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Bachelor's Thesis

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Assisted Quantum Capacity of the Quantum Depolarizing Channel

Bachelor's Thesis

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Abstract

In this thesis, we consider the problem of determining the asymptotic quantum capacity of the Quantum Depolarizing Channel in the context of the zero-capacity problem. We look for the maximal value of the depolarizing probability such that a forward-assisted quantum transmission through the Depolarizing Channel is possible. The forward-assisted transmission corresponds to a bipartite operation which is non-signalling from Bob to Alice. It is implementable by quantum communication over main and auxiliary channels and by local operations. In the communication setting, we consider a one-shot regime and the bipartite coding operation which is additionally positive-partial-transpose preserving (PPTp) and non-signalling from Alice to Bob. In the first part of this thesis we present relevant terminology and definitions from Quantum Communications which are necessary to understand our problem. Then, we reproduce the derivation of the semidefinite program for finding the optimal fidelity of an assisted transmission through a quantum channel. The optimization is performed over the aforementioned class of codes and does not assume anything additional about the type of the auxiliary channel which is the part of the bipartite operation. As a creative contribution of this thesis, we show that the semidefinite program can be transformed into the linear program for the case of the Quantum Depolarizing Channel which is generally much more efficient to solve. Based on our achievement, we present extensive numerical results obtained from solving the linear program and we conjecture the one-shot lower bound for the zero-capacity threshold of the Quantum Depolarizing Channel.

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Contents

Chapter 1

Introduction

> The lurking suspicion that something could be simplified is the world's richest source of rewarding challenges.
>
> *Edsger W. Dijkstra*

In this chapter, we present basic information regarding this thesis. We start by explaining the motivation behind our work. Then, we switch to more technical details such as mathematical notation that is used throughout this document and we define fundamental concepts and terms that are essential in the following chapters. At the end of this chapter, we specify the problem that is attempted to be solved in this thesis.

1.1 Motivation

Determining the capacity of a channel to transmit quantum information is a fundamental problem in the Quantum Information Theory. It gives us information, in terms of the optimal asymptotic rate of transmission, on how many qubits can be sent through a channel per a single use of it with assumed fidelity (usually close to perfect with a vanishing error). Despite its importance and much effort of scientists, the problem of the quantum capacity of even simple channels is not yet solved. This thesis addresses the problem of determining the quantum assisted capacity of a quantum channel which is a relaxed, yet still very insightful and relevant, variation of the quantum capacity problem. In particular, the quantum assisted capacity of the Quantum Depolarizing Channel, a fundamental and widely-applicable model of quantum transmission, will be investigated.

1.2 Preliminaries

In this thesis we adopt the notation from [1]. Its main features are summarized below.

- A quantum system Q is associated with a Hilbert space $\mathcal{H}_Q$. Any operator which acts on a system Q is denoted with a proper subscript, for example X_Q.
- To make equations more compact, tensor product symbols $\otimes$ and identity operators I_Q are omitted whenever their presence can be concluded from the context. For example $(X_Q \otimes I_R)(I_Q \otimes Y_R) = X_Q Y_R = Y_R X_Q$.
- Quantum operations (channels) are denoted with a calligraphic letter with input and output systems specified in a subscript, for example $\mathcal{N}_{B\leftarrow A}$. Their Choi matrix representations are expressed with a corresponding capital letter, i.e. N_{BA}.
- The set of all quantum operations from A to B is $\mathbf{ops}(A \to B)$.
- A trace operation on a system Q is denoted with Tr_Q which subscript, as the only exception from the notation, misses the output system since it is trivial.
- A transpose map which acts on a system Q has two possible depictions: $\mathbf{t}_{Q\leftarrow Q} X_Q$ or X_Q^T.
- Identity operation $\mathrm{id}_{\tilde{Q}\leftarrow Q} : |i\rangle\langle j|_Q \to |i\rangle\langle j|_{\tilde{Q}}$ serves as an identifier of states of Q with the states of $\tilde{Q}$ provided that systems Q and $\tilde{Q}$ are of equal dimensions.
- Applying a linear map $\mathcal{N}_{R\leftarrow Q}$ to an operator X_Q is represented as $\mathcal{N}_{R\leftarrow Q} X_Q$.
- Products of quantum operations follow the compact notation where identities and tensor symbols are skipped, for example: $\mathcal{M}_{R\leftarrow Q}\mathcal{N}_{T\leftarrow P} X_{QP} = (\mathcal{M}_{R\leftarrow Q} \otimes \mathcal{N}_{T\leftarrow P}) X_{QP}$ or $\mathcal{M}_{R\leftarrow Q} X_{QP} = (\mathcal{M}_{R\leftarrow Q} \otimes \mathrm{id}_{P\leftarrow P}) X_{QP}$.
- Order of multiplication of operators and applying linear maps which take operators to operators is such that former are performed before the latter. For example: $\mathbf{t}_{Q\leftarrow Q} X_Q Y_Q = \mathbf{t}_{Q\leftarrow Q}(X_Q Y_Q)$.

1.3 Quantum channel

1.3.1 Quantum channel definition

As defined in [2]:

Definition 1.3.1 (Quantum channel)**.** A quantum channel is a linear map $\mathcal{N}_{B\leftarrow A}$ which is trace preserving, completely positive and represents a quantum evolution.

Definition 1.3.2 (Linearity)**.** A map $\mathcal{M}_{B\leftarrow A}$ is linear if:

$$\mathcal{M}_{B\leftarrow A}(\alpha X_A + \beta Y_A) = \alpha \mathcal{M}_{B\leftarrow A}(X_A) + \beta \mathcal{M}_{B\leftarrow A}(Y_A), \tag{1.1}$$

where $\alpha, \beta \in \mathbb{C}$.

Definition 1.3.3 (Positivity)**.** A linear map $\mathcal{M}_{B\leftarrow A}$ is positive if $\mathcal{M}_{B\leftarrow A}(X_A)$ is positive semi-definite for all positive semi-definite X_A.

Definition 1.3.4 (Complete positivity)**.** A linear map $\mathcal{M}_{B\leftarrow A}$ is completely positive if $I_R \otimes \mathcal{M}_{B\leftarrow A}$ is a positive map for a reference system R whose size is arbitrary.

Definition 1.3.5 (Trace preservation)**.** A map $\mathcal{M}_{B\leftarrow A}$ is trace preserving if we have $\operatorname{Tr} \mathcal{M}_{B\leftarrow A}(X_A) = \operatorname{Tr} X_A$ for all X_A.

1.3.2 Choi matrix representation of quantum channels

Definition 1.3.6 (Choi matrix)**.** Given a unique operator $\mathcal{N}_{R\leftarrow Q}$ on $\mathcal{H}_R \otimes \mathcal{H}_Q$ and an isotropic maximally entangled state $\phi_{\tilde{Q}Q}$, the Choi matrix N_{RQ} of $\mathcal{N}_{R\leftarrow Q}$ is given by:

$$N_{RQ} = \dim{(Q)}\mathrm{id}_{Q\leftarrow\tilde{Q}}\mathcal{N}_{R\leftarrow Q}\phi_{\tilde{Q}Q}, \tag{1.2}$$

where $\phi_{\tilde{Q}Q} := |\phi\rangle\,\langle\phi|_{\tilde{Q}Q}$ and:

$$|\phi\rangle_{\tilde{Q}Q} = \frac{1}{\sqrt{\dim Q}} \sum_{i=0}^{\dim Q} |i\rangle_{\tilde{Q}}\,|i\rangle_Q \tag{1.3}$$

Theorem 1.3.1 (Choi matrix representation)**.** The Choi matrix N_{RQ} of a channel $\mathcal{N}_{R\leftarrow Q}$ determines the channel completely.

The equivalence in Theorem 1.3.1 is established by proving correspondences between $\mathcal{N}_{R\leftarrow Q}$ and N_{RQ} in: Hermiticity, complete positivity, being doubly-stochastic, unitality, trace preservation and normalization which are all inspected in [3].

1.3.3 Quantum Depolarizing Channel

Definition 1.3.7 (Quantum Depolarizing Channel)**.** A Quantum Depolarizing Channel is a channel $\mathcal{N}_{B\leftarrow A}$ acting on a qubit and of the form:

$$\mathcal{N}_{B\leftarrow A}(\rho) = p\frac{I}{2} + (1-p)\rho, \tag{1.4}$$

where $p \in \mathbb{R} : 0 \leq p \leq 1$.

1.4 Fidelity of a quantum channel

1.4.1 Fidelity

Fidelity is a distance measure that can be used as a comparator of two states in terms of a probability that one state cannot be mistaken with the other. The maximal fidelity between two states, i.e. 1, means that it is impossible to distinguish between them. Therefore, fidelity is a reasonable candidate for comparing inputs with outputs in the context of noisy quantum communication. In particular, the so called entanglement fidelity is often used to define the fidelity of the quantum channel itself. In such a case, the qubit from a maximally entangled qubit pair is fed into a channel and the other qubit is assumed to serve as a reference system and is left undisturbed. At the end of communication, the fidelity of the transmitted qubit and the reference qubit is calculated to quantify the noise introduced in the transmission.

1.4.2 Entanglement fidelity

Definition 1.4.1 (Entanglement fidelity [1])**.** Given two systems $\tilde{Q}$ and Q of equal dimensions, the entanglement fidelity F_E of a state $\sigma_{\tilde{Q}Q}$ is defined by:

$$F_E = \operatorname{Tr}_{\tilde{Q}Q} \phi_{\tilde{Q}Q} \sigma_{\tilde{Q}Q}, \tag{1.5}$$

where ϕ_{XY} denotes an isotropic maximally entangled state on systems X and Y.

1.4.3 Channel fidelity

Definition 1.4.2 (Channel fidelity [1])**.** Given two systems $\tilde{Q}$ and Q of equal dimensions and an operation $\mathcal{M}_{B'\leftarrow A}$ with $\dim A = \dim B'$ the entanglement fidelity $F(\mathcal{M}_{B'\leftarrow A})$ is defined by:

$$F(\mathcal{M}_{B'\leftarrow A}) = \operatorname{Tr}_{B'\tilde{A}} \phi_{B'\tilde{A}} \mathcal{M}_{B'\leftarrow A} \phi_{A\tilde{A}}, \tag{1.6}$$

where ϕ_{XY} denotes an isotropic maximally entangled state on systems X and Y.

1.4.4 Forward-assisted code channel fidelity

Definition 1.4.3 (Forward-assisted code channel fidelity [1])**.** A forward-assisted code fidelity $F^{\Omega}(\mathcal{N}, K)$ is the maximum channel fidelity $F(\mathcal{M}_{B'\leftarrow A})$ of quantum operations $\mathcal{M}_{B'\leftarrow A}$ with $\dim A = \dim B' = K$ which can be obtained by applying a forward-assisted code from a certain class Ω to $\mathcal{N}_{B\leftarrow A'}$.

1.5 Classes of codes

1.5.1 Forward-assisted codes

Forward-assisted codes is a vast category of codes with the property of being non-signalling from Bob to Alice. It is useful to define several subclasses which naturally emerge from the general definition of forward-assisted codes as well as those which are operationally interesting.

Definition 1.5.1 (Deterministic supermap)**.** Deterministic supermap $\mathcal{M}$ is a linear and completely positive map which maps quantum operations to quantum operations.

Definition 1.5.2 (Forward-assisted code)**.** Forward-assisted code of size K for the use of a channel $\mathcal{N}_{B'\leftarrow A}$ is a deterministic supermap $\mathcal{M}:\mathbf{ops}(A'\to B)\to\mathbf{ops}(A\to B')$ where $\dim A=\dim B'=K$ and it corresponds to a bipartite operation from the set $\mathbf{ops}(A:B\to A':B')$ which is non-signaling from Bob to Alice.

1.5.1.1 Unassisted codes

Definition 1.5.3 (Unassisted code)**.** Unassisted code is a bipartite operation $\mathcal{Z}_{A'B'\leftarrow AB}$ of the form:

$$\mathcal{Z}_{A'B'\leftarrow AB}=\mathcal{D}_{B'\leftarrow B}\mathcal{E}_{A'\leftarrow A}. \tag{1.7}$$

1.5.1.2 Forward-assisted classical codes

Definition 1.5.4 (Forward-assisted classical code)**.** Forward-assisted classical code is a bipartite operation $\mathcal{Z}_{A'B'\leftarrow AB}$ of the form:

$$\mathcal{Z}_{A'B'\leftarrow AB}=\mathcal{D}_{B'\leftarrow B}\mathcal{F}^{c}_{R\leftarrow Q}\mathcal{E}_{A'\leftarrow A}, \tag{1.8}$$

where $\mathcal{F}^{c}_{R\leftarrow Q}$ is a classical channel.

1.5.1.3 Entanglement-assisted codes

Definition 1.5.5 (Entanglement-assisted code)**.** Entanglement-assisted code is a bipartite operation $\mathcal{Z}_{A'B'\leftarrow AB}$ of the form:

$$\mathcal{Z}_{A'B'\leftarrow AB}=\mathcal{D}_{B'\leftarrow Bb}\mathcal{E}_{A'\leftarrow Aa}\psi_{ab}, \tag{1.9}$$

where ψ_{ab} is an entangled state on systems a and b which is shared between parties.

1.5.1.4 Non-signaling codes

Definition 1.5.6 (Non-signaling bipartite operation)**.** A bipartite operation $\mathcal{Z}_{A'B'\leftarrow AB}$ is non-signaling from Bob to Alice if:

$$\operatorname{Tr}_{B'}\mathcal{Z}_{A'B'\leftarrow AB}=\mathcal{Z}^{Alice}_{A'A}\operatorname{Tr}_{B}. \tag{1.10}$$

In the Choi formalism, a bipartite operation $\mathcal{Z}_{A'B'\leftarrow AB}$ is non-signaling from Bob to Alice if:

$$\mathrm{Tr}_{B'}\, Z_{A'B'AB} = Z^{Alice}_{A'A} I_B. \tag{1.11}$$

Definition 1.5.7 (Non-signaling code)**.** Non-signaling code is a bipartite operation $\mathcal{Z}_{A'B'\leftarrow AB}$ which is non-signaling from Alice to Bob and of the form

$$\mathcal{Z}_{A'B'\leftarrow AB} = \mathcal{D}_{B'\leftarrow B}\mathcal{F}_{R\leftarrow Q}\mathcal{E}_{A'\leftarrow A}. \tag{1.12}$$

1.5.1.5 Positive-partial-transpose-preserving codes

Definition 1.5.8 (A positive-partial-transpose (PPT) operator)**.** An operator X_{PQ} is PPT if:

$$\mathbf{t}_{Q\leftarrow Q} X_{PQ} \geq 0, \tag{1.13}$$

or, equivalently:

$$\mathbf{t}_{P\leftarrow P} X_{PQ} \geq 0. \tag{1.14}$$

Definition 1.5.9 (A positive-partial-transpose-preserving (PPTp) bipartite operation)**.** A bipartite operation $\mathcal{Z}_{A'B'\leftarrow AB}$ is positive-partial-transpose-preserving (PPTp) if:

$$\mathbf{t}_{B\tilde{B}\leftarrow B\tilde{B}}\rho_{A\tilde{A}B\tilde{B}} \geq 0 \implies \mathbf{t}_{B'\tilde{B}\leftarrow B'\tilde{B}}\mathcal{Z}_{A'B'\leftarrow AB}\rho_{A\tilde{A}B\tilde{B}} \geq 0, \tag{1.15}$$

where $\tilde{A}$ is an arbitrary system of Alice and $\tilde{B}$ is an arbitrary system of Bob. In the Choi matrix formalism, a bipartite operation $\mathcal{Z}_{A'B'\leftarrow AB}$ is positive-partial-transpose-preserving (PPTp) if and only if:

$$\mathbf{t}_{BB'\leftarrow BB'} Z_{A'B'AB} \geq 0. \tag{1.16}$$

Definition 1.5.10 (Positive-partial-transpose-preserving code)**.** Positive-partial-transpose-preserving code is a bipartite operation $\mathcal{Z}_{A'B'\leftarrow AB}$ which is positive-partial-transpose-preserving and of the form:

$$\mathcal{Z}_{A'B'\leftarrow AB} = \mathcal{D}_{B'\leftarrow B}\mathcal{F}_{R\leftarrow Q}\mathcal{E}_{A'\leftarrow A}. \tag{1.17}$$

1.5.1.6 Forward-Horodecki-assisted codes

Definition 1.5.11 (Forward-Horodecki-assisted code)**.** Forward-Horodecki-assisted code is a bipartite operation $\mathcal{Z}_{A'B'\leftarrow AB}$ of the form:

$$\mathcal{Z}_{A'B'\leftarrow AB} = \mathcal{D}_{B'\leftarrow B}\mathcal{F}^{H}_{R\leftarrow Q}\mathcal{E}_{A'\leftarrow A}, \tag{1.18}$$

where $\mathcal{F}^{H}_{R\leftarrow Q}$ is a Horodecki channel.

The relationship between those subclasses is presented in Fig. 1.1.

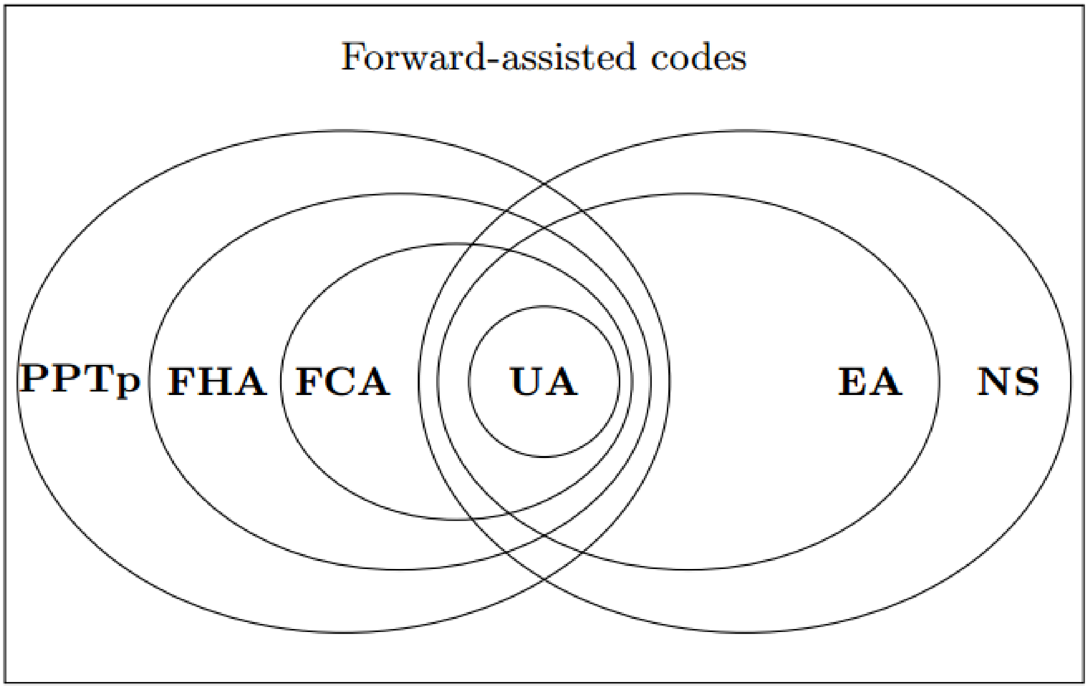


Figure 1.1: Subclasses of foward-assisted codes (graphics courtesy of the authors of [1])

1.6 Capacities of quantum channels

In the literature, different kinds of channel capacities are considered. Important classes include classical capacity [4], private classical capacity [5, 6], entanglement-assisted capacity [7] and quantum capacity [8]. In this thesis we are interested in a quantum capacity.

1.6.1 Asymptotic assisted quantum capacity

Definition 1.6.1 (Asymptotic quantum capacity)**.** The asymptotic assisted quantum capacity of a quantum channel $\mathcal{N}$ for any class of codes Ω is:

$$Q^{\Omega}(\mathcal{N}) := \sup\{r : \lim_{n\to\infty} F^{\Omega}(\mathcal{N}^{\otimes n}, K) = 1\}, \tag{1.19}$$

where K is the dimension of the system to be sent, $K = \lfloor 2^{rn} \rfloor$, r is the rate of transmission and n is the number of uses of a channel.

1.6.2 PPTp and NS-assisted asymptotic quantum capacity

Definition 1.6.2 (Asymptotic PPTp and NS-assisted quantum capacity)**.** The asymptotic PPTp and NS-assisted quantum capacity of a quantum channel $\mathcal{N}$ is defined as:

$$Q^{PPTp\cap NS}(\mathcal{N}) := \sup\{r : \lim_{n\to\infty} F^{PPTp\cap NS}(\mathcal{N}^{\otimes n}, K) = 1\} \tag{1.20}$$

where K is the dimension of the system to be sent, $K = \lfloor 2^{rn} \rfloor$, r is the rate of transmission and n is the number of uses of a channel.

1.7 Final setup for assisted quantum communication

Relying on definitions stated in preceding sections, we are ready to describe the setup for assisted quantum communication.

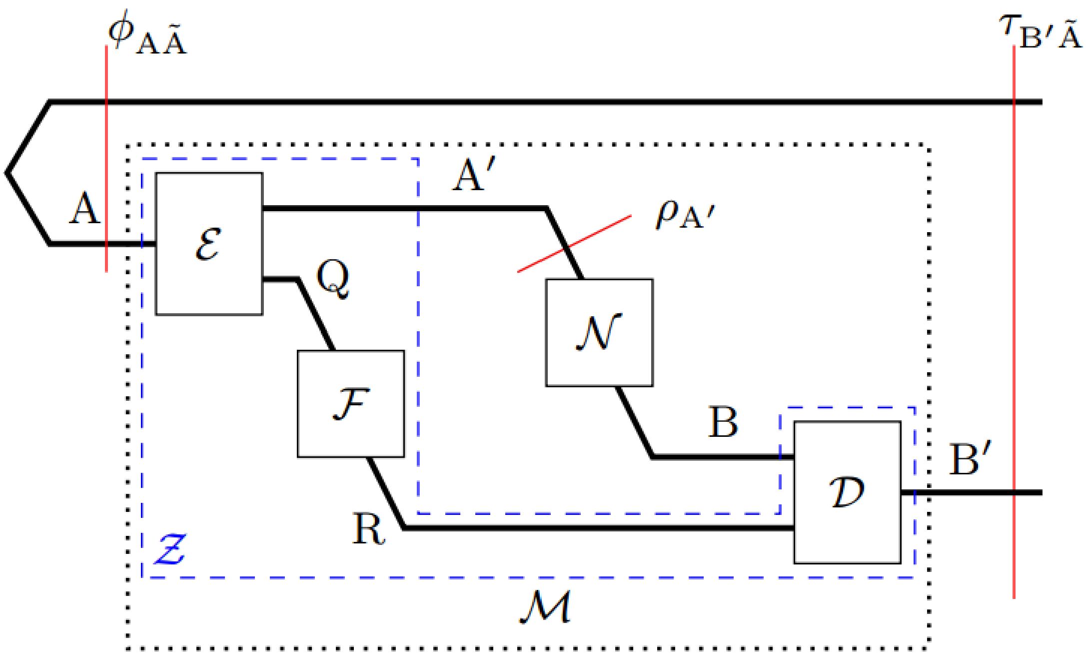


Figure 1.2: Channel setup (graphics courtesy of the authors of [1])

The considered transmission setup is depicted in Fig. 1.2. The channel itself is represented by the operation $\mathcal{N}_{B\leftarrow A'}$. However, since we allow an assistance for the transmission, the additional channel with operation $\mathcal{F}$ is introduced. The input to the whole setup undergoes an encoding operation $\mathcal{E}$ and the output is produced after a decoding operation $\mathcal{D}$. The presence of an encoding procedure means that there will be some average input state to the channel itself, denoted by $\rho_{A'}$. The part consisting of quantum operations $\mathcal{E}$, $\mathcal{F}$ and $\mathcal{D}$ are considered as a bipartite operation $\mathcal{Z}_{A'B'\leftarrow AB}$. In other words it is a deterministic supermap that takes $\mathcal{N}_{B\leftarrow A'}$ to $\mathcal{M}_{B'\leftarrow A}$. In this scenario, the average input to the channel $\mathcal{N}$ can be expressed as:

$$\rho_{A'} = \frac{\mathrm{Tr}_{ABB'} Z_{A'B'\leftarrow AB}}{\dim(A)\dim(B)} \tag{1.21}$$

The deterministic supermap $\mathcal{Z}_{A'B'\leftarrow AB}$ in Fig. 1.2 represents a class of so called 'forward-assisted codes'. However, since it includes a Quantum Identity Channel for $\mathcal{F}$, which would allow a perfect transmission even without using $\mathcal{N}$ we shall restrict the class to operationally sensible categories. In this thesis, we will focus on classes that are PPTp, NS or both (as defined in Section 1.5.1). We will consider a 'one-shot' regime which assumes no feedback in the coding protocol so that we can treat multiple channel uses as a single use of a larger channel.

1.8 Zero-capacity problem

The goal of this thesis is to solve the problem of finding the zero-assisted-capacity threshold for the Quantum Depolarizing Channel in the asymptotic regime. In other words, with reference to the definition of the channel in (1.3.7) and the definition of the asymptotic quantum assisted capacity in (1.20), the problem can be stated as follows:

Problem 1.8.1.

$$\text{Find the value of } p^\star : \quad \begin{cases} \lim_{n\to\infty} F^*(n,p,r) = 1, \text{ for } p < p^\star \text{ and } r > 0 \\ \lim_{n\to\infty} F^*(n,p,r) \neq 1, \text{ for } p \geq p^\star \text{ and } r > 0, \end{cases} \tag{1.22}$$

where $F^*(n,p,r)$ denotes the optimal fidelity that can be reached, subject to certain constraints regarding the quantum channel that are derived in the following chapters and $p^\star$ is the maximal depolarizing probability that satisfies the problem.

To investigate the zero-capacity problem we fix the rate of transmission r to be close to zero. For such a small rate of transmission through the channel, the trend of optimal fidelity converging to 1 with an increasing number of uses of the channel should be seen sooner. As a depolarizing probability p is being increased from 0, we expect to encounter a clear cutoff probability which separates smaller depolarizing probabilities which allow optimal fidelity convergence to 1 and those bigger ones which do not expose such a trend. Choosing the smallest depolarizing probability which does not expose the convergence to 1 gives us an expected upper bound on the depolarizing probability for which the channel has no asymptotic quantum assisted capacity.

Chapter 2

Semidefinite program for fidelity of the assisted transmission

In this chapter we give a brief introduction to the concept of semidefinite programming. Being equipped with its tools we present the method of stating the problem of finding the optimal fidelity of the assisted quantum communication setting from Fig. 1.2 as a semidefinite program. The method was originally described in [1].

2.1 Semidefinite programming

Definition 2.1.1 (Positive-semidefinite matrix)**.** A matrix M is a positive-semidefinite matrix if

$$x^{\star}Mx \geq 0 \tag{2.1}$$

for each $x \in \mathbb{C}^n$.

Semidefinite programming [9, 10] is a convex optimization method. It generalizes such well-known concepts as linear programming or quadratic programming for example. A person familiar with linear programming can think of a semidefinite programming as a 'linear program' where vectors are replaced with symmetric matrices and non-negativity constraints with constraints for being positive-semidefinite. More precisely, the semidefinite programming deals with optimizing a linear objective function over the intersection of an affine space with the cone of positive-semidefinite matrices.

Definition 2.1.2 (Semidefinite program)**.**

$$\text{minimize } \boldsymbol{C} \cdot \boldsymbol{X} \tag{2.2}$$

$$\text{subject to } \boldsymbol{A_i} \cdot \boldsymbol{X} = \boldsymbol{b_i} \text{ , i } = 1, \ldots, m \tag{2.3}$$

$$\boldsymbol{X} \succeq \boldsymbol{0}, \tag{2.4}$$

where $\boldsymbol{C}$ and $\boldsymbol{X}$ are symmetric matrices, $\boldsymbol{C} \cdot \boldsymbol{X} = \sum_{i=1}^{n} \sum_{j=1}^{n} C_{ij} X_{ij}$, $\boldsymbol{b}$ is a vector and the last constraint means that the matrix $\boldsymbol{X}$ is positive-semidefinite.

2.2 Derivation of the semidefinite program

We reproduce the derivation of the semidefinite program for optimal fidelity of the transmission through a quantum channel assisted by PPTp and NS codes presented in [1].

Looking at Fig. 1.2, we can see that the quantum operation $\mathcal{Z}_{A'B' \leftarrow AB}$ can be expressed mathematically as a composition of quantum operations $\mathcal{E}_{A'Q \leftarrow A}$, $\mathcal{F}_{R \leftarrow Q}$ and $\mathcal{D}_{B' \leftarrow RB}$:

$$\mathcal{Z}_{A'B' \leftarrow AB} = \mathcal{D}_{B' \leftarrow RB} \mathcal{F}_{R \leftarrow Q} \mathcal{E}_{A'Q \leftarrow A}. \tag{2.5}$$

Since the setup in Fig. 1.2 considers the operation $\mathcal{Z}_{A'B' \leftarrow AB}$ acting on a half of an entangled state $\phi_{A \prime \tilde{A}}$, it is useful to define a Choi matrix:

$$N_{RQ} = \dim(Q) \mathrm{id}_{Q \leftarrow \tilde{Q}} \mathcal{N}_{R \leftarrow Q} \phi_{\tilde{Q}Q}. \tag{2.6}$$

We note its property:

$$\mathcal{N}_{R \leftarrow Q} X_Q = \mathrm{Tr}_Q \, N_{RQ} t_{Q \leftarrow Q} X_Q = \mathrm{Tr}_Q (t_{Q \leftarrow Q} N_{RQ}) X_Q \tag{2.7}$$

and use it to express the operation $\mathcal{Z}_{A'B' \leftarrow AB}$ as:

$$Z_{A'B'AB} = \mathrm{Tr}_{QR} \, D_{B'BR} t_{R \leftarrow R} F_{RQ} t_{Q \leftarrow Q} E_{QA'A}. \tag{2.8}$$

Analogously, the deterministic supermap $\mathcal{M}_{B'A}$ can be expressed as:

$$\mathcal{M}_{B' \leftarrow A} = \mathcal{D}_{B' \leftarrow RB} \mathcal{F}_{R \leftarrow Q} \mathcal{N}_{B \leftarrow A'} \mathcal{E}_{A'Q \leftarrow A}. \tag{2.9}$$

With the Choi matrix representation as follows:

$$M_{B'A} = \mathrm{Tr}_{A'B} \, Z_{A'B'AB} N^T_{BA'}. \tag{2.10}$$

The channel fidelity is defined as:

$$f = K^{-1} \, \mathrm{Tr} \, \phi_{B'A} M_{B'A} = K^{-1} \, \mathrm{Tr} \, \phi_{B'A} Z_{A'B'AB} N^T_{BA'}. \tag{2.11}$$

It can be shown that without loss of generality we can focus on a symmetric representation of $Z_{A \prime B \prime AB}$. We make use of the property called a *transpose trick*:

$$X_Q \left|\phi\right\rangle_{\tilde{Q}Q} = X^T_{\tilde{Q}} \left|\phi\right\rangle_{\tilde{Q}Q}. \tag{2.12}$$

In the Choi formalism it would take the following form:

$$X_Q \phi_{\tilde{Q}Q} X_Q^\dagger = X_{\tilde{Q}}^T \phi_{\tilde{Q}Q} \bar{X}_{\tilde{Q}}. \tag{2.13}$$

Now, we note that by the transpose trick the following identity holds for unitary operators U_A and $U_{B'}$:

$$U_{B'}^\dagger U_A^T \phi_{B'A} \bar{U}_A U_{B'} = U_{B'}^\dagger U_{B'} \phi_{B'A} U_{B'}^\dagger U_{B'} = \phi_{B'A}. \tag{2.14}$$

We can take the unique Haar probability measure p on the unitary group $U(K)$ and express the channel fidelity as:

$$K^{-1} \operatorname{Tr} \phi_{B'A} Z_{A'B'AB} N_{BA'}^T = K^{-1} \operatorname{Tr} \int dp(U) U_{B'}^\dagger U_A^T \phi_{B'A} \bar{U}_A U_{B'} Z_{A'B'AB} N_{BA'}^T. \tag{2.15}$$

By cyclicity and linearity of the trace and commutativity of operators acting on separate systems, the equation above can be written as:

$$K^{-1} \operatorname{Tr} \phi_{B'A} Z_{A'B'AB} N_{BA'}^T = K^{-1} \operatorname{Tr} \phi_{B'A} \bar{Z}_{A'B'AB} N_{BA'}^T, \tag{2.16}$$

where

$$\bar{Z}_{A'B'AB} := \int dp(U) U_{B'} \bar{U}_A Z_{ABA'B'} U_{B'}^\dagger U_A^T. \tag{2.17}$$

This formula is named as the 'twirling' operation, i.e.:

$$\mathcal{T}_{B'A \leftarrow B'A} : X_{B'A} \to \int dp(U) U_{B'} \bar{U}_A X_{B'A} U_{B'}^\dagger U_A^T. \tag{2.18}$$

With the use of the twirling operation, we can write the following:

$$\bar{Z}_{A'B'AB} = \mathrm{id}_{BA' \leftarrow BA'} \mathcal{T}_{B'A \leftarrow B'A} Z_{A'B'AB}. \tag{2.19}$$

To conclude, we have shown, that the fidelity of a channel can be expressed as:

$$f = K^{-1} \operatorname{Tr} \phi_{B'A} \bar{Z}_{A'B'AB} N_{BA'}^T. \tag{2.20}$$

By equation (2.20) we see that considering $\bar{Z}_{A'B'AB}$ instead of $Z_{A'B'AB}$ does not change the fidelity of a channel. The new map stays PPTp and non-signalling from Alice to Bob provided that $Z_{A'B'AB}$ was so. It is the result of the fact that $Z_{A'B'AB}$ can be transformed into $\bar{Z}_{A'B'AB}$ by local operations and shared randomness. Assuming that Alice and Bob share a random variable which is classical and corresponds to an operator which belongs to a unitary group with a Haar probability measure p, it can be done in the following way: application of $U_A^\dagger$ by Alice to her input, usage of the forward assisted code ($\mathcal{Z}$), application of $U_{B'}$ to Bob's output. It corresponds to the bipartite operation of the form $\bar{\mathcal{Z}}_{A'B' \leftarrow AB}[\cdot] = \int dp(U) U_{B'} \mathcal{Z}_{A'B' \leftarrow AB}[U_A^\dagger \cdot U_A] U_{B'}^\dagger$. The form of $\bar{\mathcal{Z}}_{A'B' \leftarrow AB}$ can be obtained from equation (2.17). It can be proved by noticing that for any Choi matrix N_{RQ}, $W_Q N_{RQ} W_Q^\dagger$ corresponds to the map which conjugates the input by W^T before the action of $\mathcal{N}_{R\leftarrow Q}$ (by the transpose trick) and $W_R N_{RQ} W_R^\dagger$ corresponds to the map which conjugates by W_R after applying $\mathcal{N}_{R\leftarrow Q}$.

Based on results obtained in [11], the twirling operation can be also expressed as follows:

$$\mathcal{T}_{B'A\leftarrow B'A} : X_{B'A} \to \phi_{B'A} \operatorname{Tr} \phi_{B'A} X_{B'A} + \frac{I_{B'A} - \phi_{B'A}}{\operatorname{Tr}(I_{B'A} - \phi_{B'A})} \operatorname{Tr}(I_{B'A} - \phi_{B'A}) X_{B'A} \tag{2.21}$$

The symmetric map $\bar{Z}_{A'B'AB}$ lies in the image of $\mathrm{id}_{BA'\leftarrow BA'} \mathcal{T}_{B'A\leftarrow B'A}$ if and only if it has the form:

$$\bar{Z}_{A'B'AB} = K(\phi_{B'A}\Lambda_{A'B} + (I - \phi)_{B'A}\Gamma_{A'B}), \tag{2.22}$$

where $\Lambda_{A'B}$ and $\Gamma_{A'B}$‘ are certain operators.
The new symmetric form of $Z_{A'B'AB}$ allows us to update the expression for the average input to the channel $\mathcal{N}$ which was introduced in equation (1.21) to the following form:

$$\rho_{A'} = \frac{\operatorname{Tr}_{B'AB} \bar{Z}_{A'B'AB}}{K \dim(B)} = \frac{\Lambda_{A'} + (K^2 - 1)\Gamma_{A'}}{\dim(B)}, \tag{2.23}$$

where $\Lambda_{A'}$ and $\Gamma_{A'}$ are denoted with single subscripts to show the fact that the partial trace operation was applied.

Since the operation $\bar{Z}_{A'B'AB}$ corresponds to a physical process which is related to communication between Alice and Bob, we would like it to fulfill certain conditions which are summarized below:

- Complete positivity:
$$\bar{Z}_{A'B'AB} \geq 0. \tag{2.24}$$
- Trace preservation:
$$\operatorname{Tr}_{A'B'} \bar{Z}_{A'B'AB} = I_{AB}. \tag{2.25}$$
- Non-signalling from Bob to Alice:
$$\operatorname{Tr}_{B'} \bar{Z}_{A'B'AB} = \frac{\operatorname{Tr}_{B'B} \bar{Z}_{A'B'AB}}{\dim(B)}. \tag{2.26}$$
- PPT preservation:
$$\mathbf{t}_{BB'\leftarrow BB'} \bar{Z}_{A'B'AB} \geq 0. \tag{2.27}$$
- Non-signalling from Alice to Bob:
$$\operatorname{Tr}_{A'} \bar{Z}_{A'B'AB} = \frac{\operatorname{Tr}_{A'A} \bar{Z}_{A'B'AB}}{\dim(A)}. \tag{2.28}$$

Using equations (2.22) for $\bar{Z}_{A'B'AB}$ and (2.23) for $\rho_{A'}$, the above conditions can be expressed in terms of $\rho_{A'}$ and $\Lambda_{A'B}$ which results in an important theorem presented in the Section 2.3.

2.2.1 Fidelity of transmission

Fidelity of the channel defined in equation (3.4) can be expressed after the substitution of equation (2.22) as:

$$f_c = K^{-1} \operatorname{Tr} \phi_{B'A} \bar{Z}_{A'B'AB} N_{BA'}^T = K^{-1} \operatorname{Tr} \phi_{B'A} K(\phi_{B'A} \Lambda_{A'B} + (I-\phi)_{B'A} \Gamma_{A'B}) N_{BA'}^T. \tag{2.29}$$

By idempotency of $\phi_{B'A}$ and mutual orthogonality of $\phi_{B'A}$ and $(I-\phi)_{B'A}$ we obtain:

$$f_c = \operatorname{Tr} \phi_{B'A} \Lambda_{A'B} N_{BA'}^T = \operatorname{Tr}(\phi_{B'A} \otimes \Lambda_{A'B} N_{A'B}^T). \tag{2.30}$$

By distributive property of the trace over tensor product and cyclicity of the trace we obtain:

$$f_c = \operatorname{Tr} N_{A'B}^T \Lambda_{A'B}. \tag{2.31}$$

2.2.2 Non-signalling from Bob to Alice

We consider both sides of the non-signalling condition from equation (2.26):

$$\operatorname{Tr}_{B'} \bar{Z}_{A'B'AB} = \operatorname{Tr}_{B'} K(\phi_{B'A} \Lambda_{A'B} + (I-\phi)_{B'A} \Gamma_{A'B}) = \Lambda_{A'B} + (K^2-1)\Gamma_{A'B}, \tag{2.32}$$

$$\frac{\operatorname{Tr}_{B'B} \bar{Z}_{A'B'AB}}{\dim(B)} = \frac{\Lambda_{A'} + (K^2-1)\Gamma_{A'}}{\dim(B)}. \tag{2.33}$$

Therefore:

$$\Lambda_{A'B} + (K^2-1)\Gamma_{A'B} = \frac{\Lambda_{A'} + (K^2-1)\Gamma_{A'}}{\dim(B)}. \tag{2.34}$$

We recall equation (1.21) for the average input $\rho_{A'}$ to see that:

$$\Lambda_{A'B} + (K^2-1)\Gamma_{A'B} = \rho_{A'}. \tag{2.35}$$

Equation (2.35), as just shown, is equivalent to the condition that there is non-signalling from Bob to Alice. To reduce the number of unknown operators and since $\Gamma_{A'B}$ disappeared in the fidelity expression in equation (2.31), we will use equation (2.35) onwards to eliminate $\Gamma_{A'B}$ from other conditions. Thus, the following will be useful:

$$\Gamma_{A'B} = \frac{\rho_{A'} - \Lambda_{A'B}}{K^2-1} \tag{2.36}$$

.

2.2.3 Trace preservation

We use the condition from equation (2.25) for trace preservation together with the equation (2.22) to obtain:

$$\mathrm{Tr}_{A'B'}\,\bar{Z}_{A'B'AB} = \mathrm{Tr}_{A'B'}\,K(\phi_{B'A}\Lambda_{A'B} + (I-\phi)_{B'A}\Gamma_{A'B}) = \Lambda_{A'} + (K^2-1)\Gamma_{A'}. \quad (2.37)$$

Therefore, we also note that:

$$\Lambda_{A'} + (K^2-1)\Gamma_{A'} = I_B. \quad (2.38)$$

Equation (2.38) can be obtained by a partial trace over A' of the equation (2.26).

2.2.4 Complete positivity

As stated in equation (2.2), the map $\bar{Z}_{A'B'AB}$ must be complete positive. Based on the equation (2.22) $\bar{Z}_{A'B'AB}$ consists of $\Lambda_{A'B}$, $\Gamma_{A'B}$, $\phi_{B'A}$ and $(I-\phi)_{B'A}$. Last two are positive-semidefinite, therefore we must have:

$$\Lambda_{A'B} \geq 0, \quad (2.39)$$

$$\Gamma_{A'B} = \frac{\rho_{A'} - \Lambda_{A'B}}{K^2-1} \geq 0 \implies \Lambda_{A'B} \leq \rho_{A'}. \quad (2.40)$$

2.2.5 Non-signalling from Alice to Bob

We consider the right hand side of the non-signalling condition from equation (2.28):

$$\frac{\mathrm{Tr}_{A'A}\,\bar{Z}_{A'B'AB}}{\dim(A)} = \frac{\mathrm{Tr}_{A'A}\,K(\phi_{B'A}\Lambda_{A'B} + (I-\phi)_{B'A}\Gamma_{A'B})}{K} = \frac{\Lambda_B + (K^2-1)\Gamma_B}{K}. \quad (2.41)$$

We use equation (2.38) to obtain:

$$\frac{\mathrm{Tr}_{A'A}\,\bar{Z}_{A'B'AB}}{\dim(A)} = \frac{I_B}{K}. \quad (2.42)$$

Considering the left hand side of of the non-signalling condition from equation (2.28):

$$\mathrm{Tr}_{A'}\,\bar{Z}_{A'B'AB} = \mathrm{Tr}_{A'}\,K(\phi_{B'A}\Lambda_{A'B} + (I-\phi)_{B'A}\Gamma_{A'B}) = K(\phi_{B'A}\Lambda_B + (I-\phi)_{B'A}\Gamma_B), \quad (2.43)$$

we obtain the following condition:

$$K(\phi_{B'A}\Lambda_B + (I-\phi)_{B'A}\Gamma_B) = \frac{I_B}{K}. \quad (2.44)$$

Using equation (2.38) to substitute for Γ_B, we arrive at:

$$K^2 \left(\phi_{B'A} \Lambda_B + (I - \phi)_{B'A} \frac{I_B - \Lambda_B}{K^2 - 1} \right) = \phi_{B'A} + (I - \phi)_{B'A}, \tag{2.45}$$

where we also expanded the identity on the right hand side. Since $\phi_{B'A}$ and $(I-\phi)_{B'A}$ are orthogonal, we can compare both sides of the equation above to discover that Λ_B is equal to:

$$\Lambda_B = \frac{I_B}{K^2}. \tag{2.46}$$

2.2.6 PPT preservation

In deriving the PPT preservation condition, we will use the fact that the partial transpose of $\phi_{B'A}$ is given by the following equations that use projectors onto symmetric ($S_{B'A}$) and antisymmetric ($A_{B'A}$) subspaces:

$$\mathbf{t}_{B' \leftarrow B'} \phi_{B'A} = \frac{S_{B'A} - A_{B'A}}{K}, \tag{2.47}$$

$$S_{B'A} + A_{B'A} = I_{B'A}. \tag{2.48}$$

we substitute for the map $\bar{Z}_{A'B'AB}$ in the left hand side of the PPT preserving condition from equation (2.27):

$$\begin{aligned}
&\mathbf{t}_{BB' \leftarrow BB'} \bar{Z}_{A'B'AB} = K \mathbf{t}_{B \leftarrow B} \left[\mathbf{t}_{B' \leftarrow B'} \phi_{B'A} \Lambda_{A'B} + \mathbf{t}_{B' \leftarrow B'} (I - \phi)_{B'A} \Gamma_{A'B} \right] = \\
&K \mathbf{t}_{B \leftarrow B} \left[\frac{S_{B'A} - A_{B'A}}{K} \Lambda_{A'B} + \Gamma_{A'B} - \frac{S_{B'} - A_{B'A}}{K} \Gamma_{A'B} \right] = \\
&\mathbf{t}_{B \leftarrow B} \left[S_{B'A} \Lambda_{A'B} - A_{B'A} \Lambda_{A'B} + K \Gamma_{A'B} - S_{B'A} \Gamma_{A'B} + A_{B'A} \Gamma_{A'B} \right] = \\
&S_{B'A} (\mathbf{t}_{B \leftarrow B} \left[\Lambda_{A'B} + (K-1) \Gamma_{A'B} \right]) + A_{B'A} (\mathbf{t}_{B \leftarrow B} \left[-\Lambda_{A'B} + (K+1) \Gamma_{A'B} \right]).
\end{aligned} \tag{2.49}$$

Therefore, we obtain the condition:

$$S_{B'A} (\mathbf{t}_{B \leftarrow B} \left[\Lambda_{A'B} + (K-1) \Gamma_{A'B} \right]) + A_{B'A} (\mathbf{t}_{B \leftarrow B} \left[-\Lambda_{A'B} + (K+1) \Gamma_{A'B} \right]) \geq 0. \tag{2.50}$$

Since $S_{B'A}$ and $A_{B'A}$ are orthogonal projectors, the following conditions must be satisfied to fulfill the condition above:

$$\mathbf{t}_{B \leftarrow B} \left[\Lambda_{A'B} + (K-1) \Gamma_{A'B} \right] \geq 0, \tag{2.51}$$

$$\mathbf{t}_{B \leftarrow B} \left[-\Lambda_{A'B} + (K+1) \Gamma_{A'B} \right] \geq 0. \tag{2.52}$$

If we further evaluate the expression inside the partial transposition from equation (2.51), then by using equation (2.36) we obtain the following:

$$\begin{aligned}
&\Lambda_{A'B} + (K-1)\Gamma_{A'B} = \Lambda_{A'B} + (K-1)\frac{\rho_{A'} - \Lambda_{A'B}}{K^2-1} = \\
&\Lambda_{A'B} + \frac{\rho_{A'} - \Lambda_{A'B}}{K+1} = \frac{K\Lambda_{A'B} + \Lambda_{A'B} + \rho_{A'} - \Lambda_{A'B}}{K+1} = \\
&\frac{K\Lambda_{A'B} + \rho_{A'}}{K+1}.
\end{aligned} \tag{2.53}$$

Therefore, the condition from equation (2.51) can be written as:

$$\mathbf{t}_{B\leftarrow B}\left[\Lambda_{A'B} + (K-1)\Gamma_{A'B}\right] = \mathbf{t}_{B\leftarrow B}\left[\frac{K\Lambda_{A'B} + \rho_{A'}}{K+1}\right] \geq 0. \tag{2.54}$$

Thus, the final form of the condition from equation (2.51) is:

$$\mathbf{t}_{B\leftarrow B}\left[\Lambda_{A'B}\right] \geq \frac{-\rho_{A'}}{K}. \tag{2.55}$$

Analogously, for the condition from equation (2.52), we can show that the following condition:

$$\mathbf{t}_{B\leftarrow B}\left[\Lambda_{A'B}\right] \leq \frac{\rho_{A'}}{K} \tag{2.56}$$

must hold.

2.3 Semidefinite program - final form

To express the problem of finding the optimal fidelity of the quantum depolarizing channel as a semidefinite program we use the Theorem 3 from [1] which states:

Theorem 2.3.1. There is a forward-assisted code of size K, average channel input $\rho_{A\prime}$ and channel fidelity f_c for $N_{B_i \leftarrow A_i'}$ which is PPT preserving and/or non-signaling from Alice to Bob if and only if there exists an operator $\Lambda_{A'B}$ such that:

$$f_c = \operatorname{Tr} N^T_{A'B}\Lambda_{A'B}, \tag{2.57}$$

$$\Lambda_{A'B} \leq \rho_{A'} I_B, \tag{2.58}$$

$$\Lambda_{A'B} \geq 0, \tag{2.59}$$

$$NS : \Lambda_B = \frac{I_B}{K^2}, \tag{2.60}$$

$$PPTp: \begin{cases} t_{B\leftarrow B}\left[\Lambda_{A'B}\right] \geq -\frac{\rho_{A'}I_B}{K} \\ t_{B\leftarrow B}\left[\Lambda_{A'B}\right] \geq \frac{\rho_{A'}I_B}{K}. \end{cases} \tag{2.61}$$

The semidefinite program written above describes any quantum channel represented by the Choi matrix $N^T_{A'B}$. By deriving the Choi matrix for the quantum depolarizing channel we will show that the semidefinite program can be transformed into the linear program. NS and PPTp constraints correspond to our interest in the non-signalling and positive-partial-transpose-preserving subclasses of forward-assisted codes.

Chapter 3

Linear program for the fidelity of the assisted transmission

This chapter starts with a brief introduction to the idea of linear programming. Then, we present the method of reducing the semidefinite program stated in Chapter 2.3 to a linear program for the optimal fidelity of the assisted transmission through the Quantum Depolarizing Channel which is the contribution of this thesis. Each component of the semidefinite program is transformed to a linear programming component in a separate section and step by step. Linear programs can be solved much more efficiently than semidefinite programs, therefore it is an important step which will allow us to obtain solutions for much bigger instances of the problem.

3.1 Linear programming

Linear programming is a method of finding an optimal solution to a problem which is formulated in terms of an objective function to be optimized and constraints which both are linear with respect to the set of variables that we optimize over. More formally, the linear program can be defined in a canonical form.

Definition 3.1.1 (Linear program).

$$\text{maximize } \mathbf{c}^T\mathbf{x} \tag{3.1}$$

$$\text{subject to } \mathbf{A}\mathbf{x} \leq \mathbf{b} \tag{3.2}$$

$$\text{and } \mathbf{x} \geq \mathbf{0} \tag{3.3}$$

where $\mathbf{c}$ and $\mathbf{b}$ are vectors of coefficients, $\mathbf{A}$ is a matrix of coefficients and $\mathbf{x}$ is a vector which contains variables.

Linear programs are usually solved computationally by linear solvers which implement algorithms such as Simplex or Interior Point Method. An important feature of linear solvers is that if there exists a bounded optimal solution to the linear program, they will provide it with guarantees for being optimal (given enough computational resources).

3.2 Fidelity of transmission

We will linearize the expression for the fidelity of a transmission through a quantum channel assisted by PPTp and NS codes which is given by:

$$f_c = \operatorname{Tr} N_{A'B}^T \Lambda_{A'B} \tag{3.4}$$

for the Quantum Depolarizing Channel.
The linear map $N_{B_i \leftarrow A_i'}$ for the quantum depolarizing channel is given as follows:

$$N_{B_i \leftarrow A_i'} = p\frac{I}{2} + (1-p)\rho, \tag{3.5}$$

$$N_{B_i \leftarrow A_i'}^M = p\frac{I}{2} \operatorname{Tr} M + (1-p)M. \tag{3.6}$$

We recall that the Choi matrix representation is given by:

$$N_{A_i'B_i} = \dim(A_i') N_{B_i \leftarrow A_i'} \Phi_{A_i'B_i} = 2N_{B_i \leftarrow A_i'} \Phi_{A_i'B_i}, \tag{3.7}$$

where $\Phi_{A_i'B_i}$ is the maximally entangled state on two 2-dimensional systems A_i' and B:

$$\Phi_{A_i'B_i} = \frac{1}{2} \sum_{k,j=0}^{1} |k\rangle \langle j| \otimes |k\rangle \langle j|. \tag{3.8}$$

Substituting equations (3.6) and (3.8) into equation (3.7) we end up with the following expressions:

$$N_{A_i'B_i} = \sum_{k,j=0}^{1} |k\rangle \langle j| \otimes N_{B_i \leftarrow A_i'}(|k\rangle \langle j|) = \sum_{k,j=0}^{1} |k\rangle \langle j| \otimes \left[p\frac{I}{2} \operatorname{Tr} |k\rangle \langle j| + (1-p) |k\rangle \langle j|\right], \tag{3.9}$$

$$N_{A_i'B_i} = \sum_{k=0}^{1} |k\rangle \langle k| \otimes p\frac{I}{2} + (1-p) \dim(A_i') \Phi_{A_i'B_i} = 2\left[p\frac{I}{4} + (1-p)\Phi_{A_i'B_i}\right]. \tag{3.10}$$

Since we generally consider n uses of the channel (n channels) we need the composite Choi matrix:

$$N_{A'B} = \left[2(p\frac{I}{4} + (1-p)\Phi_{A_i'B_i})\right]^{\otimes n}. \tag{3.11}$$

To calculate the transpose of the Choi matrix the following identities are used:

$$I^T = I, \tag{3.12}$$

$$\Phi^T_{A'_iB_i} = \Phi_{A'_iB_i}, \tag{3.13}$$

We obtain:

$$N^T_{A'B} = \left[2(p\frac{I}{4} + (1-p)\Phi_{A'_iB_i})\right]^{\otimes n}. \tag{3.14}$$

By recalling that $(I - \Phi)$ and Φ are orthogonal projectors we simplify the above equation as follows:

$$N^T_{A'B} = 2^n \left[\frac{p}{4}(I-\Phi)_{A'_iB_i} + \frac{p}{4}\Phi_{A'_iB_i} + (1-p)\Phi_{A'_iB_i}\right]^{\otimes n}, \tag{3.15}$$

$$N^T_{A'B} = 2^n \left[\frac{p}{4}(I-\Phi)_{A'_iB_i} + \left(1 - \frac{3p}{4}\right)\Phi_{A'_iB_i}\right]^{\otimes n}, \tag{3.16}$$

$$N^T_{A'B} = 2^n \sum_{j=0}^{n} \left(\frac{p}{4}\right)^{n-j} \left(1 - \frac{3p}{4}\right)^j \Upsilon^n_j, \tag{3.17}$$

where Υ^n_j is the sum of all n-fold tensor products of the projectors $(I - \Phi)_{A'_iB_i}$ and $\Phi_{A'_iB_i}$ which contain exactly j copies of $\Phi_{A'_iB_i}$.

The operator $\Lambda_{A'B}$ must be a linear combination of $(n+1)$ orthogonal projectors which we choose to be Υ^n_k (defined above):

$$\Lambda_{A'B} = \sum_{k=0}^{n} x_k \Upsilon^n_k. \tag{3.18}$$

Using this definition we calculate the following product:

$$N^T_{A'B}\Lambda_{A'B} = 2^n \sum_{j=0}^{n} \left(\frac{p}{4}\right)^{n-j} \left(1 - \frac{3p}{4}\right)^j x_j \Upsilon^n_j \tag{3.19}$$

and we take its trace:

$$\mathrm{Tr}\, N^T_{A'B}\Lambda_{A'B} = 2^n \sum_{j=0}^{n} \left(\frac{p}{4}\right)^{n-j} \left(1 - \frac{3p}{4}\right)^j x_j \binom{n}{n-j} 3^{n-j}. \tag{3.20}$$

To derive the above expression we used the following properties:

$$\mathrm{Tr}\, X \otimes Y = \mathrm{Tr}\, X\, \mathrm{Tr}\, Y, \tag{3.21}$$

$$\mathrm{Tr}(I - \Phi)_{A'_iB_i} = 3, \tag{3.22}$$

$$\mathrm{Tr}\, \Phi_{A'_iB_i} = 1, \tag{3.23}$$

$$(I-\Phi)_{A_i'B_i}\Phi_{A_i'B_i}=0, \tag{3.24}$$

$$\Phi_{A_i'B_i}\Phi_{A_i'B_i}=\Phi_{A_i'B_i}. \tag{3.25}$$

The formula for the optimal fidelity of the depolarizing channel which is linear in variables x_j takes the form:

$$f_c=2^n\sum_{j=0}^{n}\left(\frac{3p}{4}\right)^{n-j}\left(1-\frac{3p}{4}\right)^{j}\binom{n}{n-j}x_j. \tag{3.26}$$

3.3 Decision variables conditions

Conditions from equations (2.58) and (2.59) transform into constraints on variables x_j themselves. We consider the condition from equation (2.59):

$$\Lambda_{A'B}\geq 0, \tag{3.27}$$

which means that the operator $\Lambda_{A'B}$ must be a positive semidefinite matrix. The form of the $\Lambda_{A'B}$ operator is given by:

$$\Lambda_{A'B}=\sum_{k=0}^{n}x_k\Upsilon_k^n. \tag{3.28}$$

Since Υ_k^n are eigenvectors of $\Lambda_{A'B}$, x_k are eigenvalues of of $\Lambda_{A'B}$ which enforces them all to be non-negative so that the $\Lambda_{A'B}$ is positive semidefinite:

$$x_k\geq 0. \tag{3.29}$$

To the condition from equation (2.58):

$$\Lambda_{A'B}\leq\rho_{A'}I_B \tag{3.30}$$

we substitute the following form of the input state:

$$\rho_{A'}=\frac{I_{A'}}{2^n}, \tag{3.31}$$

which leads to:

$$\Lambda_{A'B}\leq\frac{I_{A'B}}{2^n}, \tag{3.32}$$

$$\frac{I_{A'B}}{2^n}-\Lambda_{A'B}\geq 0. \tag{3.33}$$

We express the identity matrix as a sum of orthogonal projectors:

$$I_{A_i'B_i}=\Phi_{A_i'B_i}+(I-\Phi)_{A_i'B_i} \tag{3.34}$$

and calculate $I_{A'B}$:

$$I_{A'B} = (I_{A'_iB_i})^{\otimes n} = (\Phi_{A'_iB_i} + (I-\Phi)_{A'_iB_i})^{\otimes n} = \sum_{k=0}^{n} \Upsilon_k^n, \tag{3.35}$$

$$2^{-n}\sum_{k=0}^{n} \Upsilon_k^n - \sum_{k=0}^{n} x_k \Upsilon_k^n \geq 0. \tag{3.36}$$

Comparing coefficients standing by each term of index k (k-th eigenvalue) we obtain:

$$2^{-n} - x_k \geq 0, \tag{3.37}$$

$$x_k \leq 2^{-n}. \tag{3.38}$$

Both conditions can be written as:

$$\forall k = 0, ..., n \quad 0 \leq x_k \leq 2^{-n}. \tag{3.39}$$

3.4 PPTp conditions

We consider the condition:

$$t_{B\leftarrow B}\left[\Lambda_{A'B}\right] \geq \frac{-\rho_{A'} I_B}{K}. \tag{3.40}$$

Firstly, let's calculate the left hand side of it which is $t_{B\leftarrow B}\left[\Lambda_{A'B}\right]$. We recall the definition of $\Lambda_{A'B}$:

$$\Lambda_{A'B} = \sum_{j=0}^{n} x_j \Upsilon_j^n. \tag{3.41}$$

To calculate the partial transpose of $\Lambda_{A'B}$ we need to consider partial transposes of constituent parts of Υ_j^n which are $\Phi_{A'_iB_i}$ and $(I-\Phi)_{A'_iB_i}$:

$$t_{B_i\leftarrow B_i}\Phi_{A'_iB_i} = \frac{S-A}{2}, \tag{3.42}$$

$$t_{B_i\leftarrow B_i}(I-\Phi)_{A'_iB_i} = t_{B_i\leftarrow B_i} I_{A'_iB_i} - t_{B_i\leftarrow B_i}\Phi_{A'_iB_i} = I - \frac{1}{2}(S-A). \tag{3.43}$$

Using the completeness relation for I we obtain:

$$t_{B_i\leftarrow B_i}(I-\Phi)_{A'_iB_i} = S + A - \frac{1}{2}S + \frac{1}{2}A = \frac{1}{2}S + \frac{3}{2}A = \frac{S+3A}{2}. \tag{3.44}$$

From equations above, the following relation can be found:

$$t_{B_i \leftarrow B_i} \Upsilon_j^n = 2^{-n} \sum_{v \in S_k^n} \bigotimes_{i=1}^{n} (S + (-a)^{v_i} C_{v_i} A), \tag{3.45}$$

where S_k^n is the subset of strings in $\{0,1\}^n$ which have exactly k 0's; $C_0 = 1$, $C_1 = 3$. In binary vectors from S_k^n we associate 0 with $\Phi_{A_i'B_i}$ and 1 with $(I - \Phi)_{A_i'B_i}$.

Let's introduce a projector E_j^n which is the sum of all n-fold tensor products of the projectors S and A which contain exactly j copies of A. Then, the partial transpose of $\Lambda_{A'B}$ can be expressed in the form:

$$t_{B \leftarrow B} \Lambda_{A'B} = \sum_{k,j=0}^{n} M_{kj} E_j^n x_k. \tag{3.46}$$

Each element E_j^n is generated by elements of Υ_k^n with l zeros sharing indexes with l of j A's and $(k-l)$ zeroes sharing indexes with $(k-l)$ of $(n-j)$ S's. Therefore:

$$t_{B \leftarrow B} \Lambda_{A'B} = 2^{-n} \sum_{k=0}^{n} \sum_{j=0}^{n} \sum_{l=0}^{j} (-1)^l 3^{j-l} \binom{j}{l} \binom{n-j}{k-l} E_j^n x_k. \tag{3.47}$$

Using the equation above and the fact that $\rho_{A'}$ is the maximally mixed state of n two-dimensional systems, we obtain the following form of the PPTp condition from equation (3.40):

$$\sum_{k=0}^{n} \sum_{j=0}^{n} \sum_{l=0}^{j} (-1)^l 3^{j-l} \binom{j}{l} \binom{n-j}{k-l} E_j^n x_k \geq \frac{-I_{A'B}}{K} \tag{3.48}$$

By the completeness relation for $I_{A'B}$ we obtain:

$$\frac{\sum_{j=0}^{n} E_j^n}{K} + \sum_{k=0}^{n} \sum_{j=0}^{n} \sum_{l=0}^{j} (-1)^l 3^{j-l} \binom{j}{l} \binom{n-j}{k-l} E_j^n x_k \geq 0. \tag{3.49}$$

For the expression to be positive semidefinite, the following conditions for respective coefficients must hold:

$$\forall j = 0, ..., n \quad \frac{1}{K} + \sum_{k=0}^{n} \sum_{l=0}^{j} (-1)^l 3^{j-l} \binom{j}{l} \binom{n-j}{k-l} x_k \geq 0 \tag{3.50}$$

Analogously, we consider the second PPTp condition:

$$t_{B \leftarrow B} \left[\Lambda_{A'B}\right] \leq \frac{\rho_{A'} I_B}{K}, \tag{3.51}$$

$$\sum_{k=0}^{n} \sum_{j=0}^{n} \sum_{l=0}^{j} (-1)^l 3^{j-l} \binom{j}{l} \binom{n-j}{k-l} E_j^n x_k \leq \frac{I_{A'B}}{K}, \tag{3.52}$$

$$\frac{\sum_{j=0}^{n} E_j^n}{K} - \sum_{k=0}^{n}\sum_{j=0}^{n}\sum_{l=0}^{j}(-1)^l 3^{j-l}\binom{j}{l}\binom{n-j}{k-l}E_j^n x_k \geq 0. \tag{3.53}$$

It gives us the following conditions for respective coefficients:

$$\forall j = 0, ..., n \quad \frac{1}{K} - \sum_{k=0}^{n}\sum_{l=0}^{j}(-1)^l 3^{j-l}\binom{j}{l}\binom{n-j}{k-l}x_k \geq 0. \tag{3.54}$$

3.5 NS condition

We consider the condition for non-signaling:

$$\Lambda_B = \frac{I_B}{K^2}. \tag{3.55}$$

The operator $\Lambda_{A'B}$ can be expressed as:

$$\Lambda_{A'B} = \sum_{j=0}^{n} x_j \Upsilon_j. \tag{3.56}$$

It can be transformed into Λ_B by taking the partial trace:

$$\mathrm{Tr}_{A'}\, \Lambda_{A'B} = \Lambda_B. \tag{3.57}$$

To calculate the partial trace of $\Lambda_{A'B}$ we need to calculate the partial traces of the constituent parts of the tensor product Υ_j which are $\Phi_{A_i'B_i}$ and $(I-\Phi)_{A_i'B_i}$. We obtain:

$$\mathrm{Tr}_{A'}\, \Phi_{A_i'B_i} = \frac{I_B}{2}, \tag{3.58}$$

$$\mathrm{Tr}_{A'}(I-\Phi)_{A_i'B_i} = \frac{3I_B}{2}. \tag{3.59}$$

By those equalities it is clear that the partial trace of Υ_j is proportional to I_B:

$$\mathrm{Tr}_{A'}\, \Upsilon_j = g_j I_B. \tag{3.60}$$

Where g_j, by counting coefficients, takes the form:

$$g_j = 2^{-n}\binom{n}{j}3^{n-j}1^j = 2^{-n}\binom{n}{j}3^{n-j}. \tag{3.61}$$

To conclude, the partial trace of $\Lambda_{A'B}$ can be written as:

$$\Lambda_B = \mathrm{Tr}_{A'}\, \Lambda_{A'B} = \sum_{j=0}^{n} g_j x_j I_B. \tag{3.62}$$

It gives the following non-signaling condition:

$$\sum_{j=0}^{n} g_j x_j = \frac{1}{K^2}. \tag{3.63}$$

3.6 Depolarizing channel - linear program

By collecting equations derived in this chapter, we can write the linear program as follows:

$$\max f_c = 2^n \sum_{j=0}^{n} \left(\frac{3p}{4}\right)^{n-j} \left(1-\frac{3}{4}p\right)^{j} \binom{n}{j} x_j, \tag{3.64}$$

subject to

$$\forall j = 0, ..., n \quad 0 \le x_j \le 2^{-n}, \tag{3.65}$$

with the additional constraint

$$NS : \sum_{j=0}^{n} 2^{-n} \binom{n}{j} 3^{n-j} x_j = \frac{1}{K^2}, \tag{3.66}$$

if the code is non-signaling, and the constraint

$$PPTp : \forall j = 0...n \quad \begin{cases} \sum_{k=0}^{n} \sum_{l=0}^{j} (-1)^l 3^{j-l} \binom{j}{l} \binom{n-j}{k-l} x_k \le \frac{1}{K} \\ \sum_{k=0}^{n} \sum_{l=0}^{j} (-1)^l 3^{j-l} \binom{j}{l} \binom{n-j}{k-l} x_k \ge -\frac{1}{K}, \end{cases} \tag{3.67}$$

if the code is PPTp.

For convenience we resize the linear program such that $0 \le x_i \le 1$. By the substitution $x_i = \widetilde{x_i} 2^{-n}$ the final form of the linear program is:

$$\max f_c = \sum_{j=0}^{n} \left(\frac{3p}{4}\right)^{n-j} \left(1-\frac{3}{4}p\right)^{j} \binom{n}{n-j} x_j, \tag{3.68}$$

subject to

$$\forall j = 0, ..., n \quad 0 \le x_j \le 1, \tag{3.69}$$

with the additional constraint

$$NS : \sum_{j=0}^{n} 2^{-n} \binom{n}{j} 3^{n-j} x_j = \frac{2^n}{K^2}, \tag{3.70}$$

if the code is non-signaling, and the constraint

$$PPTp : \forall j = 0...n \quad \begin{cases} \sum_{k=0}^{n} \sum_{l=0}^{j} (-1)^l 3^{j-l} \binom{j}{l} \binom{n-j}{k-l} x_k \le \frac{2^n}{K} \\ \sum_{k=0}^{n} \sum_{l=0}^{j} (-1)^l 3^{j-l} \binom{j}{l} \binom{n-j}{k-l} x_k \ge -\frac{2^n}{K}, \end{cases} \tag{3.71}$$

if the code is PPTp.

Chapter 4
Results

In this chapter we present results produced by our Mathematica implementation of the linear program stated in Section 3.6. The linear program is solved for various set of parameters to obtain corresponding optimal fidelities. With these solutions we can address the zero-capacity problem stated in Section 1.8. We present numerical results obtained from the linear program and then we try to treat the problem in an analytic approach to prove trends that appeared in our numerical results.

4.1 Choice of parameters

When solving a linear program, we have the following degrees of freedom:

- depolarizing probability p
- rate of transmission r
- number of uses of the channel n

Ideally, we are interested in a regime where $r \to 0$. It means that we only want to transmit an infinitesimal amount of information through a channel. If the fidelity of such an infinitesimal transmission does not converge to 1 as the number of uses of a channel tends to infinity, we see that we approach the zero-capacity threshold. In a numerical case, we can only assign a fixed value to r. We started by fixing the rate of transmission r to be an arbitrary small number, chosen to be $\log_2\left(\frac{61}{60}\right) \approx 0.02385$. Since the quantum capacity in equation (1.20) is defined in the regime of $n \to \infty$, what we can do numerically is to consider cases for an increasing n, as far as our computational resources allow. We managed to compute instances for the probabilities of our interest up to $n = 200$. Biggest instances of the problem required several days of computation to be solved.
The aforementioned probabilities of interest were recognized by running small instances of the problem for a vast array of p's. We noticed that for $p \lesssim 0.5$, there was a clear trend of converging to the optimal fidelity of 1, whereas for $p \gtrsim 0.5$, there was a clear decreasing trend towards the optimal fidelity of 0. Therefore, we

focused on the range of $p \in [0.49, 0.56]$ with a step of 0.0025 and conducted extensive computations for that particular range.

4.2 Numerical results

In this section we present our numerical results regarding the optimal fidelity of the assisted transmission through the Quantum Depolarizing Channel and regarding the distribution of optimal values of decision variables.

4.2.1 Optimal fidelity of transmission

Results for optimal fidelities of the assisted transmission through the Quantum Depolarizing Channel obtained by solving the linear program for parameters described in Section 4.1 are presented in Fig. 4.1 in the logarithmic scale. Based on these results, we suspect that the cutoff depolarizing probability should be around 0.515 which is much higher than known estimates for the non-assisted communication via the Quantum Depolarizing Channel. As presented in [12], the Quantum Depolarizing Channel exhibits positive unassisted quantum capacity when the depolarizing probability $p < \frac{1}{3}$. Even assistance by one-way classical communication does not increase that threshold.
The perfect optimal fidelities obtained for the small values of n, visible in Fig. 4.1, are a consequence of the trivial dimension of the system to be sent which was defined as $K = \lfloor 2^{rn} \rfloor$. In our setting, $K = 1$ for $n \leq 41$. This region is not meaningful operationally and should be excluded from our considerations.
The chainsaw shape of the plot is a result of the sudden changes in the dimension K of the message to be sent as n changes. Because of the floor function in the definition of K, the size remains constant in a certain range of an increasing number of uses of the channel n. Naturally, having more uses available for the same size of the message, we can achieve better fidelities as n increases. Once the dimension of the system suddenly grows, the fidelity drops.

4.2.2 Distribution of the values of decision variables

By solving the linear program it is possible to obtain, apart from the optimal solution, values of the decision variables that generate it. Based on them, we create a plot of their values to check whether they form any recognizable distribution. Examples of such plots, for different values of the depolarizing probability p, can be seen in Fig. 4.2, 4.3 and 4.4. The distribution of decision variables takes quite a similar form for every instance of the problem - there is a central peak surrounded by several smaller ones (sometimes they are merged) and a more or less flat part on the right side where values are usually 1 or close to 1. However, any attempts to fit

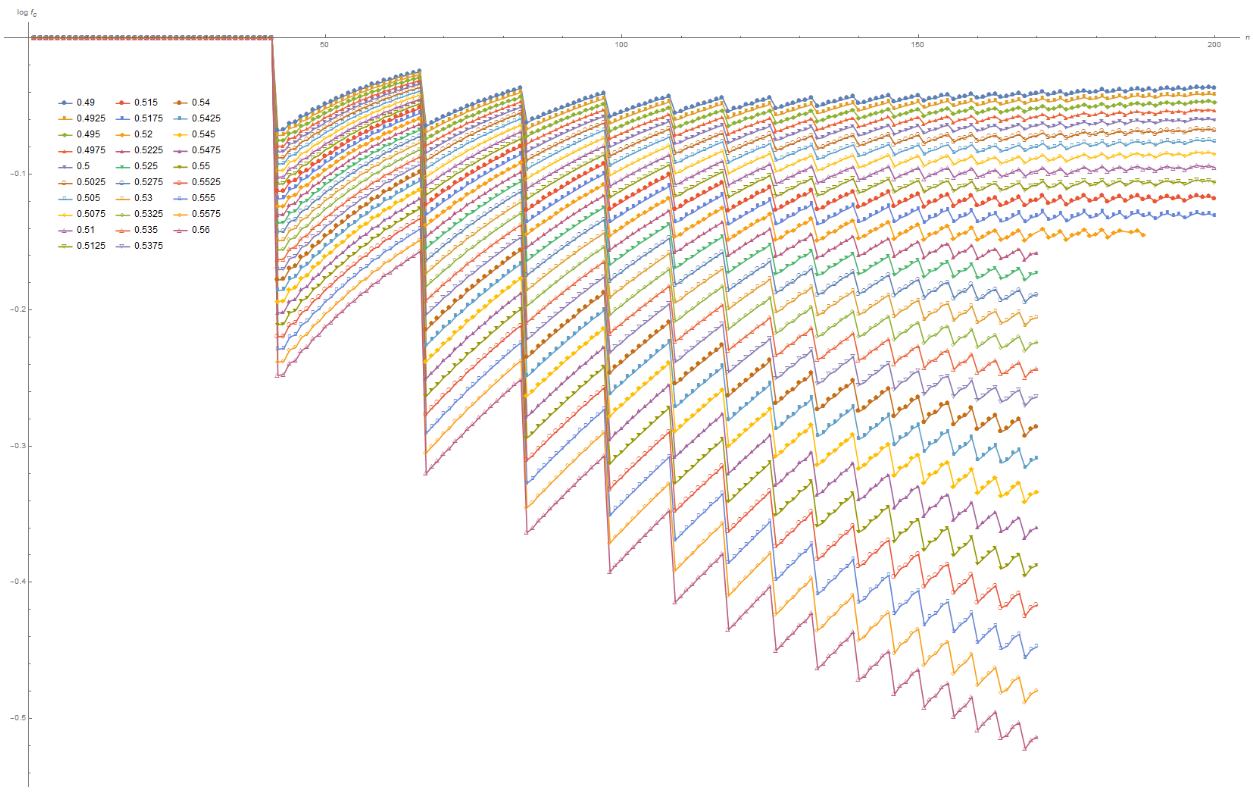
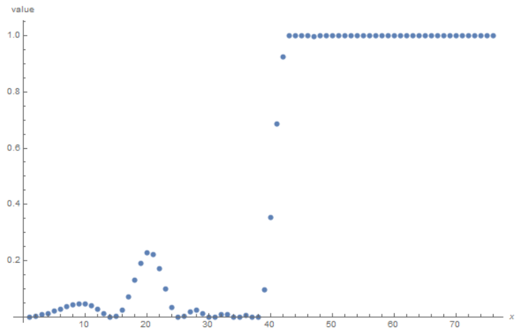
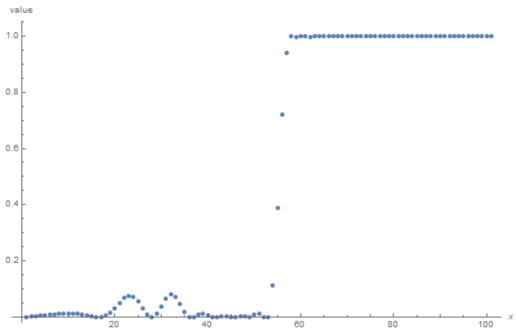


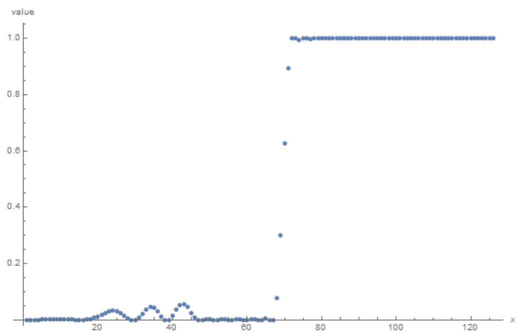
Figure 4.1: Logarithm (base-2) of the optimal channel fidelity computed by the linear program for an increasing number of uses n of the Quantum Depolarizing Channel for depolarizing probabilities $p \in [0.49, 0.56]$ and the rate of transmission $\log_2\left(\frac{61}{60}\right) \approx 0.02385$.



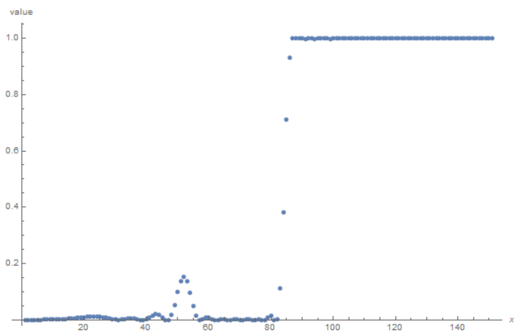
(a) $n = 75$



(b) $n = 100$

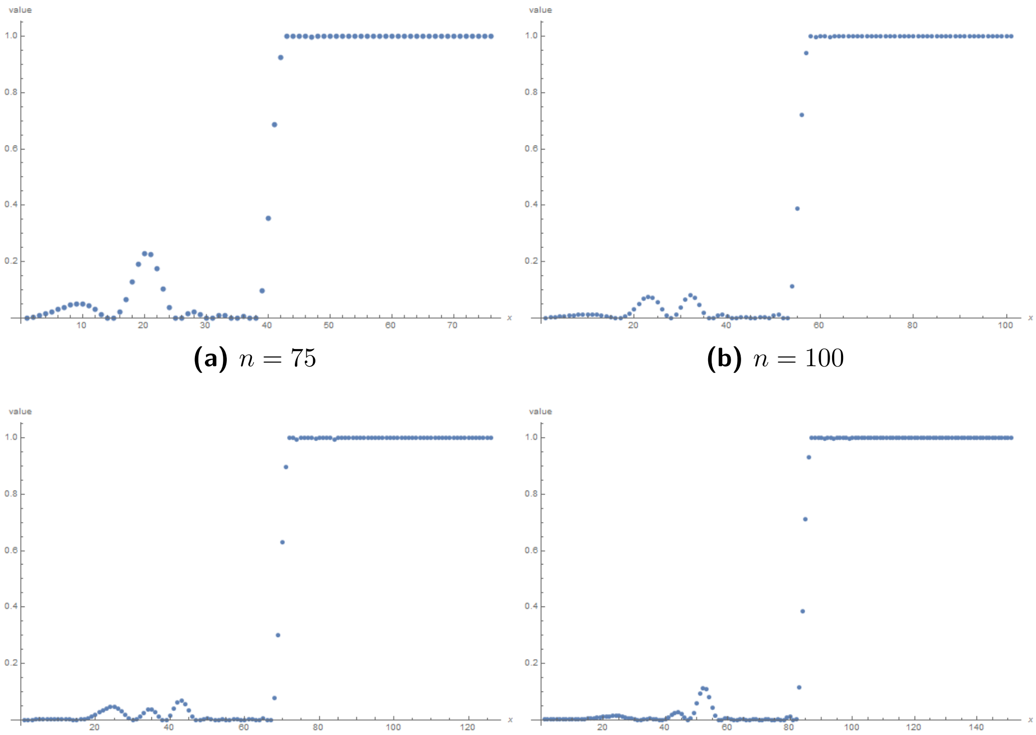


(c) $n = 125$



(d) $n = 150$

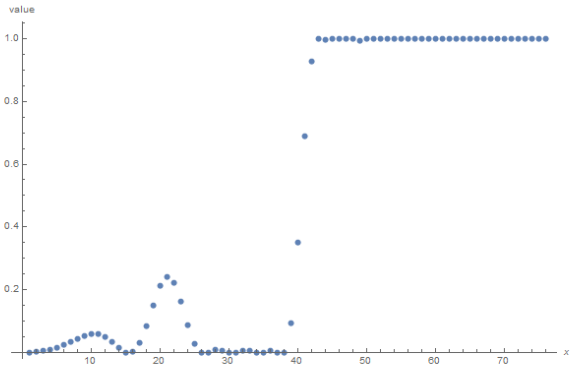
Figure 4.2: Distribution of optimal values of decision variables x_i from the linear program for the optimal fidelity of the Quantum Depolarizing Channel for $p = 0.49$.



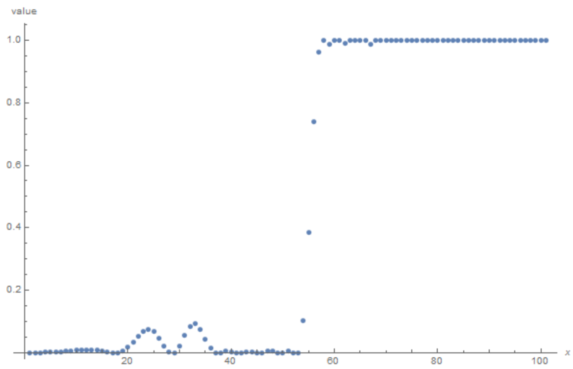
(a) $n = 75$ **(b)** $n = 100$

(c) $n = 125$ **(d)** $n = 150$

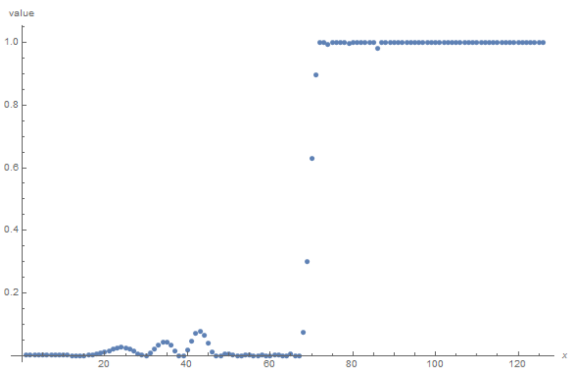
Figure 4.3: Distribution of optimal values of decision variables x_i from the linear program for the optimal fidelity of the Quantum Depolarizing Channel for $p = 0.52$.



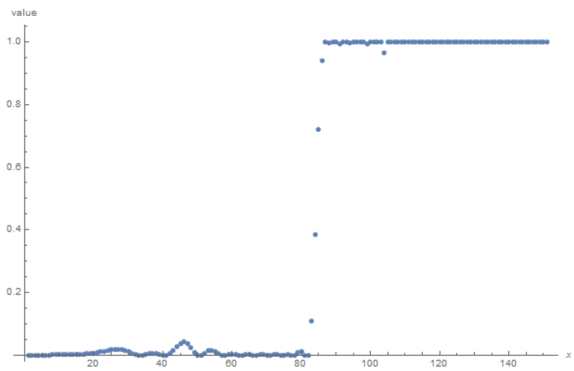
(a) $n = 75$



(b) $n = 100$



(c) $n = 125$



(d) $n = 150$

Figure 4.4: Distribution of optimal values of decision variables x_i from the linear program for the optimal fidelity of the Quantum Depolarizing Channel for $p = 0.55$.

a function to those distributions failed - shapes of smaller peaks does not seem to show any trend, whereas fitting the Gaussian-like function to account for the central peak does not satisfy all of the constraints from the linear program at the end.

Being curious of the pattern that appears in all distributions of decision variables that we obtained (examples in Fig. 4.2, 4.3 and 4.4), namely that there is a block of variables with optimal values close to 1 on the right side of the plot, we investigated our data to check whether there is any trend in where the block starts (in terms of the first decision variable which takes the value of 1). We detected these variables and plotted their indices as a function of n and for the whole depolarizing probability range that we use. It can be seen (Fig. 4.5) that the trend is linear and mostly independent of p. Fitting a linear model to our data yields results which are summarized in Tab. 4.1.

Table 4.1: Among optimal values of decision variables x_i for fixed depolarizing probability p and the number of uses of the channel n, the lowest index i^* is recorded such that $x_{i^*} = 1$. Linear fits for i^* as a function of n for various probabilities are presented in the table.

p	$i^*(n)$	p	$i^*(n)$
0.49	$-1.6213 + 0.59199n$	0.525	$-1.67728 + 0.591356n$
0.4925	$-1.67463 + 0.592311n$	0.5275	$-1.67728 + 0.591356n$
0.495	$-1.69658 + 0.592381n$	0.53	$-1.64754 + 0.591011n$
0.4975	$-1.71105 + 0.592443n$	0.5325	$-1.64754 + 0.591011n$
0.5	$-1.70809 + 0.592367n$	0.535	$-1.61631 + 0.590652n$
0.5025	$-1.72961 + 0.592433n$	0.5375	$-1.5903 + 0.59034n$
0.505	$-1.71166 + 0.592077n$	0.54	$-1.5903 + 0.59034n$
0.5075	$-1.70016 + 0.591826n$	0.5425	$-1.5903 + 0.59034n$
0.51	$-1.70016 + 0.591826n$	0.545	$-1.5903 + 0.59034n$
0.5125	$-1.70016 + 0.591826n$	0.5475	$-1.5903 + 0.59034n$
0.515	$-1.70016 + 0.591826n$	0.55	$-1.62918 + 0.590543n$
0.5175	$-1.70809 + 0.591837n$	0.5525	$-1.62478 + 0.590428n$
0.52	$-1.84776 + 0.593269n$	0.555	$-1.58983 + 0.590035n$
0.5225	$-1.74272 + 0.592101n$	0.5575	$-1.63237 + 0.590347n$

Based on the observation regarding the distribution of the optimal values of decision variables, we conjecture that the trend of having the block of decision variables with the optimal value of 1 may persist as $n \to \infty$. Then, the asymptotics of the objective function for the optimal fidelity could be determined analytically based on this conjecture. This issue will be investigated in Section 4.4.1.

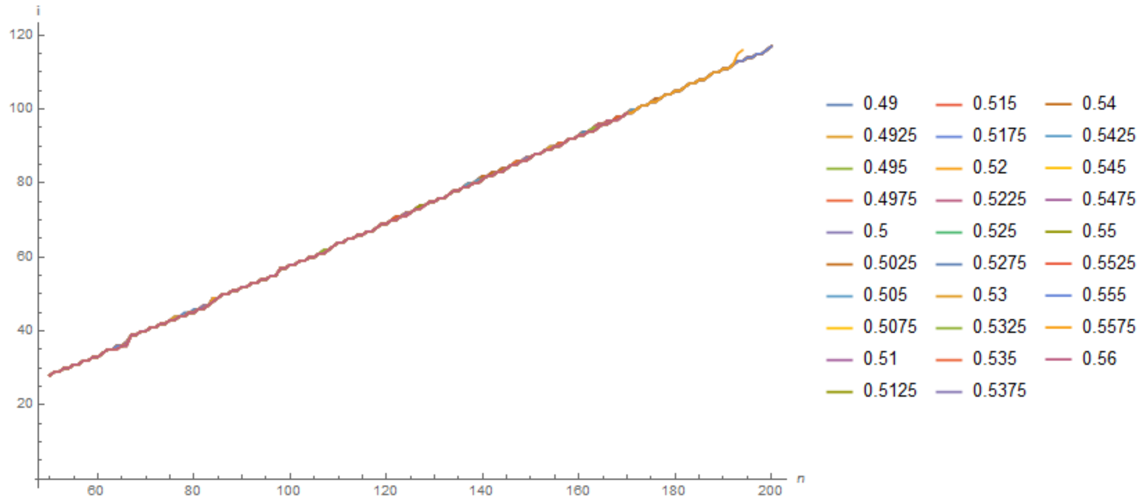


Figure 4.5: The number i of the first decision variable x_i which takes the value of 1 in the optimal solution of a linear program of size n for depolarizing probabilities $p \in [0.49, 0.56]$.

4.3 Dual problem

To gain a different perspective on the problem we proceed to consider the dual problem to the linear program. The dual linear program is given in the following way:

$$\min \sum_{k=0}^{n} y_k + \sum_{l=(n+1)}^{2n+1} \frac{2^n}{K} y_l + \sum_{m=(2n+2)}^{3n+2} \frac{2^n}{K} y_m + \frac{2^n}{K^2} y_{(3n+3)} \tag{4.1}$$

$$\begin{aligned} \forall w = 0 \dots n, \; - y_w - \sum_{i=0}^{n} \sum_{k=0}^{i} \binom{n-i}{w-k} \binom{i}{k} 3^{i-k} (-1)^k (y_{[(n+1)+i]} - y_{[(2n+2)+i]}) \\ - 2^{-n} \binom{n}{w} 3^{n-w} y_{(3n+3)} + z_w = - \binom{n}{n-w} (\frac{p}{4})^{n-w} (1 - \frac{3p}{4})^w 3^{n-w} \end{aligned} \tag{4.2}$$

$$\forall i = 0 \dots (3n+2), \; y_i \geq 0 \tag{4.3}$$

$$y_{(3n+3)} \text{ not constrained} \tag{4.4}$$

$$\forall w = 0 \dots n, \; z_w \geq 0 \tag{4.5}$$

This approach defines a clear partition among the decision variables y_i. Therefore, we define the following classes of decision variables:

- Upper-bound class (UBC): y_0 - y_n, comes from the $x_i \leq 1$ constraints

- PPT1 class (PPT1C): $y_{(n+1)}$ - $y_{(2n+1)}$, comes from the first PPTp constraints set
- PPT2 class (PPT2C): $y_{(2n+2)}$ - $y_{(3n+2)}$, comes from the second PPTp constraints set
- NS class (NSC): $y_{(3n+3)}$, comes from the NS constraint

The partition defined above lays foundations for considering distributions of decision variables y_i as 4 separate distributions.

The dual aspect of the problem can be potentially useful because of the relationship between primal and dual problems in terms of optimal and also feasible solutions [13]. However, it is left for future work.

4.4 Analytical attempt

Numerical results can only serve as hints for the analytical treatment of the problem. To prove results concerning the quantum capacity of a channel, it is necessary to consider the regime of $n \to \infty$, whereas the linear program can only be solved up to a finite value of n. Thus, in this chapter we present the analytical approach to the problem.

4.4.1 Fidelity bound

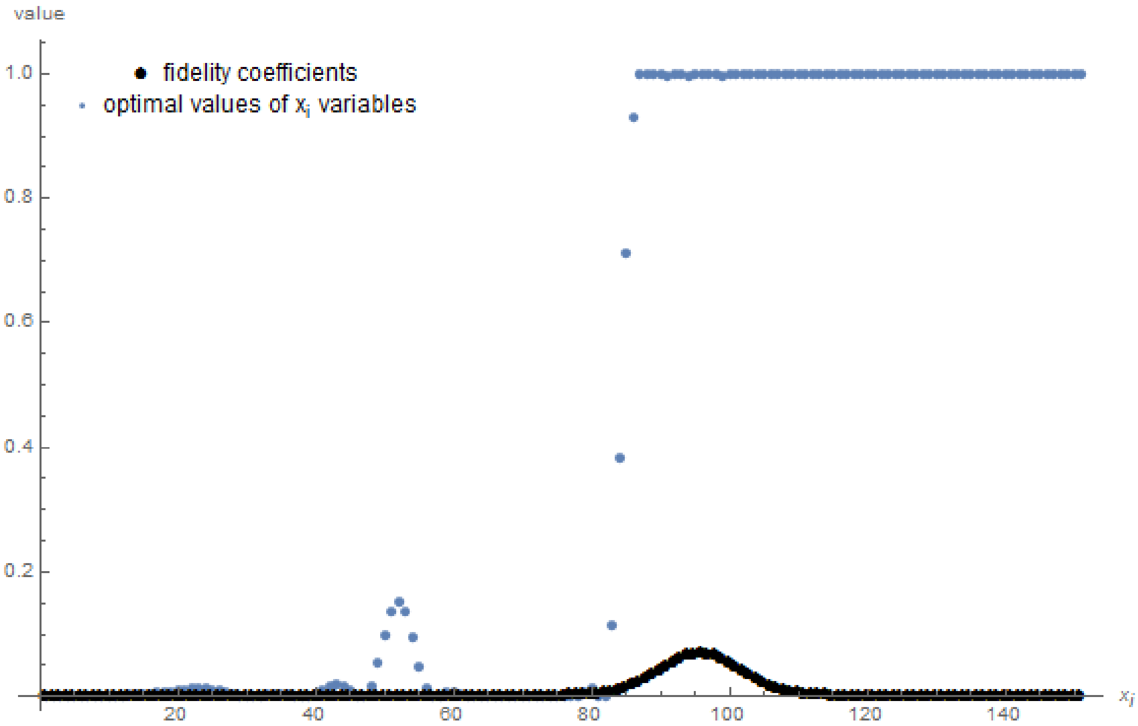
As noticed at the end of Subsection 4.2.2, distributions of the optimal values of decision variables x_i suggest the pattern for all optimal solutions which is that there is a block of decision variables which optimal values are 1 (or very close to 1) and the rest of the decision variables have generally small, fluctuating values. We notice that the coefficients of the objective function in the linear program follow a binomial distribution for the probability of the form $p' = \frac{3p}{4}$:

$$f_c = \sum_{j=0}^{n} \left(\frac{3p}{4}\right)^{n-j} \left(1 - \frac{3p}{4}\right)^{j} \binom{n}{j} x_j. \tag{4.6}$$

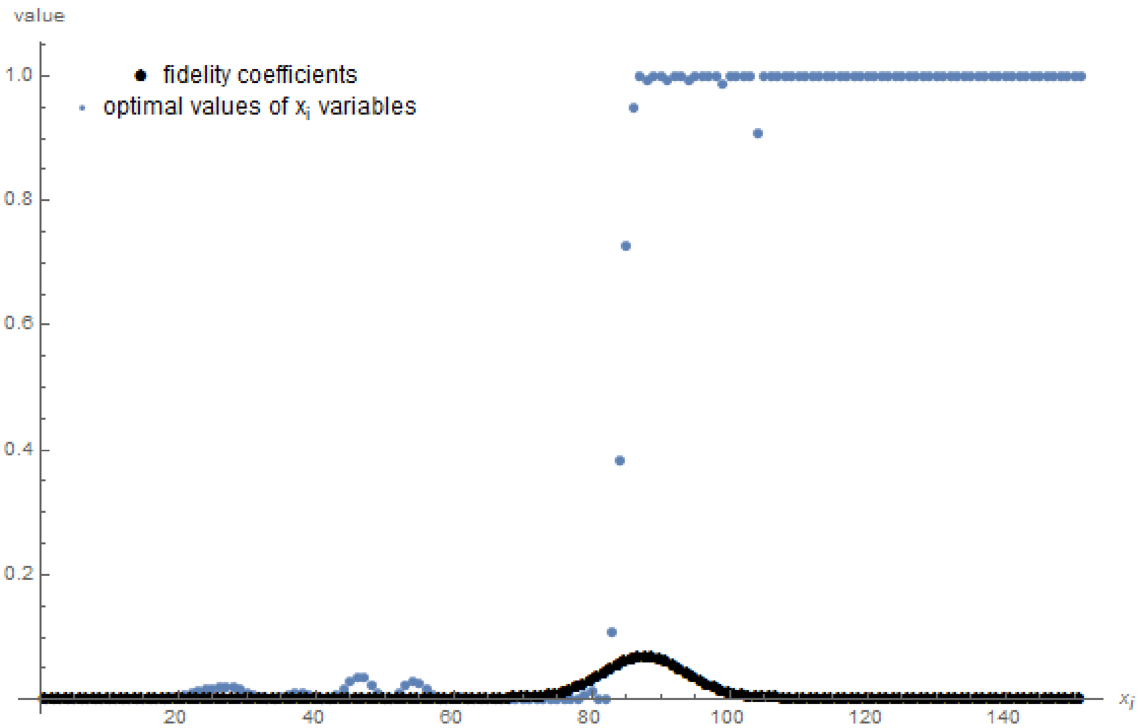
If we assumed the conjecture for the pattern, it would mean that the perfect fidelity could only be obtained if coefficients forming the binomial distribution were shifted to the region of the distribution of the decision variables where their values are 1 (see Fig. 4.6). Since the binomial distribution is normalized, we can expect the fidelity to be 1 only when all of the decision variables are equal to 1:

$$f_c = \sum_{j=0}^{n} \left(\frac{3p}{4}\right)^{n-j} \left(1 - \frac{3p}{4}\right)^{j} \binom{n}{j} = 1. \tag{4.7}$$

However, asymptotically we only require that the optimal fidelity tends to 1.



(a) $p = 0.49$ and $n = 150$



(b) $p = 0.56$ and $n = 150$

Figure 4.6: Distribution of optimal values of decision variables x_i from the linear program for the optimal fidelity of the Quantum Depolarizing Channel compared with the corresponding fidelity coefficients from the objective function in the linear program.

Thus, we may look for a bound on the left tail of the binomial distribution such that its contribution as $n \to \infty$ tends to 0. The position of the binomial distribution depends only on the depolarizing probability p, therefore finding such a bound would yield a bound on p as well.

We use the idea from [14] and consider the following partial sum:

$$\sum_{k=0}^{q} \binom{n}{k} x^k \tag{4.8}$$

where $q \leq n$. We divide the sum by $\binom{n}{q}x^q$ and bound the sum from above:

$$\begin{aligned}
&\frac{\binom{n}{q}x^q + \binom{n}{q-1}x^{q-1} + \ldots + \binom{n}{0}x^0}{\binom{n}{q}x^q} = \\
&1 + \frac{q}{(n-q+1)x} + \frac{q(q-1)}{(n-q+1)(n-q+2)x^2} + \ldots + \frac{q!}{(n)_q x^q} \leq \\
&1 + \frac{q}{(n-q+1)x} + \left[\frac{q}{(n-q+1)x}\right]^2 + \ldots + \left[\frac{q}{(n-q+1)x}\right]^q .
\end{aligned} \tag{4.9}$$

The expression on the right-hand side is the sum of the geometric series and therefore:

$$\begin{gathered}
\sum_{k=0}^{q} \binom{n}{k} x^k \leq \binom{n}{q} x^q \frac{1-r^{q+1}}{1-r}, \\
\text{where } r = \frac{q}{(n-q+1)x}.
\end{gathered} \tag{4.10}$$

To make the sum in equation (4.8) of the form of the fidelity expression, we notice the following:

$$\sum_{k=0}^{q} \binom{n}{k} x^k (1-x)^{n-k} = (1-x)^n \sum_{k=0}^{q} \binom{n}{k} \left(\frac{x}{1-x}\right)^k . \tag{4.11}$$

By proper transformation of variables, the final inequality which is useful for our problem takes the form:

$$\sum_{k=0}^{q} \binom{n}{k} \left(\frac{3p}{4}\right)^{n-k} \left(1 - \frac{3p}{4}\right)^k \leq (\frac{3p}{4})^n \binom{n}{q} \left(\frac{4-3p}{3p}\right)^q \frac{1 - \left(\frac{3pq}{(n-q+1)(4-3p)}\right)^{q+1}}{1 - \frac{3pq}{(n-q+1)(4-3p)}} . \tag{4.12}$$

It is easy to notice that:

$$\sum_{k=0}^{q} \binom{n}{k} \left(\frac{3p}{4}\right)^{n-k} \left(1 - \frac{3p}{4}\right)^k x_k \leq \sum_{k=0}^{q} \binom{n}{k} \left(\frac{3p}{4}\right)^{n-k} \left(1 - \frac{3p}{4},\right)^k \tag{4.13}$$

which gives the following:

$$
\begin{gathered}
f_c^{(p,q)} = \sum_{k=0}^{q} \binom{n}{k} \left(\frac{3p}{4}\right)^{n-k} \left(1 - \frac{3p}{4}\right)^{k} x_k \leq \\
\left(\frac{3p}{4}\right)^{n} \binom{n}{q} \left(\frac{4-3p}{3p}\right)^{q} \frac{1 - \left(\frac{3pq}{(n-q+1)(4-3p)}\right)^{q+1}}{1 - \frac{3pq}{(n-q+1)(4-3p)}} = f_c'^{(p,q)}.
\end{gathered}
\tag{4.14}
$$

In equation (4.14) we can sum the first q terms in the fidelity sum and bound it by the expression on the right hand side. It is also possible to take the limit as $n \to \infty$. Based on the conjecture regarding the distribution of optimal decision variables and data summarized in Tab. 4.1 we observe that:

$$
\begin{gathered}
\lim_{n\to\infty} f_c{}'^{(p=0.5425,q)} = 0, \\
\text{where } q = (-1.5903 + 0.115674) + (0.59034 + 0.00100226)n.
\end{gathered}
\tag{4.15}
$$

$$
\begin{gathered}
\lim_{n\to\infty}, f_c{}'^{(p=0.545,q)} = \infty \\
\text{where } q = (-1.5903 + 0.115674) + (0.59034 + 0.00100226)n.
\end{gathered}
\tag{4.16}
$$

In equations (4.15) and (4.16) we explicitly included standard errors reported by the *LinearFit* in Mathematica in the expression for q. We added them so that the expression is overestimated and hence it gives us a lower bound for the zero-capacity threshold. Equation (4.15) means that decision variables which do not belong to the block of decision variables with values close to 1, do not contribute to the asymptotic value of the optimal fidelity of the channel. Since the binomial distribution of fidelity coefficients is normalized to 1 and values of decision variables beyond the summed range have values very close to 1, the optimal fidelity for $p = 0.5425$ should approach 1. It means that the Quantum Depolarizing Channel has a nonzero quantum assisted capacity for this parameter. Equation (4.16) shows the opposite for $p = 0.545$ what suggests that the Quantum Depolarizing Channel has no assisted quantum capacity for this parameter. We shall emphasize again that this result relies on the conjecture of distribution of optimal decision variables that should be proved.

Chapter 5

Summary

In this thesis we approached the problem of finding the zero-capacity threshold of the Quantum Depolarizing Channel. The linear program for optimal fidelity of the transmission turned out to be a useful tool for obtaining interesting numerical solutions. Since solutions could only be computed up to a finite number of uses of the channel, it was not possible to solve the given problem completely because it is defined in an asymptotic regime. However, numerical solutions that were obtained give much insight into the structure and mathematical behavior of the problem. It was possible to notice trends in the monotonicity of the optimal fidelity of transmission as well as patterns in the distribution of the optimal values of decision variables from the linear program. Based on the data, we were able to indicate the range of depolarizing probabilities p in which there is probably the threshold value for which the assisted quantum capacity of the Quantum Depolarizing Channel vanishes. As we have shown, it is also possible to approach the problem in the analytical way and there exist tools that are capable of analyzing the asymptotic behavior as well. By using them, we obtained the specific value regarding the zero-capacity threshold which was that for $p = 0.545$ and the rate of transmission $r = \log_2\left(\frac{61}{60}\right)$ the Quantum Depolarizing Channel has no assisted quantum capacity. Since ultimately we are interested in the regime as $r \to 0$, we can treat this value as a hypothetical lower bound for p. Although our reasoning is based on the conjecture for the distribution of the optimal values of decision variables, it is justified by clear numerical trends. The distribution of the optimal values of decision variables is of course enforced by the constraints that we derived for the linear program, namely PPTp and NS constraints. Therefore, there should be an analytical way to grasp them and rigorously prove what kind of distribution do they enforce. Nevertheless, we found this to be challenging and this aspect must have been left for future work.

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