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Stationary Response Probability Distribution of SDOF Nonlinear Stochastic Systems

There has been no significant progress in developing new techniques for obtaining exact stationary probability density functions (PDFs) of nonlinear stochastic systems since the development of the method of generalized probability potential in 1990s. In this paper, a general technique is proposed for constructing approximate stationary PDF solutions of single degree of freedom (SDOF) nonlinear systems under external and parametric Gaussian white noise excitations. This technique consists of two novel components. The first one is the introduction of new trial solutions for the reduced Fokker–Planck–Kolmogorov (FPK) equation. The second one is the iterative method of weighted residuals to determine the unknown parameters in the trial solution. Numerical results of four challenging examples show that the proposed technique will converge to the exact solutions if they exist, or a highly accurate solution with a relatively low computational effort. Furthermore, the proposed technique can be extended to multi degree of freedom (MDOF) systems. [DOI: 10.1115/1.4036307]

Keywords: stationary probability density function, weighted residue method, iterative, nonlinear system, FPK equation

1 Introduction

The nonlinear stochastic dynamics has been of great interests to engineering and science research communities. For example, high-rise buildings and long-span bridges can have severe nonlinear random vibrations due to seismic loads or gusty winds. Vehicles traveling on highway with rough road surface also experience random vibrations. Scientists have known for a long time that noise could play a positive role in the nonlinear dynamics governing chemical and biological processes. The stochastic resonance is among the most typical examples. The statistic properties of the response of nonlinear stochastic systems are fully described by the probability density function governed by the FPK equation. To obtain analytical solutions of the FPK equation has been a challenge. There has been no significant progress in developing new techniques for obtaining exact stationary probability density functions (PDFs) of nonlinear stochastic systems since the development of the method of generalized probability potential in 1990s. In this paper, a general technique is proposed for constructing approximate stationary PDF solutions of single degree of freedom (SDOF) nonlinear systems under external and parametric Gaussian white noise excitations.

The most general procedures developed by Yong and Lin [1] and Lin and Cai [2] are the detailed balance principle and the method of generalized stationary potential for obtaining analytical solutions of nonlinear systems under parametric and/or external white noise excitations. Zhu et al. [3–5] obtained five classes of exact stationary solutions of MDOF Hamiltonian systems with the property of nonequipartition energy. While these solutions represent significant contributions on this topic, they fall into the category of the generalized probability potential. However, the solvability conditions for the existence of exact solutions are often not satisfied. This paper presents a novel method for constructing analytical solutions of the reduced FPK equation, which are highly accurate and can converge to the exact solutions when the solvability conditions are satisfied.

Many other numerical and approximate methods for the analysis of nonlinear stochastic systems have been developed in the literature. Excellent reviews on the methods can be found in Refs. [5–13]. Among the existing methods, the equivalent linearization method [14,15], equivalent nonlinear system method [16–19], Gaussian closure procedure [20], non-Gaussian closure procedure [21,22], stochastic averaging method [23–26], Monte Carlo simulation [27,28], path integration [29–31], and generalized cell mapping technique [32] are most frequently used.

Recently, the present authors proposed an iterative method of weighted residuals to solve the reduced FPK equation for SDOF strongly nonlinear systems [33]. The solution of the reduced FPK equation is assumed to be in the form of an exponential polynomial with a logarithmic term to account for parametric excitations. Such an assumed solution together with the iterative procedure can lead to the exact stationary PDF of some systems whose exact solution exists. On the other hand, the assumed solution still has its limitations. We have found that the assumed solution may not be appropriate for some nonlinear stochastic systems. Hence, a more general technique is needed to assist us in constructing appropriate assumed solutions for a wider class of nonlinear stochastic systems.

In the present paper, such a general technique for the stationary PDF of nonlinear SDOF systems subject to external and parametric Gaussian white noise excitations is developed. In particular, we make use of the concept of the probability flow when constructing the assumed solutions. The assumed solution has three parts. The first part is associated with the circulatory probability flow, which is determined by the energy of the conservative part of the system. The second part is related to the potential probability flow, which satisfies the compatibility conditions. The third part is in the form of the exponential polynomial of the state variables. The iterative method of weighted residuals is then employed to compute the undetermined parameters in the assumed solution.

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Contributed by the Applied Mechanics Division of ASME for publication in the Journal of Applied Mechanics. Manuscript received November 26, 2016; final manuscript received March 17, 2017; published online April 5, 2017. Assoc. Editor: Daining Fang.

The rest of the paper is outlined as follows. Section 2 reviews the method of detailed balance briefly, which provides the mathematical basis for the circulatory and potential probability flows. Section 3 presents the new technique to construct analytical solutions of the reduced FPK equation. A versatile trial solution of the reduced FPK equation of nonlinear stochastic systems is first proposed, and then the iterative method of weighted residuals is introduced to find the undetermined parameters of the trial solution. In Sec. 4, four examples of varying degree of complexities are studied to demonstrate the proposed method. The accuracy of the solutions is examined with the available exact solutions or the results from Monte Carlo simulations. Section 5 concludes the paper.

2 Detailed Balance

Consider a SDOF nonlinear oscillator under external and parametric Gaussian white noise excitations in the state space form as

$$\begin{aligned}\frac{dX}{dt} &= Y \\ \frac{dY}{dt} &= -g_1(X,Y)Y - g_2(X) + \sum_{i=1}^{l}[h_i(X,Y)]W_i(t)\end{aligned} \tag{1}$$

where $g_1(X, Y)Y$ denotes the linear or nonlinear damping, $g_2(X)$ represents restoring forces, $h_i(X, Y)$ are linear or nonlinear functions of X and Y, and $W_i(t)$ are the Gaussian white noises with zero mean and the correlation matrix $E[W_i(t)W_j(t-\tau)] = 2D_{ij}\delta(\tau)\,(i,j = 1,2,\ldots,l)$.

The reduced FPK equation of the system reads

$$0 = -\frac{\partial}{\partial x}[pm_1] + \frac{\partial}{\partial y}[pm_2] + \frac{1}{2}\frac{\partial^2}{\partial y^2}[pb_{22}] \tag{2}$$

where the stationary PDF $p = p(x, y)$, the drift and diffusion terms m_1, m_2, and b_{22} are given by

$$\begin{aligned} m_1(x,y) &= y \\ m_2(x,y) &= g_1(x,y)y + g_2(x) - \sum_{i=1}^{l}\sum_{j=1}^{l}\left[D_{ij}\frac{\partial h_i(x,y)}{\partial y}h_j(x,y)\right] \\ b_{22}(x,y) &= \sum_{i=1}^{l}\sum_{j=1}^{l}\left[2D_{ij}h_i(x,y)h_j(x,y)\right]\end{aligned} \tag{3}$$

It is usually difficult to obtain the exact solution of the reduced FPK equation (2). In the late 1980s, Lin and coworkers [1,2] have developed the detailed balance procedure to obtain the exact stationary solutions for the class of nonlinear systems under parametric and/or external white noise excitations. According to the procedure, Eq. (2) is separated into two parts

$$-y\frac{\partial p}{\partial x} + g_2(x)\frac{\partial p}{\partial y} = 0 \tag{4}$$

and

$$\frac{\partial}{\partial y}\left\{p[m_2 - g_2(x)]\right\} + \frac{1}{2}\frac{\partial^2}{\partial y^2}[pb_{22}] = 0 \tag{5}$$

Equation (4) describes the equilibrium of the circulatory probability flow, and Eq. (5) suggests vanishing the potential probability flow. Since p and its first derivatives vanish as $|x| + |y| \to \infty$ for stable systems, Eq. (5) implies

$$p[m_2 - g_2(x)] + \frac{1}{2}\frac{\partial}{\partial y}[pb_{22}] = 0 \tag{6}$$

The exact stationary solution, if it exists, must satisfy both Eqs. (4) and (6). However, the solvability condition of the detailed balance for the reduced FPK equation is usually not satisfied, indicating the nonexistence of an exact solution. In Sec. 3, we will propose a novel method for constructing approximate PDF solutions of the reduced FPK equation.

3 The Proposed Method

The method for constructing approximate PDF solutions of the reduced FPK equation of nonlinear stochastic systems consists of two steps. The first step involves the construction of a novel trial solution for the reduced FPK equation. The second step computes the unknown parameters in the trial solution by means of iterative applications of the method of weighted residuals.

3.1 Trial Solution.

It is assumed that the solution of the reduced FPK equation (2) is of the form

$$\bar{p}(x,y) = C_0\exp[-\varphi(x,y)] \tag{7}$$

where C_0 is a normalization constant. $\varphi(x,y)$ is the probability potential assumed to be in the following form:

$$\varphi(x,y) = \sum_{i,j\geq 0}\sum_{0<i+j\leq n} c_{ij}x^i y^j + k_1\psi_1(x,y) + k_2\psi_2(x,y) \tag{8}$$

where c_{ij}, k_1 and k_2 are the undetermined parameters, ψ_1 and ψ_2 are the circulatory probability flow and potential probability flow, respectively, which can be obtained from Eqs. (4) and (6). They are given by

$$\begin{aligned}\psi_1 &= \frac{y^2}{2} + \int_0^x g_2 dx \\ \psi_2 &= \int_0^y \frac{2m_2(x,y) - 2g_2(x)}{b_{22}(x,y)}dy + \ln b_{22}(x,y)\end{aligned} \tag{9}$$

The conditions for the existence of $\bar{p}(x,y)$ are stated as

$$\begin{aligned}&\varphi(x,y) \to \infty, \quad \text{as} \quad r \to \infty \\ &\alpha > -1, \text{ if } \varphi(x,y) \propto r^\alpha \quad \text{as} \quad r \to 0\end{aligned} \tag{10}$$

where $x = r\cos\theta$ and $y = r\sin\theta$. The second condition guarantees the integrability of $\bar{p}(x,y)$ at the origin.

Note that the equilibrium of the circulatory probability flow determined by Eq. (4) is related to the conservative part of the system (1). The potential probability flow determined by Eq. (6) describes the balance of energy input and dissipation. Therefore, ψ_1 and ψ_2 have already captured the main characteristics of the nonlinear system. Furthermore, when the solvability conditions of the method of detailed balance are met, ψ_1 and ψ_2 become compatible solutions and the polynomial part of the solution in Eq. (8) vanishes. Then, the trial solution in Eq. (7) contains the solutions that are obtainable with the method of detailed balance [1]. When the solvability conditions of the detailed balance are not met, ψ_1 and ψ_2 are not compatible. The polynomials of the assumed solution in Eq. (8) compensate the incompatibility of ψ_1 and ψ_2. In this case, the order of the polynomials may need to be sufficiently large. However, our experience indicates that a limited order of the polynomials may still provide a high precision of the approximate solution since the terms ψ_1 and ψ_2 have already captured the main characteristics of the nonlinear system. When ψ_1 and ψ_2 are also in the form of polynomials, these two terms are automatically merged to the polynomial part of the solution in Eq. (8). Then, the trial solution (8) is reduced to the one of the exponential polynomial method due to Er [34,35]. In the following, we shall show that the proposed trial functions are highly versatile and can accurately describe the stationary probability distributions of a wide range of nonlinear stochastic systems with external and parametric excitations.

3.2 Iterative Method of Weighted Residuals.

Substituting $\bar{p}(x,y)$ for $p(x, y)$ in Eq. (2) leads to a residual error, $r(x,y,c_{ij},k_1,k_2)\bar{p}(x,y)$ where

$$r(x,y,c_{ij},k_1,k_2) = -m_1\frac{\partial\varphi}{\partial x} + \frac{\partial m_2}{\partial y} - m_2\frac{\partial\varphi}{\partial y} + \frac{b_{22}}{2}\left[\left(\frac{\partial\varphi}{\partial y}\right)^2 - \frac{\partial^2\varphi}{\partial y^2}\right] - \frac{\partial\varphi}{\partial y}\frac{\partial b_{22}}{\partial y} + \frac{1}{2}\frac{\partial^2 b_{22}}{\partial y^2} \tag{11}$$

$r(x,y,c_{ij},k_1,k_2)$ is used as a measure of error of the approximate solution $\bar{p}(x,y)$ from the unknown true solution $p(x, y)$. A weak solution for $\bar{p}(x,y)$ can be obtained in the sense that the weighted averages of the residual error are zero. This leads to a set of algebraic equations as

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} M_l(x,y)r(x,y,c_{ij},k_1,k_2)dxdy = 0 \tag{12}$$

where $M_l(x,y)$ $(l=1,2,\ldots,N+2)$ are a set of weighting functions. N is the number of polynomial terms x^iy^j such that $0<i+j\le n$. This is known as the method of weighted residuals.

It should be noted that the success of the method of weighted residuals depends highly on the choice of the weighting function. Let us select a set of weighting functions in the following form:

$$\begin{aligned} M_k(x,y) &= p_m(x,y)x^iy^j, \quad k=1,2,\ldots N \\ M_{N+1}(x,y) &= p_m(x,y)\psi_1(x,y) \\ M_{N+2}(x,y) &= p_m(x,y)\psi_2(x,y) \end{aligned} \tag{13}$$

where $p_m(x, y)$ can be initially chosen, for example, as a Gaussian PDF or an approximate PDF of the reduced FPK obtained with the equivalent linearization technique or the stochastic averaging method. In this work, we choose the following function to start the solution process:

$$p_m(x,y) = \exp[-\psi_1(x,y) - \psi_2(x,y)] \tag{14}$$

By substituting the weighting functions in Eq. (13) into Eq. (12), we obtain a set of nonlinear algebraic equations. Solving these equations numerically, we may obtain multiple sets of solutions for the parameters c_{ij}, k_1 and k_2. The parameters such that the resulting PDF $\bar{p}(x,y)$ satisfies the existence conditions (10) are accepted. Otherwise, they will be discarded.

Note that the application of the method of weighted residuals just once may not obtain sufficiently accurate solution of the PDF $\bar{p}(x,y)$. A very effective way to improve the accuracy of the solution is to introduce an iterative procedure [33,36]. Let $\bar{p}^{(k)}$ be the approximation of the steady-state PDF after applying the method of weighted residuals k times. We use $\bar{p}^{(k)}$ in place of p_m in Eq. (13) for all $k>0$ to compute the next approximation of the PDF denoted as $\bar{p}^{(k+1)}$ with the method of weighted residuals. We repeat these steps until the following convergence criterion is satisfied:

$$\varepsilon = \sqrt{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left(\bar{p}^{(k)} - \bar{p}^{(k+1)}\right)^2 dxdy} \le \varepsilon_0 \tag{15}$$

where ε_0 is a preset tolerance. In every iteration, the existence condition (10) must be satisfied. Numerical computations indicate that the convergence of the iterative procedure is quite fast, although a rigorous proof is still elusive at this time. Note that other convergence criteria can also be considered. For example, the following residual error can be considered:

$$\varepsilon = \sqrt{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left(r\bar{p}^{(k)}\right)^2 dxdy} \le \varepsilon_0 \tag{16}$$

In order to evaluate the accuracy of the converged solution, we define the error of the computed PDF with respect to a reference solution denoted as $p_R(x, y)$

$$\Delta p_k = \sqrt{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left(\bar{p}^{(k)} - p_R\right)^2 dxdy} \tag{17}$$

The reference solution can be the available exact steady-state PDF or the steady-state PDF from Monte Carlo simulations.

When the trial solution $\bar{p}(x,y)$ and the part of weighting function $p_m(x, y)$ are far away from the true solution, the iterative procedure may not converge to a sufficiently accurate solution, particularly for strongly nonlinear systems. In Ref. [33], a progressive iteration to compute the steady-state PDFs of strongly nonlinear systems has been developed. By choosing the converged solution of a weaker nonlinear system as $p_m(x, y)$, and slowly varying the nonlinear parameter when applying the iterative method of weighted residuals to search the PDF solution, we can usually obtain the steady-state PDFs of strongly nonlinear systems with sufficient accuracy.

4 Examples

Before we present the numerical examples, we note that the domain of double integrations in the previous discussions is chosen to be finite such that the truncation error is negligible.

Example 1. Consider a nonlinear oscillator under parametric Gaussian white noise excitation. The equation of motion in the state space is given by

$$\begin{aligned} \frac{dX}{dt} &= Y \\ \frac{dY}{dt} &= -2Y\left(\beta X^2 - \frac{4\alpha X^2}{X^2+Y^2}\right) - X + XW_1(t) \end{aligned} \tag{18}$$

where β and α are positive constants. $W_1(t)$ denotes the Gaussian white noise excitation with intensity $2D_1$. The reduced FPK equation of the system has the following drift and diffusion coefficients:

$$\begin{aligned} m_1 &= y \\ m_2 &= 2y\left(\beta x^2 - \frac{4\alpha x^2}{x^2+y^2}\right) + x \\ b_{22} &= 2D_1x^2 \end{aligned} \tag{19}$$

We have obtained $\psi_1(x,y) = x^2/2 + y^2/2$ and $\psi_2(x,y) = \beta/2D_1y^2 - 2\alpha/D_1\ln(x^2+y^2)$. Let $\beta=0.1$, $\alpha=0.1$, $D_1=0.1$, and $n=4$. By using the iterative method of weighted residuals, the stationary PDF is obtained after one iteration as follows:

$$\begin{aligned} \bar{p}(x,y) = C_0\exp[&-1.00000005x^2 + 1.112083576\times10^{-9}xy \\ &-1.00000004y^2 - 4.363002637\times10^{-10}x^4 \\ &-5.360006185\times10^{-10}x^3y + 4.612177970\times10^{-10}x^2y^2 \\ &-1.100109403\times10^{-10}xy^3 - 2.33152422\times10^{-10}y^4 \\ &+4.000000000\ln(x^2+y^2)] \end{aligned} \tag{20}$$

After neglecting the terms with coefficients of the order 10^{-9} and higher, the solution in Eq. (20) is in agreement with the exact steady-state PDF

$$p_{\text{ext}}(x, y) = C_0 (x^2 + y^2)^4 \exp\left[-(y^2 + x^2)\right] \tag{21}$$

This exact solution has not been obtained by other methods for approximate solutions in the literature.

Example 2. The second example deals with a dry friction nonlinear system. The equation of motion in the state space is given by

$$\begin{aligned} \frac{dX}{dt} &= Y \\ \frac{dY}{dt} &= -\beta_0 \text{sgn}(Y) + \beta_1 Y - \beta_3 Y^3 - \omega^2 X - \alpha X^3 + W_1(t) \end{aligned} \tag{22}$$

where $\beta_0,\ \beta_1,\ \beta_3 \geq 0,\ \omega,\ \alpha > 0$, and $W_1(t)$ denotes the Gaussian white noise excitation with intensity $2D_1$. The drift and diffusion coefficients of the FPK equation are given by

$$\begin{aligned} m_1 &= y \\ m_2 &= \beta_0 \text{sgn}(y) - \beta_1 y + \beta_3 y^3 + \omega^2 x + \alpha x^3 \\ b_{22} &= 2D_1 \end{aligned} \tag{23}$$

We obtain

$$\psi_1 = \frac{y^2}{2} + \frac{\omega^2 x^2}{2} + \frac{\alpha x^4}{4}$$

and

$$\psi_2 = \frac{1}{4} \frac{4\beta_0 |y| - 2\beta_1 y^2 + \beta_3 y^4}{D_1}$$

Let $\beta_0 = \beta_1 = \beta_3 = 0.1,\ \omega = 1.0,\ D_1 = 0.1,\ \varepsilon_0 = 10^{-3},\ \alpha = 1.0$ and $n = 6$. The iterative method of weighted residuals after five iterations gives the following solution:

$$\begin{aligned} \bar{p}(x, y) = 0.3401744499 \exp(&-1.14528773x^2 - 0.0221861356xy \\ &+ 0.3264513986y^2 + 0.5738843175x^4 \\ &+ 0.0203959728x^3 y + 0.4799681599x^2 y^2 \\ &+ 0.0955446233xy^3 - 0.1981457145y^4 \\ &- 0.419215276x^6 + 0.0000116282x^5 y \\ &- 0.3667505829x^4 y^2 - 0.037389776x^3 y^3 \\ &- 0.1415524532x^2 y^4 - 0.0306704806xy^5 \\ &- 0.003101293307y^6 - 0.9555878956|y|) \end{aligned} \tag{24}$$

Figure 1 shows the analytical solution in Eq. (24) and the data obtained from Monte Carlo simulations with 4×10^6 points. The error Δp is 0.011. It is clear that the analytical solution not only has high accuracy, but also catches the characteristic of the dry friction completely.

Example 3. The third example is a nonlinear system subject to both external and parametric Gaussian white excitations. The system is governed by the following stochastic differential equation:

$$\begin{aligned} \frac{dX}{dt} &= Y \\ \frac{dY}{dt} &= -Y\left(\alpha + \beta X^2\right) - X + W_1(t) + XW_2(t) \end{aligned} \tag{25}$$

where α and β are constants, $W_i(t)$ $(i = 1, 2)$ denote the independent Gaussian white noise excitations with intensity $2D_i$. The reduced FPK equation of the system has the following drift and diffusion terms:

$$\begin{aligned} m_1 &= y \\ m_2 &= y(\alpha + \beta x^2) + x \\ b_{22} &= 2D_1 + 2D_2 x^2 \end{aligned} \tag{26}$$

It is easy to obtain that $\psi_1 = x^2/2 + y^2/2$ and $\psi_2 = (1/2) y^2(\alpha + \beta x^2)/D_1 + D_2 x^2$. For this example, two cases are studied.

Case 1: $D_1/D_2 = \alpha/\beta$

In this case, the solvability conditions are satisfied so that the reduced FPK equation has an exact solution

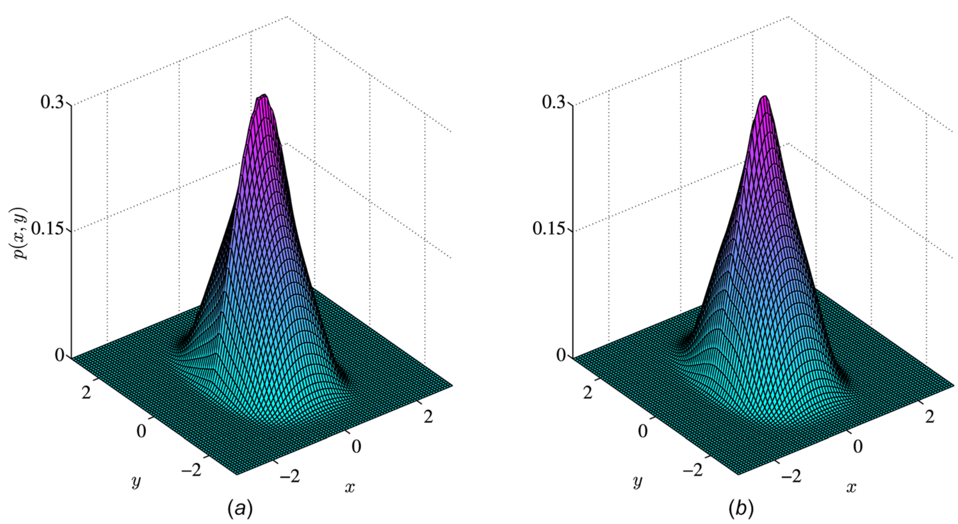

Fig. 1 The stationary PDF of example 2 with parameters $\beta_0 = \beta_1 = \beta_3 = 0.1$, $\omega = 1.0$, and $D_1 = 0.1$. (*a*) The closed-form PDF in Eq. (24). (*b*) The PDF obtained with Monte Carlo simulations.

$$p_{\text{ext}}(x,y)=C_0\exp\left[-\frac{\beta(x^2+y^2)}{2D_1}\right] \tag{27}$$

Let $\alpha=1.0$, $\beta=1.0$, $D_1=D_2=0.1$, $n=4$ and $\varepsilon_0=10^{-3}$. The iterative method of weighted residuals leads to the stationary solution after two iterations. The result is given below

$$\begin{aligned}\bar{p}(x,y)=C_0\exp(&-4.999999992x^2-3.361487619\times10^{-9}xy\\&-4.999999998y^2-1.76706273\times10^{-9}x^4\\&+1.227932558\times10^{-10}x^3y-2.210250080\times10^{-9}x^2y^2\\&-6.093308925\times10^{-10}xy^3-2.820583132\times10^{-11}y^4)\end{aligned} \tag{28}$$

It is evident that the solution in Eq. (27) matches with the exact stationary PDF after neglecting the terms with coefficients of the order 10^{-9} and higher.

Case 2: $D_1/D_2\neq\alpha/\beta$

In this case, the solvability conditions are not satisfied and the reduced FPK equation has no exact solution available. Let $\alpha=-0.1$, $\beta=1.0$, $D_1=D_2=0.25$, $n=4$ and $\varepsilon_0=10^{-3}$. We should point out that when $p_m(x,y)=\exp[-\psi_1(x,y)-\psi_2(x,y)]$, i.e.,

$$p_m(x,y)=\exp\left[-\left(\frac{x^2}{2}+\frac{y^2}{2}\right)-\frac{1}{2}\frac{y^2(\alpha+\beta x^2)}{D_1+D_2x^2}\right] \tag{29}$$

the iterative method of weighted residuals cannot converge to an accurate solution of the reduced FPK equation. Alternatively, we start with the solution $\bar{p}(x,y)$ in Eq. (28) for case 1 such that $p_m(x,y)=\bar{p}(x,y)$ and apply the iterative method of weighted residuals to compute the stationary PDF $\bar{p}(x,y)$ for this case by setting $\alpha=0.9$, for example. After we obtain the converged solution, we use the new solution for $p_m(x,\ y)$ and decrease α by another factor until it reaches $\alpha=-0.1$. Progressively, the approximate stationary PDF $\bar{p}(x,y)$ for $\alpha=-0.1$ is obtained as follows:

$$\begin{aligned}\bar{p}(x,y)=0.0924126265\exp(&0.5883044670x^2-0.5158361910xy\\&-0.04345295515y^2-0.2460809073x^4\\&+0.1046008655x^3y-0.3104263776x^2y^2\\&+0.1959744613xy^3-0.06248245755y^4\\&+0.2628283964\frac{(x^2-0.1)y^2}{x^2+1})\end{aligned} \tag{30}$$

Next, we compare the proposed method with the existing methods. We apply the exponential polynomial approximation of order $n=6$ denoted as $\bar{p}_0(x,y)$, and obtain

$$\begin{aligned}\bar{p}_0(x,y)=0.1095945870\exp(&-0.001665564232x^6\\&-0.0118049672x^5y+0.0089775857x^4y^2\\&-0.0317590517x^3y^3+0.016965593x^2y^4\\&-0.01085065231xy^5-0.007349317y^6-0.22575097x^4\\&+0.1655182494x^3y-0.4156116936x^2y^2\\&+0.2439961186xy^3-0.06403555909y^4\\&+0.4301646101x^2-0.5227219184xy-0.1124081301y^2)\end{aligned} \tag{31}$$

We then apply the method of stochastic averaging and find the approximate PDF given by

$$\begin{aligned}p_{\text{ave}}(x,y)=0.1176319073\exp[&-2(x^2+y^2)\\&+8.8\ln(0.25(x^2+y^2)+1)]\end{aligned} \tag{32}$$

To check the accuracy of all the approximate solutions, we carry out Monte Carlo simulations. Figure 2(*a*) shows the solution in Eq. (30) by the proposed method. The error Δp is 0.012. Figure 2(*b*) shows the stationary PDF obtained from Monte Carlo simulations with 9×10^6 points. Figure 2(*c*) plots the solution in Eq. (32) by the method of stochastic averaging. The error Δp is 0.098. Figure 2(*d*) draws the exponential polynomial solution in Eq. (31). The error Δp is 0.015. In Fig. 3, the marginal distribution densities of $p_1(x)$ and $p_2(y)$ from the Monte Carlo simulations and closed-form solutions in Eqs. (30) and (32) are depicted. The root mean squared (RMS) error between the analytical results and Monte Carlo data is also given in the figure. It is observed that both analytical solutions in Eqs. (30) and (31) are in good agreement with the Monte Carlo simulations and capture the nonlinear feature of the system, while the solution by the method of stochastic averaging completely fails to capture the distinct feature of the system. Table 1 shows the convergence of the iterative method for every α of the solutions that lead to the results in Eqs. (30) and (31). It is seen that the convergence of both solutions is fast, but the exponential polynomial method takes much longer computational time.

Our numerical experiments have found that the solution by the exponential polynomial method in Eq. (31) must be of order $n=6$ with 15 terms in order to be comparable to the solution by the proposed method in Eq. (30), which consists of a polynomial of order

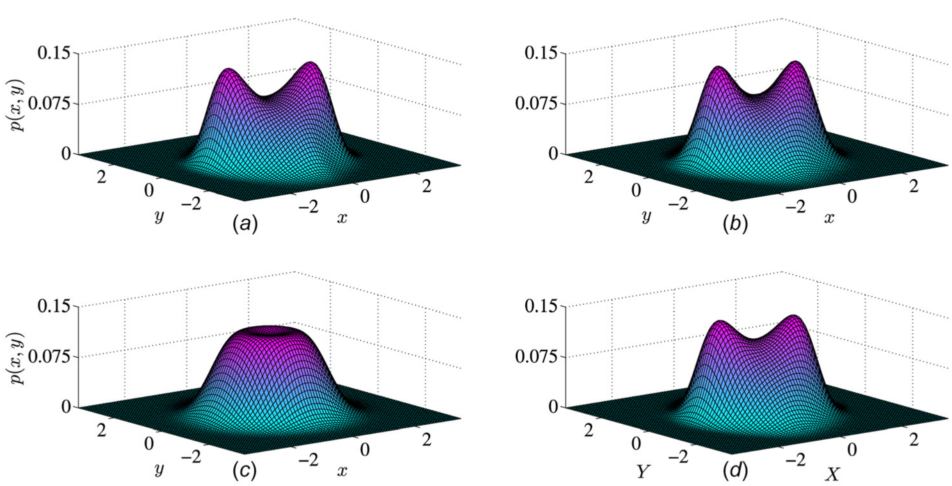

Fig. 2 The stationary PDF of example 3 with parameters $\alpha=-0.1$, $\beta=1.0$, and $D_i=0.25$. (*a*) The closed-form PDF in Eq. (30). (*b*) The PDF obtained with Monte Carlo simulations. (*c*) The averaged PDF in Eq. (32). (*d*) The exponential polynomial approximation in Eq. (31).

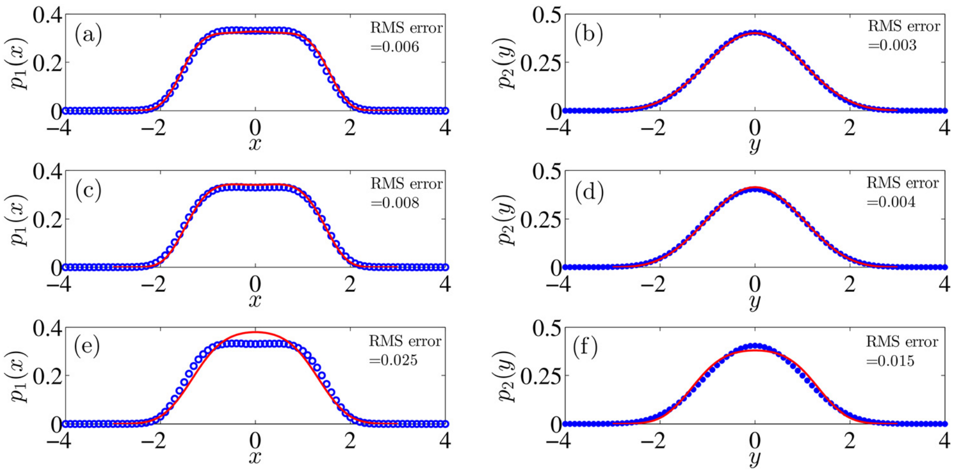

Fig. 3 The marginal probability densities of $p_1(x)$ and $p_2(y)$ of example 3 obtained from the corresponding joint PDF. Symbols (◦, ⋆) denote the Monte Carlo data. Solid line denotes the analytical solutions. (*a*) and (*b*): the closed-form PDF in Eq. (30). (*c*) and (*d*): the exponential polynomial approximation in Eq. (31). (*e*) and (*f*): the averaged PDF in Eq. (32).

Table 1 Number of iterations of the method of weighted residuals to convergence for each α of example 3, case 2

A	0.9	0.5	0.2	0.1	−0.1
Iterations for $\bar{p}(x, y)$	4	2	2	1	2
Iterations for $\bar{p}_0(x, y)$	5	2	3	2	4

$n = 4$ and has nine terms. This is an indirect proof that the proposed solution in Eqs. (7) and (8) for the stationary PDF of the reduced FPK equation can capture the key features of the probability distribution of nonlinear stochastic systems with a small number of terms.

Example 4. The fourth example deals with the motion of ship rolling. The equation of motion in the state space is expressed by

$$\begin{aligned}\frac{dX}{dt} &= Y \\ \frac{dY}{dt} &= -\beta_1 Y - \beta_2 Y|Y| - \alpha_1 X - \alpha_3 X^3 - \alpha_5 X^5 + W_1(t)\end{aligned} \tag{33}$$

where X denotes the ship roll angle, Y denotes the ship roll angular velocity, β_1, β_2, α_1, α_3 and α_5 are constants, and $W_1(t)$ denotes the Gaussian white noise excitation with intensity $2D_1$. The drift and diffusion coefficients of the reduced FPK equation associated with the system are given by

$$\begin{aligned}m_1 &= y \\ m_2 &= \beta_1 y + \beta_2 y|y| + \alpha_1 x + \alpha_3 x^3 + \alpha_5 x^5 \\ b_{22} &= 2D_1\end{aligned} \tag{34}$$

We obtain

$$\psi_1 = \frac{y^2}{2} + \frac{\alpha_1 x^2}{2} + \frac{\alpha_3 x^4}{4} + \frac{\alpha_5 x^6}{6}$$

and

$$\psi_2 = \frac{1}{6}\frac{2\beta_0|y|^3 + 3\beta_1 y^2}{D_1}$$

Let $\beta_1 = \beta_2 = 0.1$, $\alpha_1 = \alpha_5 = 1$, $\alpha_3 = -2.5$, $D_1 = 0.1$, $\varepsilon_0 = 10^{-3}$ and $n = 6$. By using the iterative method of weighted residuals after one iteration, the approximation solution is obtained as follows:

$$\begin{aligned}\bar{p}(x, y) = 0.1456834853\exp(&-0.8894128289x^2 + 0.0188697569xy \\ &-0.4890760220y^2 + 1.122325624x^4 + 0.0008617269x^3y \\ &-0.0496059394x^2y^2 - 0.0307025130xy^3 + 0.51785608y^4 \\ &-0.2987434359x^6 - 0.008823367x^5y \\ &+0.03922444499x^4y^2 - 0.0015618021x^3y^3 \\ &-0.0051500990x^2y^4 + 0.0072121576xy^5 \\ &-0.0024612358y^6 - 0.3251447260|y^3|)\end{aligned} \tag{35}$$

As a comparison, the solution obtained with the exponential polynomial method of order $n = 8$ is cited here

$$\begin{aligned}p_e(x, y) = 0.1520954086\exp(&0.3519745166x^2 + 0.0565388158xy \\ &- 0.8625034331y^2 + 0.270273262x^2y^2 \\ &- 1.076865438x^4 - 0.0749544486x^3y \\ &+ 0.0085327567xy^3 - 0.1695902301y^4 \\ &+ 0.817486505x^6 + 0.0304065384x^5y \\ &+ 0.0152299705x^4y^2 - 0.0104914665x^3y^3 \\ &- .1019452006x^2y^4 + 0.0268234634xy^5 \\ &+ 0.0365176382y^6 - 0.0031009986y^8 \\ &- 0.0076127034xy^7 - 0.0043307825x^2y^6 \\ &-0.0023423795x^3y^5 + 0.0314369945x^4y^4 \\ &+ 0.0004808876x^5y^3 - 0.0383307589x^6y^2 \\ &- 0.035907154x^7y - 0.1730116870x^8)\end{aligned} \tag{36}$$

and the exponential polynomial solution of order $n = 8$ is obtained with the iterative method of weighted residuals after 15 iterations as

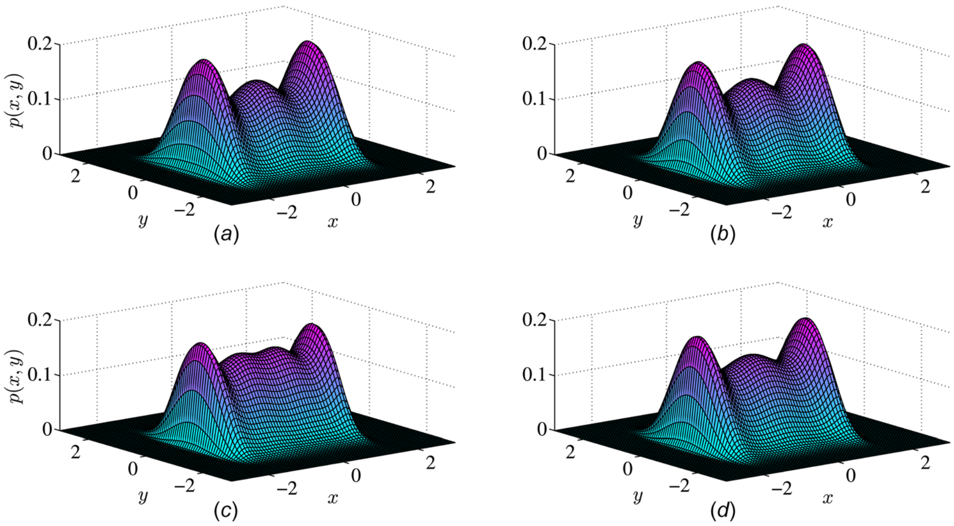

Fig. 4 The stationary PDF of example 4 with parameters $\alpha_1 = \alpha_5 = 1$, $\alpha_3 = -2.5$, $\beta_1 = \beta_2 = 0.1$, and $D_1 = 0.1$. (*a*) The closed-form PDF in Eq. (35). (*b*) The PDF obtained with Monte Carlo simulations. (*c*) The exponential polynomial solution in Eq. (36). (*d*) The iterated exponential polynomial approximation in Eq. (37).

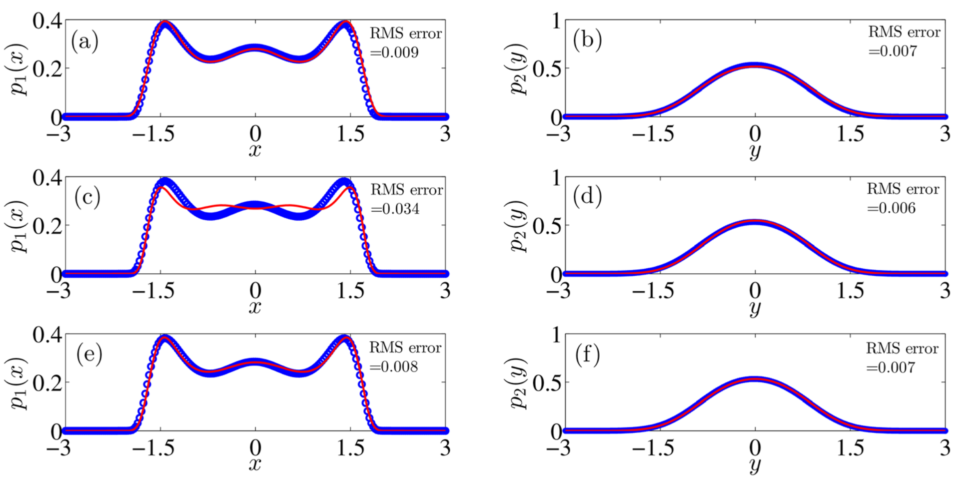

Fig. 5 The marginal probability densities of $p_1(x)$ and $p_2(y)$ of example 4 obtained from the corresponding joint PDF. Symbols (◦, ⋆) denote the Monte Carlo data. Solid line denotes the analytical solutions. (*a*) and (*b*): the closed-form PDF in Eq. (35). (*c*) and (*d*): the exponential polynomial solution in Eq. (36). (*e*) and (*f*): the iterated exponential polynomial approximation in Eq. (37).

$$
\begin{aligned}
\bar{p}_0(x,y) = 0.149226769\exp(&-0.4723516221x^2 - 0.0289345360xy \\
&-0.6363951347y^2 + 0.3895831417x^4 + 0.006659186x^3y \\
&-0.1521405710x^2y^2 + 0.0418280032xy^3 \\
&-0.2339090354y^4 + 0.0804603322x^6 + 0.0029816869x^5y \\
&+0.1821743997x^4y^2 - 0.0129367447x^3y^3 \\
&+0.0158244554x^2y^4 - 0.0101555046xy^5 \\
&+0.0363150697y^6 - 0.0604634934x^8 - 0.0004315259x^7y \\
&-0.0532812541x^6y^2 - 0.0007167168x^5y^3 \\
&+0.0031705827x^4y^4 + 0.0027817148x^3y^5 \\
&-0.0035589969x^2y^6 + 0.0003134506xy^7 \\
&-0.0026343131y^8)
\end{aligned} \tag{37}
$$

Figure 4 shows the analytical solutions in Eqs. (35)–(37) and the Monte Carlo simulation results with 4×10^6 points. It can be seen from the figures that all the analytical solutions capture the multimode characteristic of the PDF. The error Δp of the analytical solutions in Eqs. (35)–(37) are 0.0141, 0.0379, and 0.0132, respectively. Figure 5 shows the marginal probability densities of $p_1(x)$ and $p_2(y)$ from the Monte Carlo simulations and the corresponding closed-form solutions. The RMS error of the analytical results as compared with the Monte Carlo data is also given in the figure.

We must note that the order of polynomials in all the solutions has been determined by numerical experiments. We have found that the exponential polynomial method must be of order $n = 8$ with 24 terms in order to be comparable to the proposed method, and still gives the stationary PDF with four peaks in Eq. (36), while the Monte Carlo simulations show only three peaks. The exponential polynomial method combined with the iterative procedure must also be of order $n = 8$ with 24 terms in order to be comparable and delivers the solution in Eq. (37) with the best accuracy among the three closed-form solutions. However, the computational time to update the coefficients in this solution is the largest, since it involves solving 24 nonlinear algebraic equations 15 times. The solution by the proposed method in Eq. (35) has 16 terms, whose coefficients are determined by the iterative method of weighted residuals after solving 16 nonlinear algebraic equations only once, and represents an excellent balance of accuracy and efficiency.

5 Conclusions

A general approach has been proposed for obtaining analytical solutions of the stationary PDF distribution of SDOF nonlinear systems under external and (or) parametric Gaussian white noise excitations. In particular, an innovative approach to construct trial solutions is proposed. An iterative method of weighted residuals is then used to determine the coefficients in the trial solution. Numerical experiments have shown that the proposed procedure can converge to the exact solution when it exists. Highly accurate solutions with smaller number of terms than that by other methods can be obtained with the proposed method resulting a much lower computational effort. Effort is on the way to extend the proposed procedure to multidegree-of-freedom nonlinear stochastic systems.

Acknowledgment

This work is supported by the Natural Science Foundation of China through the Grant Nos. 11172197, 11332008, 11572215, and 11672111, by the Research Award Fund for Outstanding Young Researcher in Higher Education Institutions of Fujian Province and by the Research Fund for Excellent Young Scientific and Technological Project of Huaqiao University under the Grant No. ZQN-YX307.

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