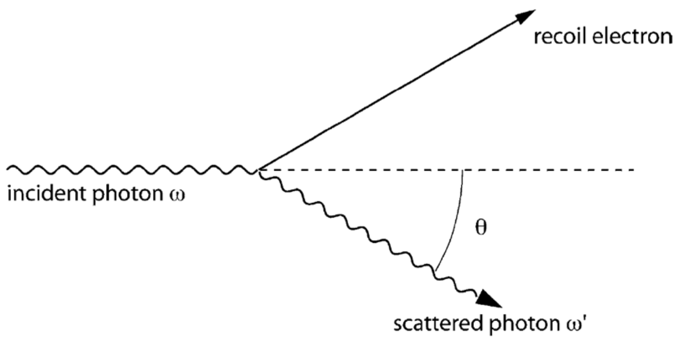

Figure 1: The outgoing photon (frequency, ω') is emitted at an angle θ relative to the direction of the incoming photon (frequency ω). The angle of electron's trajectory relative to the incoming photon's direction is not necessary for the prof but if you like you may call it ϕ

Given the initial conditions set in Fig.1, with the electron initially at rest we have a system with initial energy

$$E_i = \hbar\omega + mc^2 \tag{1}$$

where m is the rest mass of the electron. After the interaction, the electron will have energy $E_e = \sqrt{p^2c^2 + m^2c^4}$ the system has a final energy

$$E_f = \hbar\omega' + E_e = \hbar\omega' + \sqrt{p^2c^2 + m^2c^4} \tag{2}$$

where p^2 is the squared magnitude of the electron's final momentum, $\vec{p}$.

Because this is an elastic collision, we can set Equations 1 and 2 equal to each other and solve for the electron's momentum

$$\begin{aligned}
\hbar\omega + mc^2 &= \hbar\omega' + \sqrt{p^2c^2 + m^2c^4} \\
p^2c^2 + m^2c^4 &= (\hbar\omega - \hbar\omega' + mc^2)^2 \\
p^2c^2 + \cancel{m^2c^4} &= \hbar^2\left[\omega^2 + \omega'^2 - 2\omega\omega' + \frac{2mc^2}{\hbar}(\omega - \omega')\right] + \cancel{m^2c^4} \\
p^2c^2 &= \hbar^2\left[\omega^2 + \omega'^2 - 2\omega\omega' + \frac{2mc^2}{\hbar}(\omega - \omega')\right]. \qquad (3)
\end{aligned}$$

Next we take advantage of momentum conservation. The system's initial momentum, $\vec{p_i} = \frac{\hbar\omega}{c}$ is equal to the sum of the outgoing photon's momentum

$\vec{p_f}$ and $\vec{p}$.

$$\begin{aligned}\vec{p_i} &= \vec{p_f} + \vec{p} \\ \vec{p} &= \vec{p_f} - \vec{p_i} \end{aligned} \tag{4}$$

Using the law of cosines we get

$$\begin{aligned} p^2 &= p_i^2 + p_f^2 - 2p_i p_f \cos\theta \\ &= \frac{\hbar^2\omega^2}{c^2} + \frac{\hbar^2\omega'^2}{c^2} - \frac{2\hbar^2}{c^2}\omega\omega'\cos\theta \\ p^2c^2 &= \hbar^2(\omega^2 + \omega'^2 - 2\omega\omega'\cos\theta). \end{aligned} \tag{5}$$

Now we can set Equations 3 and 5 equal to each other.

$$\begin{aligned} \hbar^2\left[\omega^2 + \omega'^2 - 2\omega\omega' + \frac{2mc^2}{\hbar}(\omega - \omega')\right] &= \hbar^2(\omega^2 + \omega'^2 - 2\omega\omega'\cos\theta) \\ \frac{mc^2}{\hbar}(\omega - \omega') &= \omega\omega'(1 - \cos\theta) \\ \frac{mc^2}{\hbar}\omega &= \omega'\left[\frac{mc^2}{\hbar}\omega(1 - \cos\theta)\right] \qquad (6) \\ \hbar\omega' &= \frac{mc^2\omega}{\left[\frac{mc^2}{\hbar} + \omega(1 - \cos\theta)\right]} \\ &= \frac{\hbar\omega}{1 + \frac{\hbar\omega}{mc^2}(1 - \cos\theta)} \qquad (7) \end{aligned}$$

Quod erat demonstrandum