RESEARCH STATEMENT

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1. Introduction

My research lies within the area of *stable homotopy theory*. The field of *homotopy theory* studies those properties of spaces which are preserved by continuous deformation. *Stable* homotopy theory is that part of homotopy which studies algebraic invariants of spaces which do not change under continuous deformation, which are known as *generalized (co)homology theories*. Examples of such invariants include singular (co)homology, and topological K-theory (which is defined using vector bundles), and various flavors of cobordism theory. There is a universal homology theory, known as the *stable homotopy groups of spheres* π_*^s. These groups have deep connections to manifold theory, for example they arise from framed cobordism between framed manifolds. Because π_*^s is the universal homology theory, it is rather complicated and seemingly chaotic and it is unlikely a full description will ever be obtained. On the other hand, a systematic framework for studying π_*^s has been developed which decomposes π_*^s into certain *v_n-periodic families* for $n \in \mathbb{N}$. This approach to studying π_*^s is known as *chromatic homotopy theory*. Moreover, it relates these periodic families to the algebraic geometry of formal groups. The v_1-periodic family has been completely calculated and has surprising connections to algebraic K-theory and the Riemann ζ-function. Quite a bit is known about the v_2-periodic families, though some important questions remain.

Much of my research aims to study these periodic families arising in chromatic homotopy theory. I have also done work on analogues of these questions in motivic stable homotopy theory. Motivic stable homotopy theory arises in the same way as stable homotopy theory does, but with algebraic varieties replacing topological spaces. I study these families by considering Adams type spectral sequences based on interesting multiplicative cohomology theories, such as *topological modular forms* or *Hermitian K-theory*. An *Adams spectral sequence* is a way of giving an algebraic approximation to π_*^s and then reconstructing π_*^s from this approximation. By using these Adams spectral sequences, I hope to describe these elaborate patterns in terms of invariants of elliptic curves, quadratic forms, and other well-known objects in mathematics.

2. Background

A important tool for approaching the stable homotopy groups of spheres is the classical *Adams spectral sequences* (ASS). In general, for a space X, the mod p cohomology $H^*(X;\mathbb{F}_p)$ is not just a graded group, but in fact a module over the algebra of operations, known as the mod p *Steenrod algebra* $\mathscr{A}$. Luckily, this algebra has an explicit (albeit complicated) presentation. The Adams spectral sequence is a device that takes as its input the Ext-groups of $H^*(X)$ as a module over $\mathscr{A}$ and outputs the p-torsion components of the stable homotopy groups of X. In the case when $X = S^0$, the input is then $\mathrm{Ext}_{\mathscr{A}}(\mathbb{F}_p, \mathbb{F}_p)$. One should view this Ext-input as an algebraic approximation to π_*^s and the spectral sequence itself as an error term. This input is amenable to computer computation and extensive computer run calculations have been carried out. Finding differentials in the spectral sequence, on the other hand, requires topological insight. This spectral sequence is particularly good for computing stable homotopy groups in a finite range, and many people have made great strides in expanding our knowledge of these groups. However, the mod p ASS is not well-equipped to studying systemic patterns in the stable homotopy groups of spheres.

More generally, given any ring spectrum E,[1] one can always construct an ASS based on E. Under suitable conditions, this spectral sequence takes the form

$$\mathrm{Ext}_{E_*E}(E_*, E_*) \implies \pi_* L_E S^0.$$

Here $E_*E = \pi_*(E \wedge E)$ and $L_E S^0$ is a certain *localization* of the sphere. When E is $H\mathbb{F}_p$, the spectrum representing mod p homology, we recover the ASS. In the case when $E = MU$, the complex cobordism spectrum, we get the *Adams-Novikov spectral sequence* (ANSS). Usually one localizes at one prime at a time, and replaces MU with a summand called the Brown-Peterson spectrum BP. The ANSS takes the form

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{BP_*BP}(BP_*, BP_*) \implies \pi_{t-s} S^0 \otimes \mathbb{Z}_{(p)}.$$

A classical theorem of Quillen tells us that the ring

$$BP_* := \pi_* BP \cong \mathbb{Z}_{(p)}[v_1, v_2, v_3, \ldots]$$

is the ring which carries the *universal p-typical formal group law* and that the pair (BP_*, BP_*BP) classifies p-typical formal group laws and isomorphisms between them. It is a deep insight of Quillen and Morava that this pair gives a presentation of the moduli stack $\mathscr{M}_{pFG}$ of p-typical formal groups and that the E_2-term of the ANSS can be described as the cohomology of this stack, i.e.

$$\mathrm{Ext}_{BP_*BP}(BP_*, BP_*) \cong H^*(\mathscr{M}_{pFG}).$$

There is a filtration on $\mathscr{M}_{pFG}$ by an invariant called the *height*, which turns out to be controlled by inverting the elements v_i in $\pi_* BP$. This filtration leads to the *algebraic chromatic spectral sequence* which converges to $\mathrm{Ext}_{BP_*BP}(BP_*, BP_*)$. The E_1-term of this spectral sequence consists of elements which are *periodic* with respect to some v_n, and shows that each element of the ANSS E_2-term belongs to some v_n-periodic family. From this perspective, the stable homotopy groups have a lot of structure and contain many intricate, but systematic, patterns. Computations for $n = 1, 2, 3$ and for sufficiently large primes have been done in [36].

This led Doug Ravenel to ask if this algebraic structure has its origins in topology, leading him to formulate several conjectures ([37]). All but the now infamous *telescope conjecture* have been proven. In particular, it is now understood that there are certain homology theories $K(n)$, the nth *Morava K-theory*, so that the homotopy groups of the sphere spectrum S^0 are built out of the localizations $L_{K(n)} S^0$. The elements of $\pi_* L_{K(n)} S^0$ are periodic with respect to v_n. For example, the homotopy of $L_{K(1)} S^0$ corresponds to the image of J, which was intensely studied by Adams in the 60s. In this case, there is a periodicity arising from the Bott element in topological K-theory and v_1 corresponds to the Bott element. The higher v_n-periodic families are analogues of the image of J with periodicity arising from v_n. In recent years, the focus has moved to computing $L_{K(2)} X$ for various X and to settling the *telescope conjecture*. The latter says, roughly, that for appropriate X, we have an isomorphism $v_n^{-1} \pi_* X \cong \pi_* L_{K(n)} X$. The importance of this conjecture stems from the fact that the left hand side is used to define the *chromatic filtration* in stable homotopy, which is

[1] or multiplicative cohomology theory

what decomposes π_*^s into v_n-periodic families, whereas the right hand side can be calculated in terms of the algebraic geometry of height n formal group laws.

My research is concerned with better understanding these periodic families at height $n = 2$ using the spectrum tmf of *topological modular forms* and studying analogous problems in motivic homotopy theory.

3. Current and Future Research

3.1. **Chromatic homotopy theory.**

3.1.1. *A brief overview of height 1 chromatic homotopy theory.* A major problem in homotopy theory in the mid-twentieth century was the study of the image of the J-homomorphism. This is a homomorphism

$$J : \pi_* SO \to \pi_*^s$$

from the infinite special orthogonal group to the stable homotopy groups of the sphere. By Bott periodicity, it was known that $\pi_i SO$ is infinite cyclic when i is congruent to -1 modulo 8. On the other hand, it was known from Serre's work that π_i^s is a finite abelian group. Whitehead asked what the order of the image of J was in degrees -1 modulo 8. This was later solved by Adams up to the Adams conjecture, and later Quillen proved the Adams conjecture.

The J-homomorphism is a part of height 1 chromatic homotopy theory because the element $v_1 \in \mathrm{BP}_*$ corresponds to the Bott element giving rise to Bott periodicity in topological K-theory. In particular, the image of J is part of the v_1-family in stable stems. In this case, the *telescope conjecture* asks if this accounts for all of the v_1-periodic pattern.

In [33] and [34], Mahowald intensely studied the image of J through the lens of the Adams spectral sequence based on the connective real K-theory spectrum, denoted bo. Using it he was able to prove the height 1 telescope conjecture and perform computations of the stable homotopy groups of spheres.

On the other hand, it was shown that the $K(1)$-local sphere sits in a fiber sequence

$$L_{K(1)}S \longrightarrow KO \xrightarrow{\psi^3 - 1} KO \tag{3.1}$$

where ψ^3 is an Adams operation and KO denotes real topological K-theory. It was also known that $\pi_* L_{K(1)}S \cong \operatorname{im} J_*$ for $* \geq 0$. In other words, the connective cover of $L_{K(1)}S$ is the connective J-spectrum j. Mahowald and Milgram showed that this fiber sequence has a connective analog

$$j \longrightarrow \Sigma^4 bsp \xrightarrow{\psi^3 - 1} bo \tag{3.2}$$

where j denotes the *connective image of J spectrum* (i.e. $\pi_* j \cong \operatorname{im} J$), and where bsp denotes the 4-connected cover of KO. It was observed by Mahowald that the second map appears as a d_1-differential in the bo-resolution for the sphere, thereby showing that the image of J appears in the first two rows of the bo-resolution. Mahowald's proof of the height 1 telescope conjecture ([33]) relies on showing that there are no v_1-periodic families above these first two rows.

3.1.2. *tmf-resolutions.* In [33] and [34], Mahowald intensely studied the image of J through the lens of the Adams spectral sequence based on the connective real K-theory spectrum, denoted *bo*. Using it he was able to prove the height 1 telescope conjecture and perform computations of the stable homotopy groups of spheres. In recent years, the spectrum *tmf* of topological modular forms has entered the scene, and is a height 2 version of *bo*. One of my main research problems is to carry out Mahowald's program for *tmf*. This leads to three subgoals for the *tmf*-based ASS:

(1) settle the height 2 telescope conjecture,
(2) compute homotopy groups, and
(3) study/compute $K(2)$-local homotopy groups.

The first step in approaching each of these problems is to study the homotopy groups of $tmf \wedge tmf$. The 2-primary version of this problem has been studied in [11] and [35]. In previous work, I studied a more approachable version of the problem at all primes. In particular, I computed the $BP\langle 2\rangle$ co-operations. Here

$$BP\langle 2\rangle_* = BP_*/(v_3, v_4, \dots) = \mathbb{Z}_{(p)}[v_1, v_2]$$

Theorem 3.3 (Culver, [18], [19])**.** *There is an algorithm for computing v_2-torsion free component of $BP\langle 2\rangle \wedge BP\langle 2\rangle$ at all primes. Moreover, the v_2-torsion in $\pi_* BP\langle 2\rangle \wedge BP\langle 2\rangle$ is simple torsion and concentrated in Adams filtration 0.*

In the case when $p = 2$, then $BP\langle 2\rangle$ is equivalent to $tmf_1(3)$, a variant of *tmf* built out of elliptic curves with a chosen point of order three, and so this theorem can viewed as a calculation of $tmf_1(3) \wedge tmf_1(3)$. In a different direction, Paul VanKoughnett and I have computed the $K(1)$-local homotopy groups of $tmf \wedge tmf$ using Hopkins' original construction of $tmf_{K(1)}$ [32]. In Hopkins' original construction, he relates the homotopy of $tmf_{K(1)}$ to the homotopy groups of real K-theory KO_* and the inverse of the modular j-function, which also appears in the following theorem.

Theorem 3.4 (Culver-VanKoughnett, [21])**.** *For the primes $p = 2, 3$, the $K(1)$-local homotopy of $tmf \wedge tmf$ is given by*

$$\pi_* L_{K(1)}(tmf \wedge tmf) \cong \left(KO_*[j^{-1}, \overline{j^{-1}}] \otimes \mathbb{T}(\lambda)/(\psi^p(\lambda) - \lambda - j^{-1} + \overline{j^{-1}})\right)^{\wedge}_p$$

where $\mathbb{T}$ refers to the free ϑ-algebra.

In joint work with Beaudry, Behrens, Bhattacharya, and Xu, [4], I have studied the *bo*-based ASS, and we have extended previous computations of Mahowald ([33]) and Mahowald-Lellmann ([34]). This project is a warm-up to computing the *tmf*-resolution. In particular, we have

Computation 1 (Beaudry-Behrens-Bhattachary-C.-Xu, [9])**.** *There are various spectral sequences which allow us to compute the E_2-term of the bo-based ASS through a range. Moreover, the bo-based ASS collapses at the E_2-term up through the 40-stem.*

We hope that the techniques we developed for the *bo*-ASS will extend to the *tmf*-ASS. We have already had success in applying them to the *tmf*-ASS for the Bhattacharya-Egger spectrum Z ([14]). In particular, we have showed the following.

Theorem 3.5 (Beaudry-Behrens-Bhattacharya-C.-Xu, [5])**.** *The tmf-ASS for Z is computable using similar techniques to those in* [4]*, and there are several possible counter examples to the telescope conjecture at the E_2-page. Furthermore, a previously unresolved potential*

d_3-differential in the $K(2)$-local ANSS for Z ([15]) does not occur, thereby determining the entire spectral sequence.

At the moment we cannot show that the potential counter examples survive the spectral sequence; if they did survive this would show the telescope conjecture is false. One of my goals is to study the analogous problem at the prime $p = 3$. In this case, the homotopy groups of *tmf* are computationally simpler than at $p = 2$, while still having a large Hurewicz image. This leads to the first of my current goals.

Goal 1. *Compute the homotopy groups of $tmf \wedge tmf$, at least enough so that we can begin studying the tmf-based Adams spectral sequence at the prime 3.*

Once we have computed the *tmf*-cooperations at $p = 3$, we ought to be able to find potential counter examples to the height 2 telescope conjecture at $p = 3$, and hopefully show that they survive the *tmf*-based ASS.

3.1.3. *Modular descriptions of the $K(2)$-local sphere.* Chromatic homotopy theory tells us that homotopy groups of the $K(n)$-local spheres $L_{K(n)}S$ can be reassembled to give us a calculation of the stable homotopy groups of spheres. Thus, calculating these $K(n)$-local homotopy groups can be quite valuable.

Techniques to compute the homotopy of $S^0_{K(2)}$ were developed by Goerss-Henn-Mahowald-Rezk (GHMR) in [27] and later shown in [6],[7],[8] to have an interpretation in terms of elliptic curves and modular forms. In particular, Behrens shows that for any prime ℓ there is a spectrum $Q(\ell)$, which is built out of isogenies of elliptic curves of degree ℓ^k, and shows at $p = 3$ that there is a cofiber sequence

$$L_{K(2)}S \to Q(2) \to \Sigma D_{K(2)}Q(2). \tag{3.6}$$

The second map in this sequence is often referred to as the *middle map*. This is the height 2 analogue of the Adams-Baird resolution (3.1). There is also a connective version of this resolution (3.2). It would be interesting to know whether or not there is a height 2 analogue of (3.2), i.e. is there a connective version of $Q(2)$ and $\Sigma D_{K(2)}Q(2)$ which fit into a fiber sequence with the connective cover of $L_{K(2)}S$. This leads to the following question.

Question 3.7. Is there a connective version of the resolution (3.6) which sits within the *tmf*-based ASS. By this, we mean that when we $K(2)$-localize, we recover (3.6).

In the height 1 case, (3.2) allowed Mahowald to locate the v_1-periodic elements in π_*^s arising from $L_{K(1)}S$ and played a key part of his proof of the telescope conjecture. If this conjecture is true, we would be able to similarly locate the v_2-periodic families in π_*S arising from $L_{K(2)}S$. This would be tremendously helpful in resolving the height 2 telescope conjecture at $p = 3$.

In the height 1 case (3.2), some care was needed to give the correct "connective model" of (3.1), in particular it was not enough to simply take connective covers. It is likely that similar considerations will required in the height 2 case. This leads to another of my current goals.

Goal 2. *Construct the "correct" connective versions of the spectra $Q(\ell)$.*

A periodic version has already been constructed by Behrens. It would seem that the work of Lawson-Hill ([30]) and recent work of Jack Davies [23] will be relevant here to extend the construction over the cusps of the relevant moduli spaces.

While we can compute the homotopy groups of $L_{K(2)}S$ for primes $p \geq 5$, this calculation is very complicated. While the spectra $Q(\ell)$ exist at all primes, it is currently unknown whether or not a version of the fiber sequence (3.6) holds at primes $p \geq 5$. Such a fiber sequence could potentially give us a much more conceptual understanding of the calculation of $\pi_* L_{K(n)}S$.

Part of Behrens' proof of (3.6) relied on explicit computations of $\pi_*(L_{K(2)}Q(2))$ and $\pi_*(L_{K(2)}V(1))$ ([26]), however because of the abundance of non-isomorphic supersingular elliptic curves at primes $p > 5$ makes it impossible to give a *uniform* computation at all large primes. Thus, a more conceptual argument is required.

Goal 3 (joint with Behrens, Bobkova, and VanKoughnett)**.** *Provide a modular description of $L_{K(2)}S$ for primes $p \geq 5$.*

Behrens' result (3.6), as mentioned, provides a modular interpretation of the $K(2)$-local sphere. At higher heights, there are analogues of $Q(\ell)$ and tmf which are constructed using Shimura varieties and automorphic forms ([10]). On the other hand, obtaining explicit computations of the $K(n)$-local sphere $L_{K(n)}S$ for $n > 2$ by generators and relations seems implausible. Instead we chould try to produce *automorphic* interpretations of $L_{K(n)}S$. If we can achieve Goal 3, this could be a potential first step towards giving an automorphic interpretation of $L_{K(n)}S$ for larger heights.

3.1.4. *Higher height chromatic homotopy theory and Picard groups.* Chromatic homotopy theory provides us with a way to reconstruct the stable homotopy groups of spheres from the homotopy groups of the $K(n)$-localizations of the sphere $\pi_* L_{K(n)}S$. Furthermore, it was shown by [25] that there is a spectral sequence

$$E_2 = H_c^*(\mathbb{G}_n; E_n[u^{\pm 1}]) \implies \pi_* L_{K(n)}S \tag{3.8}$$

where $\mathbb{G}_n$ is the *Morava stabilizer group*, defined as the automorphism of a height n formal group, E_n denotes the *Lubin-Tate deformation space* of a height n formal group, and the E_2-page is the *continuous group cohomology* of $\mathbb{G}_n$.

It has been observed by several authors that there are self-dual patterns arising in the E_2-term of (3.8). It was discovered by Gross–Hopkins ([31]) that this can be explained more conceptually by relating it to certain dualities on Lubin–Tate space; this duality behaves much like Grothendieck–Serre duality in that it requires certain "dualizing sheaves" on Lubin–Tate space. This duality can even be expressed in the category of $K(n)$-local spectra. Unfortunately, this is complicated by the fact that for certain p and n, there are certain "dualizing spectra" that are seen to be trivial sheaves after applying Lubin–Tate homology; such spectra are called *exotic* and they form a subgroup κ_n (under the smash product) of the $K(n)$-local Picard group Pic_n (the group of invertible[2] objects in the category of $K(n)$-local spectra).

At height two, a great deal of work has been done to determine the group of these exotic elements ([28, 29]). Beyond height 2, little is known about these exotic elements. Given an exotic element X, its Adams-Novikov SS will take a form similar to (3.8),

$$E_2 = H_c^*(\mathbb{G}_n; E_n[u^{\pm 1}]) \implies \pi_* L_{K(n)}X, \tag{3.9}$$

[2]by an invertible spectrum, we mean a $K(n)$-local spectrum X such that there is a Y so that $X \wedge Y \simeq L_{K(n)}S^0$.

in particular the E_2-terms are the same. This allows one to relate the group κ_n to differentials in (3.9) originating on $H_c^0(\mathbb{G}_n; E_n) \cong \mathbb{Z}_p$. In the case when $2p-1 = n^2$, it turns out the only possible target is the group $H^{n^2}(\mathbb{G}_n; E_n \cdot \{u^{p-1}\})$.[3] In particular there is an embedding of κ_n into this cohomology group in this case. The first case when $2p-1 = n^2$ is when $p = 5$ and $n = 3$, which is just outside the range of known computations. With Ningchuan Zhang, I have studied conditions under which the cohomology group $H^{n^2}(\mathbb{G}_n; E_n \cdot \{u^{p-1}\})$ is trivial. In particular, we show the following.

Theorem 3.10 (C.–Zhang, [22]). *At $p = 5$ and $n = 3$, the exotic elements in κ_3 are not detected by the Smith-Toda complex $V(1) = S/(5, v_1)$. In other words, if $X \in \kappa_3$ then*

$$X \wedge V(1) \simeq V(1).$$

In addition, if the Reduced Homological Vanishing Conjecture (RHVC) *holds, i.e. if the natural map*

$$\mathbb{F}_5 \cong H_0(\mathbb{G}_3; \mathbb{F}_{5^3}) \to H_0(\mathbb{G}_3; E_3/5)$$

is an isomorphism, then the group κ_3 is trivial.

Zhang and I have also investigated general conditions which would imply that the reduced homological vanishing conjecture holds or that the relevant cohomology group is trivial. Our current approach uses the explicit formulas involving $\mathrm{BP}_*\mathrm{BP}$ and techniques developed in [36]. While we cannot determine the relevant cohomology group at the moment, we can prove the following.

Theorem 3.11. *[C.–Zhang, [22]] There are two separate sets of bounds on the invariant ideals of* $\mathrm{BP}_*\mathrm{BP}$ *which would imply the Reduced Homological Vanishing Conjecture or that $H^{n^2}(\mathbb{G}_n; E_n\{u^{p-1}\}) = 0$. The bounds for the former are stricter than the latter.*

The proof of this theorem requires relating the desired calculation to a calculation of the invariants of E_n twisted by the determinant via Gross-Hopkins duality and then relating it to [36] via the Morava change of rings theorem. The main drawback of this approach is that the relevant $\mathrm{BP}_*\mathrm{BP}$ formulas are quite elaborate and difficult to manipulate. On the other hand, the Morava stabilizer group $\mathbb{G}_n$ is a very special kind of group, its a p-adic Lie group. Zhang and I are investigating if we can reprove the above theorem, and even strengthen it, by using the properties of $\mathbb{G}_n$ as a p-adic Lie group, its representation theory, and its connections to Dieudonne theory of formal groups. Much work along these lines at general heights has been carried out by Gross–Hopkins ([31]) and Devinatz–Hopkins ([24]) , and height 2 specific work has been done by Kohlhaase and Goerss–Henn–Mahowald–Rezk. But this avenue of exploration still has much to offer.

3.2. **Motivic homotopy theory.** Voevodsky and others have constructed a category of motivic spectra, which is built from schemes over a base field rather than topological spaces. It turns out that this category has a lot of similar properties as the classical stable homotopy category. In particular, in [16],[17], motivic analogs of the Morava K-theories $K(n)$ are constructed. In recent work with J.D. Quigley, I have been studying v_1-periodicity in the motivic setting. In particular, we have carried out Mahowald's program on *bo*-resolutions in the $\mathbb{C}$-motivic category, using the motivic analog of *bo*, denoted as *kq* ([1]). We have shown the following.

[3] amusingly, whether or not there are infinitely many such primes of this form is a special case of an open problem of Euler. Using Sage, one can show there are at least $\sim 10^5$ such primes.

Theorem 3.12 (Culver-Quigley,[20])**.** *In the kq-based ASS, the* 0*- and* 1*-lines of the* E_∞*-page can be explicitly computed and display a pattern similar to the image of* J*. Moreover, the* E_∞*-page has a vanishing line of slope 1/5, which implies all the* v_1*-periodic elements are detected in the* 0*- and* 1*-line.*

In motivic homotopy theory, the first Hopf invariant one element η is non-nilpotent, and an important question in the subject is to compute the η-inverted homotopy of the motivic sphere spectrum. Over $\mathbb{C}$, this has been computed by Andrews-Miller ([2]). There is ongoing research to compute $\pi_{**}\eta^{-1}S$ over $\mathbb{R}$. Quigley and I have shown the following:

Theorem 3.13 (Culver-Quigley, [20])**.** *We can recover* $\eta^{-1}\pi_{**}S^{0,0}$ *over* $\mathbb{C}$ *from the* 0*- and* 1*-line of the* E_∞*-page of the kq-based ASS.*

This suggests that the kq-based ASS is a useful tool for studying $\eta^{-1}\pi_{**}S^{0,0}$. Quigley and I have begun to consider the analogous situation over other base fields. The spectra KQ and kq are defined over any field F whose characteristic is not 2, and hence the kq-resolution can be studied over such bases. We are especially interested in the case $F = \mathbb{Q}, \mathbb{R}, \mathbb{Q}_p$, and $\mathbb{F}_q$ for q odd. More specifically, we are interested in the following question.

Question 3.14. Can we use the kq-based ASS over $F = \mathbb{F}_q, \mathbb{Q}_p, \mathbb{R}$ to compute the v_1-periodic elements? Can we use it to compute some new 2-torsion in the motivic homotopy groups of spheres? Can we use it to compute $\eta^{-1}\pi_{**}S^0$, the stable homotopy groups of spheres where we have inverted η.

The first natural starting point would be to generalize Mahowald's splitting result. This leads to the following goal.

Goal 4. *Construct motivic Brown-Gitler spectra over general base fields and derive a motivic version of Mahowald's splitting for* $kq \wedge kq$*.*

There are several strategies for accomplishing this. Currently, Quigley and I are investigating how to generalize the original construction due to Brown and Gitler to the motivic setting. This relies on knowing what the motivic Λ-algebra is. In joint work with William Balderrama and J.D. Quigley, we have given an explicit presentation of such an algebra.

Theorem 3.15 (Balderrama-C-Quigley)**.** *There is an explicit subcomplex of the cobar complex for the dual motivic Steenrod algebra,* Λ^{mot}*, whose cohomology is* $\mathrm{Ext}_A(\mathbb{M}_p, \mathbb{M}_p)$*, where* $\mathbb{M}_p$ *denotes the motivic cohomology of the base field with mod* p *coefficients.*

Classically, the Λ-algebra has been used to study low lines in the Adams spectral sequence, and notably leads to a shorter proof of the Hopf invariant 1 problem (cf. [38]). The computation over $\mathbb{R}$ is by far the most complicated, this is because of of the presence of a non-nilpotent element ρ in the motivic cohomology of a point. Belmont-Isaksen [12] have made tremendous progress in computing the $\mathbb{R}$-motivic Adams spectral sequence in a range, and their work shows just how complex these calculations are. With our Lambda we have been able to give an infinite calculation along the lines of Wang's work [38].

Theorem 3.16 (Balderrama-C-Quigley, [3])**.** *One can explicitly compute the the first three rows of the* $\mathbb{R}$*-motivic Adams* E_2*-term. In particular we can give a complete set of generators and relations of these low lines as a module over* $\mathbb{F}_2[\rho]$*.*

As a consequence of this calculation, we are able to derive an infinite family of d_2-differentials in the $\mathbb{R}$-motivic Adams spectral sequence.

Theorem 3.17 (Balderrama-C-Quigley, [3])**.** *We have the following d_2-differentials in the $\mathbb{R}$-motivic ASS*

(1) *for all $n \geq 4$, $d_2(h_n) = (h_0 + \rho h_1)h_{n-1}^2$,*
(2) *For all $n \geq 4$ and $k \geq 0$, we have*

$$d_2(\tau^{2^n+2^{n+2}k}h_{n+1}) = (h_0 + \rho h_1)(\tau^{2^{n-1}+2^{n+1}k}h_n)^2.$$

Using a Hasse type principal, we obtain similar d_2-differentials in the motivic ASS over $\mathbb{Q}_2$?

The Λ-algebra also plays an interesting role in classical unstable homotopy theory. There are certain subalgebras $\Lambda(n)$ which are the E_1-term of an *unstable Adams spectral sequence* converging to the (unstable) homotopy groups $\pi_* S^n$. Furthermore, there is a short exact sequence relating the $\Lambda(n)$ which, in cohomology, gives an algebraic approximation of the EHP sequence relating the unstable homotopy groups of spheres. I have become interested in the motivic analogues of these classical applications.

Question 3.18. Can we produce a motivic unstable Adams spectral sequence which calculates $\pi_{**}S^{p,q}$ and can we identify its E_2-term with a certain subalgebra $\Lambda^{mot}(p,q)$ of Λ^{mot}? Are there short exact sequences relating the $\Lambda^{mot}(p,q)$ which give the simplicial EHP sequence of Wickelgren-Williams ([39])?

A construction of unstable Adams spectral sequences based on more general cohomology theories was developed by Bendersky-Curtis-Miller ([13]). A natural starting point for the above question would be to generalize [13] to motivic categories. This work does not generalize immediately and part of doing this would require incorporating the $\mathbb{P}^1$-loops functor.

Classically, in order to identify the E_1-term of the unstable Adams spectral sequence with the subalgebras $\Lambda(n)$ of Λ one needs to show that the free objects of the unstable module category over the Steenrod algebra can be realized by the cohomology of the Eilenberg-MacLane spaces. So we are naturally led to the following questions.

Question 3.19. What is the correct notion of an "unstable module over the motivic Steenrod algebra" and what are the properties of the category of such objects. What is the motivic cohomology of motivic Eilenberg-MacLane spaces? What are they as unstable modules over the motivic Steenrod algebra?

References

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