is comprised of 3 columns which are unit vectors along 3 axes. These unit vectors are orthogonal to each other

$$R = [\hat{x} \quad \hat{y} \quad \hat{z}]$$

$$||\hat{x}|| = ||\hat{y}|| = ||\hat{z}|| = 1$$

$$\hat{x}^T \hat{y} = \hat{x}^T \hat{z} = \hat{y}^T \hat{z} = 0$$

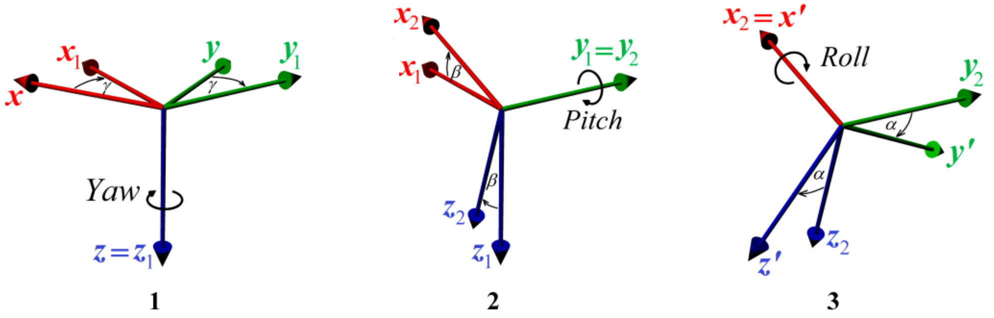


Euler Angles – Roll, Pitch Yaw

Rotational matrix can be formed from 3 sequential rotations about the axes:

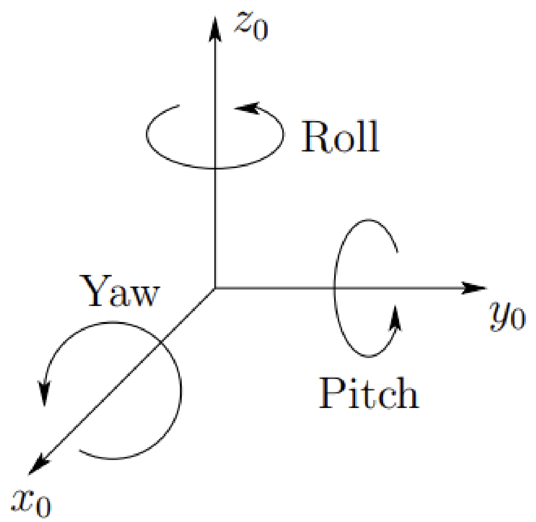
Aerospace Convention

$$\phi(Roll) - rotation\ about\ X$$
$$\theta(Pitch) - rotation\ about\ Y$$
$$\Psi(Yaw) - rotation\ about\ Z$$



Robotics Convention

$$\phi(Roll) - rotation\ about\ Z$$
$$\theta(Pitch) - rotation\ about\ Y$$
$$\Psi(Yaw) - rotation\ about\ X$$



$$
\begin{aligned}
R_{XYZ} &= R_{z,\phi} R_{y,\theta} R_{x,\psi} \\
&= \begin{bmatrix} c_\phi & -s_\phi & 0 \\ s_\phi & c_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_\theta & 0 & s_\theta \\ 0 & 1 & 0 \\ -s_\theta & 0 & c_\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\psi & -s_\psi \\ 0 & s_\psi & c_\psi \end{bmatrix} \\
&= \begin{bmatrix} c_\phi c_\theta & -s_\phi c_\psi + c_\phi s_\theta s_\psi & s_\phi s_\psi + c_\phi s_\theta c_\psi \\ s_\phi c_\theta & c_\phi c_\psi + s_\phi s_\theta s_\psi & -c_\phi s_\psi + s_\phi s_\theta c_\psi \\ -s_\theta & c_\theta s_\psi & c_\theta c_\psi \end{bmatrix}
\end{aligned}
\quad (2.39)
$$

$$
R(\phi, \theta, \psi) = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}
$$

$$\phi = atan2(r_{32}, r_{33})$$

$$\psi = atan2(r_{21}, r_{11})$$

$$
\theta = \begin{cases} atan2\left(-r_{31}, \dfrac{r_{21}}{\sin(\psi)}\right) & if\ \cos(\psi) = 0 \\ atan2\left(-r_{31}, \dfrac{r_{11}}{\cos(\psi)}\right) & otherwise \end{cases}
$$

Euler Angle: Order of rotations

1. XYZ
2. XZY
3. YXZ
4. YZX
5. ZXY
6. ZYX
7. ZXZ
8. ZYZ
9. XYX
10. XZX
11. YXY
12. YZY

VERY IMPORTANT

- Rotation can be in any sequences but not the same axis in succession.
- Total of 12 combinations of rotations.

$$R = R_3 R_2 R_1$$

Formulate Rotational Matrix depending on combinations

$$R_{3,2,1} = R_3 R_2 R_1$$

$$\begin{aligned} R_{XYZ} &= R_{z,\phi} R_{y,\theta} R_{x,\psi} \\ &= \begin{bmatrix} c_\phi & -s_\phi & 0 \\ s_\phi & c_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_\theta & 0 & s_\theta \\ 0 & 1 & 0 \\ -s_\theta & 0 & c_\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\psi & -s_\psi \\ 0 & s_\psi & c_\psi \end{bmatrix} \\ &= \begin{bmatrix} c_\phi c_\theta & -s_\phi c_\psi + c_\phi s_\theta s_\psi & s_\phi s_\psi + c_\phi s_\theta c_\psi \\ s_\phi c_\theta & c_\phi c_\psi + s_\phi s_\theta s_\psi & -c_\phi s_\psi + s_\phi s_\theta c_\psi \\ -s_\theta & c_\theta s_\psi & c_\theta c_\psi \end{bmatrix} \end{aligned} \tag{2.39}$$

Easiest way to follow:

Last rotation always goes first

$$R_{\phi,Z} = \begin{bmatrix} \cos(\phi) & -sin(\phi) & 0 \\ \sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_{\theta,Y} = \begin{bmatrix} \cos(\theta) & 0 & \cos(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \sin(\theta) \end{bmatrix}$$

$$R_{\psi,X} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\psi) & -\sin(\psi) \\ 0 & \sin(\psi) & \cos(\psi) \end{bmatrix}$$

Formulate Rotational Matrix depending on combinations - Example

$$R_{\phi,Z} = \begin{bmatrix} \cos(\phi) & -sin(\phi) & 0 \\ \sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_{\theta,Y} = \begin{bmatrix} \cos(\theta) & 0 & \cos(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \sin(\theta) \end{bmatrix}$$

$$R_{\psi,X} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\psi) & -\sin(\psi) \\ 0 & \sin(\psi) & \cos(\psi) \end{bmatrix}$$

- 1. XYZ = $R_{Z,Y,X}$ = $R_{\phi,Z}R_{\theta,Y}R_{\psi,X}$
- 2. XZY = $R_{Y,Z,X}$ = $R_{\theta,Y}R_{\phi,Z}R_{\psi,X}$
- 3. YXZ = $R_{Z,X,Y}$ = $R_{\phi,Z}R_{\psi,X}R_{\theta,Y}$
- 4. YZX = $R_{X,Z,Y}$
- 5. ZXY = $R_{Y,X,Z}$
- 6. ZYX = $R_{X,Y,Z}$
- 7. ZXZ = $R_{Z,X,Z}$
- 8. ZYZ = $R_{Z,Y,Z}$
- 9. XYX = $R_{X,Y,X}$
- 10. XZX = $R_{X,Z,X}$
- 11. YXY = $R_{Y,X,Y}$
- 12. YZY = $R_{Y,Z,Y}$

And so on....

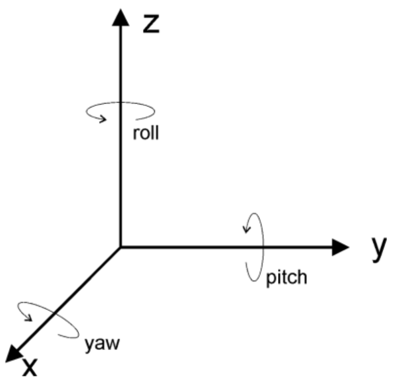
Resolving Axes Conventions

- Remember before when we showed you two different types of conventions (Robotics and Aerospace).
- The following formula shows how to transform a frame rotational matrix to another:

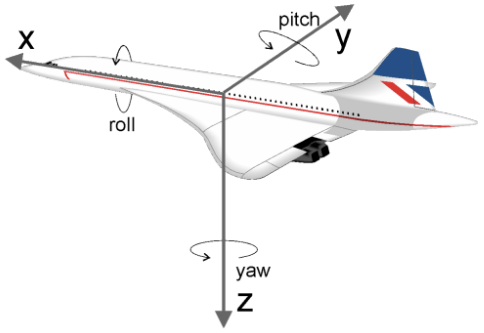
$$R_{new} = T_{old\rightarrow new} R_{old}\, T_{old\rightarrow new}^{-1}$$

- $T_{old\rightarrow new}$ is essentially the rotational matrix of robotics frame to aerospace frame or from an old frame to new frame in general. To reverse this, take the inverse.

- In many Kinematics References:



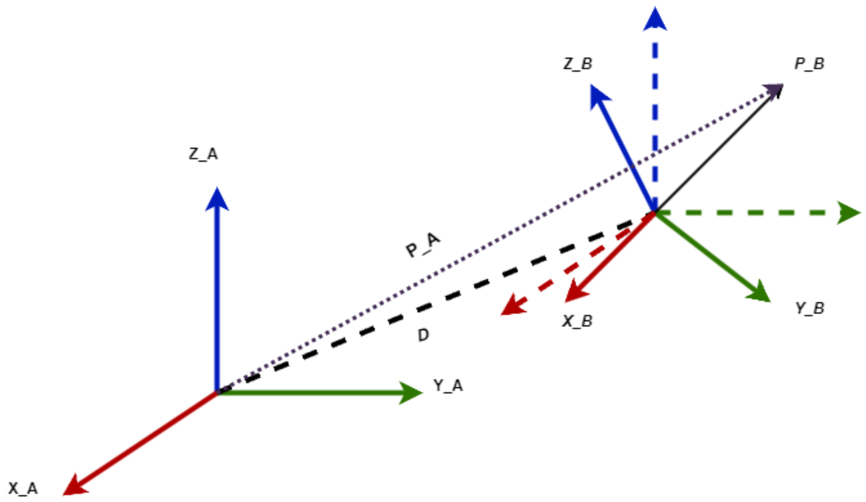
- In many Engineering Applications:



1. Translation	$\begin{bmatrix}1&0&0\\0&1&0\\t_x&t_y&1\end{bmatrix}$ or $\begin{bmatrix}1&0&t_x\\0&1&t_y\\0&0&1\end{bmatrix}$
2. Scaling	$\begin{bmatrix}S_x&0&0\\0&S_y&0\\0&0&1\end{bmatrix}$
3. Rotation (clockwise)	$\begin{bmatrix}\cos\theta&-\sin\theta&0\\\sin\theta&\cos\theta&0\\0&0&1\end{bmatrix}$
4. Rotation (anti-clock)	$\begin{bmatrix}\cos\theta&\sin\theta&0\\-\sin\theta&\cos\theta&0\\0&0&1\end{bmatrix}$
5. Reflection against X axis	$\begin{bmatrix}1&0&0\\0&-1&0\\0&0&1\end{bmatrix}$
6. Reflection against Y axis	$\begin{bmatrix}-1&0&0\\0&1&0\\0&0&1\end{bmatrix}$
7. Reflection against origin	$\begin{bmatrix}-1&0&0\\0&-1&0\\0&0&1\end{bmatrix}$
8. Reflection against line Y=X	$\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}$
9. Reflection against Y= –X	$\begin{bmatrix}0&-1&0\\-1&0&0\\0&0&1\end{bmatrix}$
10. Shearing in X direction	$\begin{bmatrix}1&0&0\\Sh_x&1&0\\0&0&1\end{bmatrix}$
11. Shearing in Y direction	$\begin{bmatrix}1&Sh_y&0\\0&1&0\\0&0&1\end{bmatrix}$
12. Shearing in both x and y direction	$\begin{bmatrix}1&Sh_y&0\\Sh_x&1&0\\0&0&1\end{bmatrix}$

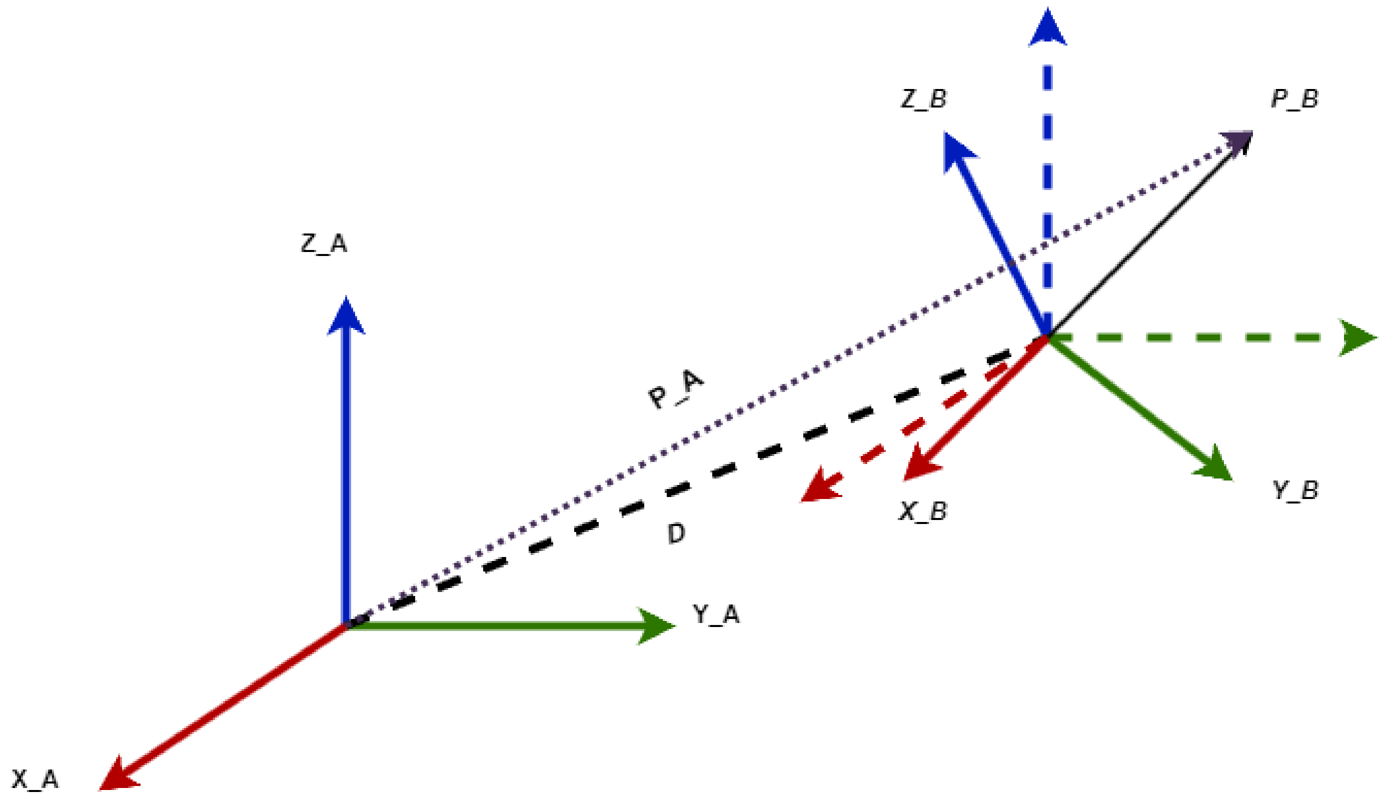
Frame Mapping

- This combines both rotational and translational transformations.
- It’s kinda cooked
- Essentially transforming point in one frame to another frame



$$H = \left[\begin{array}{c|c} R_{3\times3} & d_{3\times1} \\ \hline f_{1\times3} & s_{1\times1} \end{array}\right] = \left[\begin{array}{c|c} Rotation & Translation \\ \hline perspective & scale\ factor \end{array}\right]$$

Homogenous Transformations



- **Translation:** Displacement of a frame to referenced frame
- **Rotation:** orientation of a frame relative to referenced frame

Homogenous Transformations

- *Frame translation:*

$$D = \begin{bmatrix} d_x \\ d_y \\ d_z \end{bmatrix}$$

Remember the rotational transformation matrix

$$R(\phi,\theta,\psi) = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

$$H = \left[\begin{array}{c|c} R_{3\times3} & d_{3\times1} \\ \hline f_{1\times3} & s_{1\times1} \end{array}\right] = \left[\begin{array}{c|c} Rotation & Translation \\ \hline perspective & scale\ factor \end{array}\right]$$

Formulate transformation from frame B to A

$$H_B^A = [R_B^A \quad D] = \begin{matrix} r_{11} & r_{12} & r_{13} & d_x \\ r_{21} & r_{22} & r_{23} & d_y \\ r_{31} & r_{32} & r_{33} & d_z \\ 0 & 0 & 0 & 1 \end{matrix}$$

$$\begin{bmatrix} P_B \\ 1 \end{bmatrix} = H_B^A \begin{bmatrix} P_A \\ 1 \end{bmatrix}$$

https://modernrobotics.northwestern.edu/nu-gm-book-resource/3-3-1-homogeneous-transformation-matrices/

Homogenous Transformations

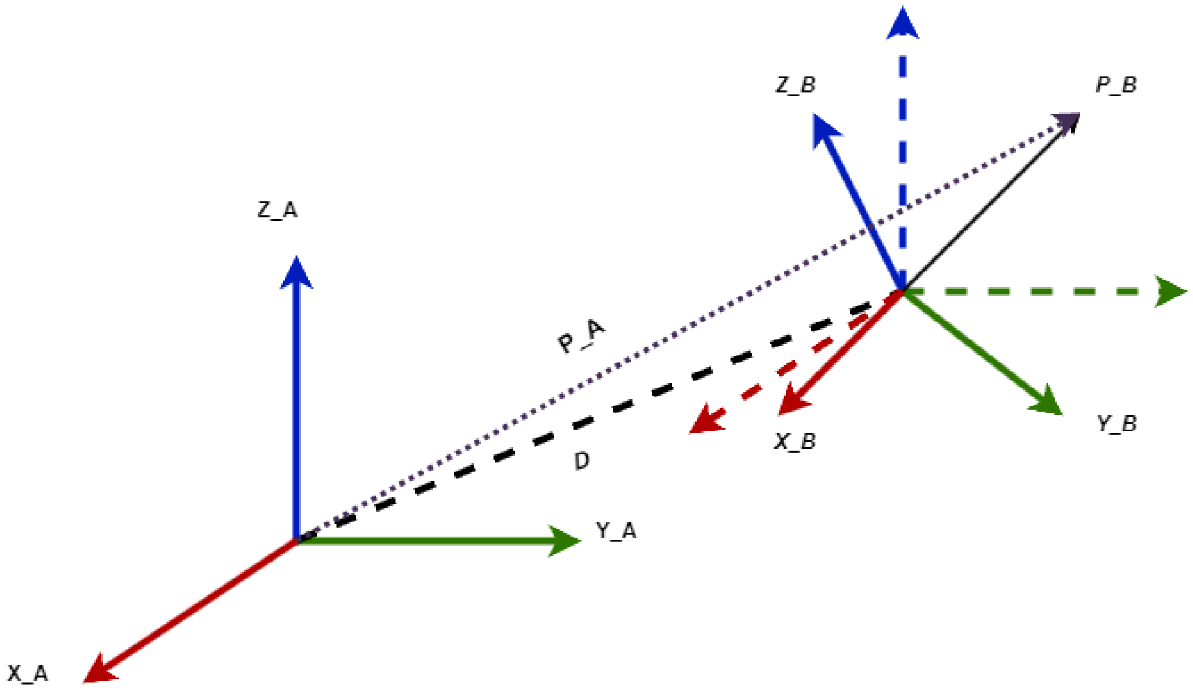
Transforming a Point in frame B to frame A

$$\begin{bmatrix} P_A \\ 1 \end{bmatrix} = H_B^A \begin{bmatrix} P_B \\ 1 \end{bmatrix}$$

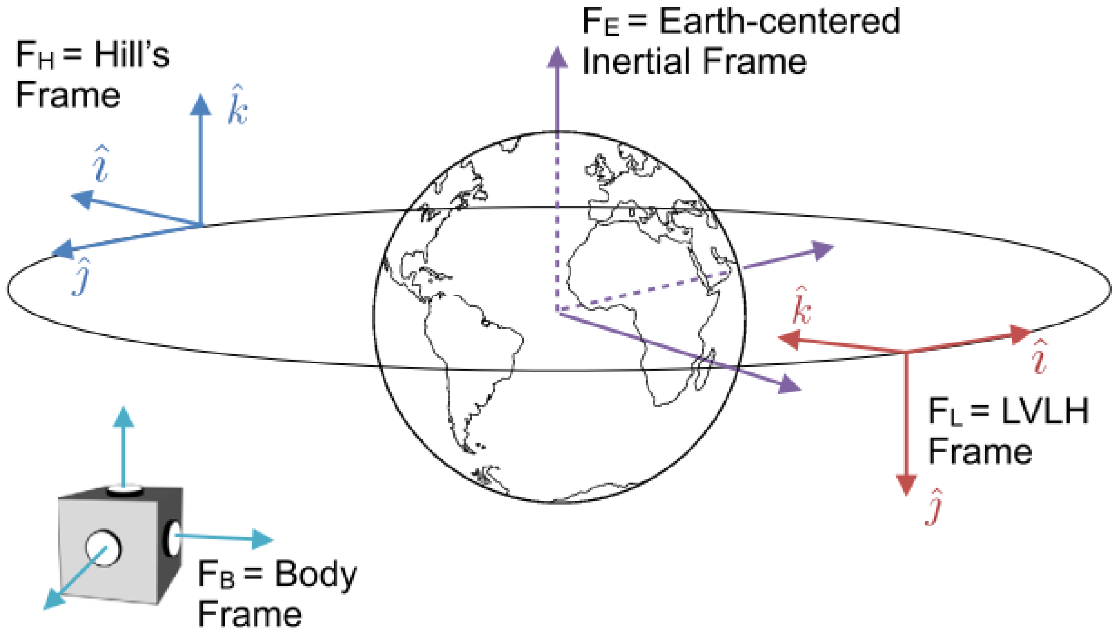
$$\begin{bmatrix} x_A \\ y_A \\ z_A \\ 1 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & d_x \\ r_{21} & r_{22} & r_{23} & d_y \\ r_{31} & r_{32} & r_{33} & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_B \\ y_B \\ z_B \\ 1 \end{bmatrix}$$

Transforming a Point in frame A to frame B

$$\begin{bmatrix} P_B \\ 1 \end{bmatrix} = H_A^B \begin{bmatrix} P_A \\ 1 \end{bmatrix} = \begin{bmatrix} \left(R_B^A\right)^T & -\left(R_B^A\right)^T D \\ 0 \quad 0 \quad 0 & 1 \end{bmatrix} \begin{bmatrix} P_A \\ 1 \end{bmatrix}$$



Can you get a trajectory from the data ?



Resources

- https://motion.cs.illinois.edu/RoboticSystems/CoordinateTransformations.html
- Robot Modelling and Control 1st Edition - Mark W. Spong, Seth Hutchinson, and M. Vidyasagar