An Accelerated Macroscopic Forcing Method (MFM) for Determining Eddy Viscosity Operators

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Introduction

MFM is a rheometer-like method for numerically determining closure operators in turbulent flows [1]. We can use MFM to determine the generalized eddy viscosity [2]:

$$-\overline{u_j' u_i'}(\mathbf{x}) = \int D_{jilk}(\mathbf{x},\mathbf{y}) \left.\frac{\partial U_k}{\partial x_l}\right|_{\mathbf{y}} \mathbf{dy}$$

We do this by specifying certain forcings s to the Generalized Momentum Transport (GMT) equation:

$$\frac{\partial v_i}{\partial t} + \frac{\partial}{\partial x_j} u_j v_i = -\frac{\partial q}{\partial x_i} + \frac{1}{Re}\frac{\partial^2 v_i}{\partial x_j \partial x_j} + s_i$$

We can approximate the GMT closure using the leading order moment:

$$\int D_{jilk}(\mathbf{x},\mathbf{y}) \left.\frac{\partial V_k}{\partial x_l}\right|_{\mathbf{y}} \mathbf{dy} \approx D^0_{jilk}(\mathbf{x}) \frac{\partial V_k}{\partial x_l}$$

$$D^0_{jilk}(\mathbf{x}) = \int D_{jilk}(\mathbf{x},\mathbf{y}) \mathbf{dy}$$

In MFM, we choose s for the GMT eq. to achieve a certain $\frac{\partial V_k}{\partial x_l}$ that allows us to retrieve D^0_{jilk} by measuring $\overline{u_j' v_i'}$. Computing all 81 D^0_{jilk} requires 9 high-fidelity DNS (1 per $\frac{\partial V_k}{\partial x_l}$).

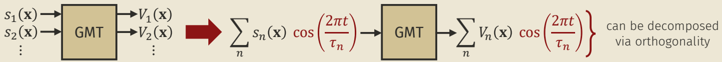
In this work, **we combine 9 DNS computing D^0_{jilk} in channel flow into just 1 DNS**, reducing the cost of MFM.

Accelerated MFM

In standard MFM, we solve the GMT equation in several separate DNS for different $s_n(\mathbf{x})$ with responses $V_n(\mathbf{x})$.

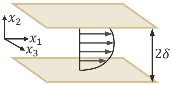
Instead, let us choose $s(\mathbf{x},t) = s_n(\mathbf{x}) \cos\left(\frac{2\pi t}{\tau_n}\right)$ and τ_n s.t. $\tau_{response} \ll \tau_n \le \tau_{sim}$, where $\tau_{response}$ is the timescale of the flow and τ_{sim} is the simulation time. At this τ_n, we make the statistically quasi-steady assumption. The response becomes $V_n(\mathbf{x}) \cos\left(\frac{2\pi t}{\tau_n}\right)$.

Choose $\tau_n = \frac{2\tau_{sim}}{n}$ to generate temporally orthogonal response modes. Assign modes to several s_n and superpose. Decompose responses in postprocessing.

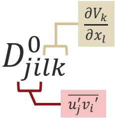


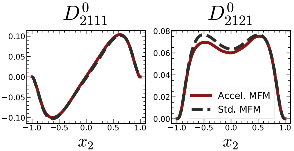
Model Problem: Channel Flow Leading Order Eddy Viscosity

Simulation Setup:



- $Re_\tau = \frac{u_\tau \delta}{\nu} = 180$
- $\tau_{response} = 17 \frac{\delta}{u_\tau}$
- $\tau_{sim} = 500 \frac{\delta}{u_\tau}$
- Fastest mode: $\tau = 125 \frac{\delta}{u_\tau}$





- **D^0_{jilk} measured using accelerated MFM agree w/ those measured using standard MFM** (2 examples shown above)
- **MFM cost reduction by a factor of 9**

Conclusion

- By superposing DNS, we reduce cost of MFM by an order of magnitude
- Future work: apply accelerated MFM to higher order moments & nonlocal eddy viscosity

Acknowledgements

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References

[1] A. Mani and D. Park. (2019). arXiv:1905.08342.
[2] F. Hamba. (2005). Physics of Fluids, 17(11), 115102. doi:10.1063/1.2130749.

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Macroscopic Forcing Method (MFM)

MFM[1] is a non-intrusive, rheometer-like method for numerically determining closure operators in turbulent flows. We can use MFM to determine the **generalized eddy viscosity**[2]

$$-\overline{u_j' u_i'}(\mathbf{x}) = \int D_{jilk}(\mathbf{x}, \mathbf{y}) \left.\frac{\partial U_k}{\partial x_l}\right|_{\mathbf{y}} \mathbf{dy},$$

which we use to model the RANS closure operator. The eddy viscosity kernel D_{jilk} is anisotropic and nonlocal.

[1] Mani and Park 2019
[2] Hamba 2005

Generalized Momentum Transport Equation

We determine D_{jilk} by specifying certain forcings s to the **Generalized Momentum Transport (GMT) equation**,

$$\frac{\partial v_i}{\partial t} + \frac{\partial}{\partial x_j} u_j v_i = -\frac{\partial q}{\partial x_i} + \frac{1}{\mathrm{Re}} \frac{\partial^2 v_i}{\partial x_j \partial x_j} + s_i,$$

whose closure operator is equal to the RANS closure operator.

We use the GMT equation instead of the Navier-Stokes equation, because the GMT equation is linear, and MFM can only be done with linear differential equations.

Leading Order Eddy Viscosity Tensor

D_{jilk} is *very* computationally expensive to determine. Instead, we can approximate the generalized eddy viscosity for the GMT equation using the **leading order moment** D^0_{jilk}:

$$-\overline{u'_j v'_i}(\mathbf{x}) = \int D_{jilk}(\mathbf{x}, \mathbf{y}) \left.\frac{\partial V_k}{\partial x_l}\right|_{\mathbf{y}} \mathbf{dy} \approx D^0_{jilk}(\mathbf{x}) \frac{\partial V_k}{\partial x_l}$$

$$D^0_{jilk}(\mathbf{x}) = \int D_{jilk}(\mathbf{x}, \mathbf{y}) \mathbf{dy}$$

D^0_{jilk} is the anisotropic eddy viscosity tensor and consists of 81 coefficients.

Leading Order Eddy Viscosity Tensor Example

For example, to compute D^0_{2121}, choose s in the GMT equation such that

$$V_1 = x_2, \quad V_2 = V_3 = 0 \Rightarrow \frac{\partial V_k}{\partial x_l} = \begin{cases} 1 & k = 1, \quad l = 2 \\ 0 & \text{otherwise} \end{cases}$$

In postprocessing, we obtain D^0_{2121} by measuring the Reynolds stress:

$$-\overline{u_2' v_1'}(\mathbf{x}) = \int D_{2121}(\mathbf{x}, \mathbf{y}) \frac{\partial V_1}{\partial x_2} \mathbf{dy} = D^0_{2121}(\mathbf{x})$$

Determining D^0_{jilk} is Still Expensive!

All other D^0_{ji21} can be obtained by measuring each of the Reynolds stresses $\overline{u'_j v'_i}$. This gives 9 coefficients of the eddy viscosity tensor.

Do this for all 9 average velocity gradients to obtain all 81 coefficients:

$$\begin{aligned} V_1 &= x_1, \quad V_2 = V_3 = 0 \Rightarrow D^0_{ji11} \\ V_2 &= x_1, \quad V_1 = V_3 = 0 \Rightarrow D^0_{ji12} \\ &\vdots \\ V_3 &= x_3, \quad V_1 = V_2 = 0 \Rightarrow D^0_{ji33} \end{aligned}$$

Thus, 9 high-fidelity DNS are needed to determine all D^0_{jilk}.

$\Rightarrow$ **Goal:** reduce cost of MFM by combining these 9 DNS into just one DNS.

Temporal Modulation

In standard MFM, we solve the GMT equation in several separate DNS for different $s_n(\mathbf{x})$ with responses $V_n(\mathbf{x})$.

Instead, let us choose $s(\mathbf{x}, t) = s_n(\mathbf{x}) \cos\left(\frac{2\pi}{\tau_n} t\right)$ and τ_n such that

$$\tau_{\text{response}} << \tau_n \leq \tau_{\text{sim}},$$

where τ_{response} is the timescale of the flow (i.e., timescale for flow to be fully developed) and τ_{sim} is the simulation time. At this τ_n, we make the **statistically quasi-steady assumption**, so the response becomes $V(\mathbf{x}, t) = V_n(\mathbf{x}) \cos\left(\frac{2\pi}{\tau_n} t\right)$.

Orthogonal Response Modes

Choosing $\tau_n = \frac{2\tau_{\text{sim}}}{n}$ generates **temporally orthogonal response modes**. We can assign each of these temporal modes to a spatial forcing and superpose. By leveraging orthogonality, we can decompose the responses in postprocessing.

Standard MFM:

$$\begin{matrix} s_1(\mathbf{x}) \rightarrow \\ s_2(\mathbf{x}) \rightarrow \\ \vdots \end{matrix} \boxed{\text{GMT}} \begin{matrix} \rightarrow V_1(\mathbf{x}) \\ \rightarrow V_2(\mathbf{x}) \\ \vdots \end{matrix}$$

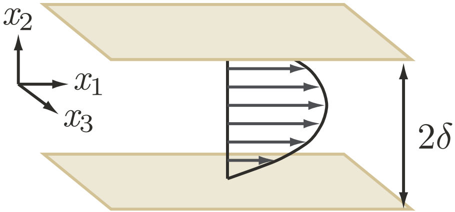
$$\boxed{\frac{\partial v_i}{\partial t} + \frac{\partial}{\partial x_j} u_j v_i = -\frac{\partial q}{\partial x_i} + \frac{1}{\text{Re}} \frac{\partial^2 v_i}{\partial x_j \partial x_j} + s_i}$$

Accelerated MFM:

$$\sum_n s_n(\mathbf{x}) \cos\left(\frac{2\pi}{\tau_n} t\right) \rightarrow \boxed{\text{GMT}} \rightarrow \sum_n V_n(\mathbf{x}) \cos\left(\frac{2\pi}{\tau_n} t\right)$$

Model Problem: Channel Flow

We demonstrate accelerated MFM with channel flow, combining 9 DNS calculating all D^0_{jilk} into just 1 DNS.



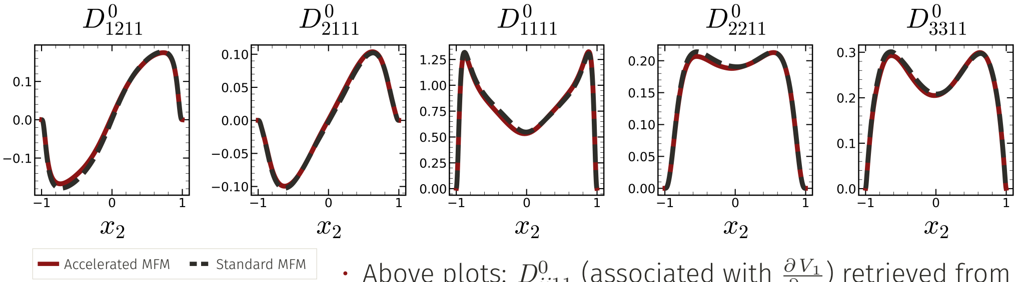
Simulation parameters:

- $Re_\tau = \frac{u_\tau \delta}{\nu} = 180$
- $L_x \times L_y \times L_z = 2\pi \times 2 \times \pi$
- $N_x \times N_y \times N_z = 144 \times 144 \times 144$

Timescales:

- $\tau_{\text{response}} \approx 17 \frac{\delta}{u_\tau}$ $\rightarrow$ 1/2 channel height $\rightarrow$ friction velocity
- $\tau_{\text{sim}} = 500 \frac{\delta}{u_\tau}$
- Fastest mode: $\tau = 125 \frac{\delta}{u_\tau}$

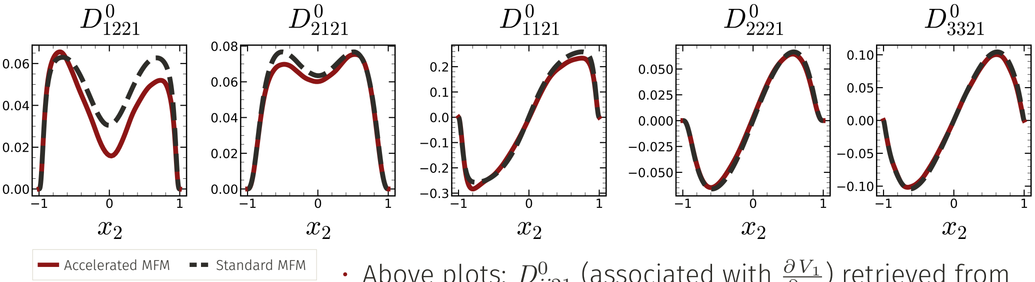
Results

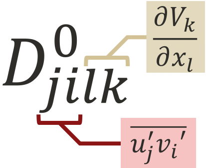


$$D^{0}_{\underbrace{ji}_{\overline{u_j'v_i'}}\overbrace{lk}^{\frac{\partial V_k}{\partial x_l}}}$$

- Above plots: D^0_{ji11} (associated with $\frac{\partial V_1}{\partial x_1}$) retrieved from decomposed responses
- D^0_{jilk} **measured using accelerated MFM agree with those measured using standard MFM in separate simulations** ✓

Results





- Above plots: D^0_{ji21} (associated with $\frac{\partial V_1}{\partial x_2}$) retrieved from decomposed responses
- Noticeable errors for components of small magnitudes (D^0_{1221}, D^0_{2121})
  - Noise from other modes contaminate results
- Ultimately, **cost of MFM reduced by a factor of 9**

Conclusion

- Presented an **accelerated MFM** for determining closure operators
  - Cost of MFM for full characterization of eddy viscosity operator is reduced by an order of magnitude
  - Showed that, under **statistically quasi-steady** conditions, temporally orthogonal response modes can be assigned to different spatial forcings and superposed
  - Through **orthogonality**, responses can be separated

$$\sum_n s_n(\mathbf{x}) \cos\left(\frac{2\pi}{\tau_n} t\right) \rightarrow \boxed{\frac{\partial v_i}{\partial t} + \frac{\partial}{\partial x_j} u_j v_i = -\frac{\partial q}{\partial x_i} + \frac{1}{\mathrm{Re}} \frac{\partial^2 v_i}{\partial x_j \partial x_j} + s_i} \rightarrow \sum_n V_n(\mathbf{x}) \cos\left(\frac{2\pi}{\tau_n} t\right)$$

- Demonstrated method w/ turbulent channel flow
  - **Combined 9 DNS computing D^0_{jilk} into just 1 DNS**, greatly reducing cost of MFM
- Future work: apply accelerated MFM to higher-order moments & nonlocal eddy viscosity

*This work is supported by Boeing and the Charles H. Kruger Stanford Graduate Fellowship.

References

[1] A. Mani and D. Park. Macroscopic forcing method: a tool for turbulence modeling and analysis of closures. 2019. arXiv: `1905.08342 [physics.flu-dyn]`.

[2] F. Hamba. Nonlocal analysis of the reynolds stress in turbulent shear flow. *Physics of Fluids*, 17(11):115102, 2005. DOI: `10.1063/1.2130749`.