RESEARCH STATEMENT

D. CULVER

1. Introduction

My research lies in the area of *stable homotopy theory*. This is the part of homotopy theory which studies invariants of spaces that are stable under the suspension functor. My research primarily focuses on studying patterns that arise in the stable homotopy groups of finite complexes, with a particular emphasis on spheres. I am especially interested in computing *infinite periodic families* in the stable homotopy groups of spheres. Most of my research can be described as studying these families by using Adams type spectral sequences based on interesting ring spectra. The primary case of interest is the ring spectrum tmf of *topological modular forms*. I have also become interested in studying the analogous problems in motivic and equivariant homotopy theory. In addition, I am computing invariants of ring spectra such as *topological Hochschild homology, topological cyclic homology, and algebraic K-theory*.

2. Background

A natural tool for approaching the stable homotopy groups of spheres is the classical *Adams spectral sequences* (ASS). This is a spectral sequence of the form

$$\mathrm{Ext}^{s,t}_{A_*}(\mathbb{F}_p,\mathbb{F}_p) \Longrightarrow \pi_{t-s}S^0\otimes\mathbb{Z}_p,$$

where p is a prime number. This spectral sequence is particularly good for computing the stable homotopy groups in a finite range, but obscures infinite patterns in the stable homotopy groups of spheres.

However, given any ring spectrum E[1], one can always construct an ASS based on E. Under suitable conditions, this spectral sequence takes the form

$$\mathrm{Ext}_{E_*E}(E_*,E_*) \Longrightarrow \pi_*L_ES^0.$$

Here $E_*E=\pi_*(E\wedge E)$ and L_ES^0 is a certain *localization* of the sphere. When E is $H\mathbb{F}_p$, the spectrum representing mod p homology, then we recover the ASS. In the case when $E=MU$, the complex cobordism spectrum, we get the *Adams-Novikov spectral sequence* (ANSS). Usually one

[1] or multiplicative cohomology theory

localizes at one prime at a time, and replaces MU with a summand called the Brown-Peterson spectrum BP. The ANSS takes the form

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{BP_*BP}(BP_*, BP_*) \implies \pi_{t-s}S^0 \otimes \mathbb{Z}_{(p)}.$$

A classical theorem of Quillen tells us that the ring

$$BP_* := \pi_* BP \cong \mathbb{Z}_{(p)}[v_1, v_2, v_3, \ldots]$$

is the ring which carries the *universal p-typical formal group law* and that the pair (BP_*, BP_*BP) classifies p-typical formal group laws and isomorphisms between them. It is a deep insight of Quillen and Morava that this pair gives a presentation of the moduli stack $\mathcal{M}_{pFG}$ of p-typical formal groups and that the E_2-term of the ANSS can be described as the cohomology of this stack, i.e.

$$\mathrm{Ext}_{BP_*BP}(BP_*, BP_*) \cong H^*(\mathcal{M}_{pFG}).$$

There is a well-known filtration on $\mathcal{M}_{pFG}$ by an invariant called the *height*, which turns out to be controlled by the elements v_i in $\pi_* BP$. This filtration leads to the *algebraic chromatic spectral sequence* which converges to $\mathrm{Ext}_{BP_*BP}(BP_*, BP_*)$. The E_1-term of this spectral sequence consists of elements which *periodic* with respect to some v_n, and shows that each element of the ANSS E_2-term belongs to some v_n-periodic family. From this perspective, the stable homotopy groups have a lot of structure and contain many intricate, but systematic, patterns. Computations for $n = 1, 2, 3$ and for sufficiently large primes have been done in [25].

This led Doug Ravenel to ask if this algebraic structure has its origins in topology, leading him to formulate several conjectures ([28]). All but the now infamous *telescope conjecture* have been proven. In particular, it is now understood that there are certain homology theories $K(n)$, the nth *Morava K-theory*, so that the homotopy groups of the sphere spectrum S^0 are built out of the localizations $L_{K(n)}S^0$. The elements of $\pi_* L_{K(n)}S^0$ are periodic with respect to v_n. For example, the homotopy of $L_{K(1)}S^0$ corresponds to the image of J, which was intensely studied by Adams in the 60s. In this case, there is a periodicity arising from the Bott element in topological K-theory and v_1 corresponds to the Bott element. The higher v_n-periodic families are analogues of the image of J with periodicity arising from v_n. In recent years, the focus has moved to computing $L_{K(2)}X$ for various X.

Given a finite CW complex X (of type n), one can associate a spectrum, the *telescope* $\widehat{X}$ of X, so that the stable homotopy groups of $\widehat{X}$ consist of v_n-periodic families for X. In the 1970s, Doug Ravenel posed the **telescope conjecture**, which asks if the natural map $\widehat{X} \to L_{K(n)}X$ is a homotopy equivalence. This conjecture has become extremely important since the homotopy groups of $L_{K(n)}X$ can be calculated using the geometry of height n formal groups. However, it is believed that this conjecture is generally false.

Broadly speaking, each part of my research relates to this story in one way or another and falls within the following three branches:

(1) Study v_2-periodic homotopy theory with a particular emphasis on computations;
(2) Develop and compute the motivic and equivariant analogues of the v_1-periodic theory, again with an emphasis on computations;
(3) Study invariants of ring spectra, such as topological Hochschild homology and algebraic K-theory, which see pieces of the v_n-periodic families.

3. Current and Future Research

3.1. **v_1 and v_2-periodic homotopy theory.** In [21] and [22], Mahowald intensely studied the image of J through the lens of the Adams spectral sequence based on the connective real K-theory spectrum, denoted bo. Using it he was able to prove the height 1 telescope conjecture and perform computations of the stable homotopy groups of spheres. In recent years, the spectrum tmf of topological modular forms has entered the scene, and is a height 2 version of bo. One of my main research problems is to carry out Mahowald's program for tmf. This leads to three subgoals for the tmf-based ASS:

(1) settle the height 2 telescope conjecture,
(2) compute homotopy groups, and
(3) study/compute $K(2)$-local homotopy groups.

The first step in approaching each of these problems is to study the homotopy groups of $\mathrm{tmf}\wedge\mathrm{tmf}$. The 2-primary version of this problem has been studied in [11] and [23]. In previous work, I studied a more approachable version of the problem at all primes. In particular, I computed the $BP\langle 2\rangle$ co-operations. Here

$$BP\langle 2\rangle_* = BP_*/(v_3, v_4, \ldots) = \mathbb{Z}_{(p)}[v_1, v_2]$$

Theorem 3.1 (Culver, [17], [16]). *There is an algorithm for computing $BP\langle 2\rangle\wedge BP\langle 2\rangle$ at all primes. Moreover, the v_2-torsion in $\pi_*BP\langle 2\rangle\wedge BP\langle 2\rangle$ is simple torsion and concentrated in Adams filtration 0.*

In the case when $p=2$, then $BP\langle 2\rangle$ is equivalent to $\mathrm{tmf}_1(3)$, a variant of tmf built out of elliptic curves with a chosen point of order three, and so this theorem can viewed as a calculation of $\mathrm{tmf}_1(3) \wedge \mathrm{tmf}_1(3)$. In a different direction, Paul VanKoughnett and I have computed the $K(1)$-local homotopy groups of $\mathrm{tmf} \wedge \mathrm{tmf}$ using Hopkins' original construction of $\mathrm{tmf}_{K(1)}$.

Theorem 3.2 (Culver-VanKoughnett, [18]). *For the primes $p = 2, 3$, the $K(1)$-local homotopy of $\mathrm{tmf} \wedge \mathrm{tmf}$ is given by*

$$\pi_* L_{K(1)}(\mathrm{tmf} \wedge \mathrm{tmf}) \cong \left(KO_*[j^{-1}, \overline{j^{-1}}] \otimes \mathbb{T}(\lambda)/(\psi^p(\lambda) - \lambda - j^{-1} + \overline{j^{-1}}) \right)^{\wedge}_p$$

where $\mathbb{T}$ refers to the free ϑ-algebra.

I am also computing the unlocalized homotopy groups of $\mathrm{tmf} \wedge \mathrm{tmf}$ at the prime $p = 3$. I am currently using an analogue of the Adams spectral sequence developed by Hill and Henriques, which takes the form

$$\mathrm{Ext}_{A(1)_* \otimes E(a_2)}(\mathbb{F}_3, H_* \mathrm{tmf}) \implies \pi_*(\mathrm{tmf} \wedge \mathrm{tmf}) \otimes \mathbb{Z}_3.$$

I am currently on track to give a complete computation of this spectral sequence. Once armed with this computation, we ought to be able to show that $\mathrm{tmf} \wedge \mathrm{tmf}$ splits in terms of $\mathrm{tmf}_0(2^j)$, which is a variant of tmf built out of elliptic curves with level $\Gamma_0(2^j)$.

In joint work with Beaudry, Behrens, Bhattacharya, and Xu, [5], I have intensely studied the bo-based ASS, and we have extended previous computations of Mahowald ([21]) and Mahowald-Lellmann ([22]). This project is a warm-up to computing the tmf-resolution. In particular, we have

Computation 1 (Beaudry-Behrens-Bhattachary-C.-Xu, [10]). *There are various spectral sequences which allow us to compute the E_2-term of the bo-based ASS through a range. Moreover, the bo-based ASS collapses at the E_2-term up through the 40-stem.*

One of the difficulties with the bo-based ASS are classes which are v_1-torsion on the E_1-page; we refer to such classes as *evil*. Classes which are v_1-periodic on the E_1-page are called *good*. It turns out that evil classes are very difficult to account for on the E_∞-page of the bo-based ASS, despite the fact that many important elements in $\pi_* S$ are detected by such elements. We prove the following result.

Theorem 3.3 (Beaudry-Behrens-Bhattacharya-C.-Xu [5]). *Suppose x is a non-trivial class of $\mathrm{Ext}^{s,t}_{A_*}(\mathbb{F}_2, \mathbb{F}_2)$, then,*

1. *if x is v_1-torsion, it is evil; and*
2. *if x lies above the $1/3$-line ($t - s < 3s$), then it is good and v_1-periodic,*
3. *Suppose x is v_1-periodic. There is a $M \gg 0$ is so that $y = v_1^M x$ lies above the $1/3$-line. Suppose that y has bo-filtration n. The class x is good if and only if $s \geq n$.*

This theorem reduces the question of when there is an evil class in the bo-based ASS to questions about $\mathrm{Ext}_{A_*}(\mathbb{F}_2,\mathbb{F}_2)$, which is approachable via digital computation.

We hope that our techniques will extend to the tmf-ASS, and we have been able to apply them to the tmf-ASS at the prime $p=2$ for the Bhattacharya-Egger spectrum Z, instead of the sphere ([12]). In particular, we can show the following.

Theorem 3.4 (Beaudry-Behrens-Bhattacharya-C.-Xu, [6]). *The* tmf*-ASS for Z is computable using similar techniques to those in* [5]*, and there are several possible counter examples to the telescope conjecture at the E_2-page. Furthermore, a previously unresolved potential d_3-differential in the $K(2)$-local $ANSS$ for Z (*[13]*) does not occur.*

At the moment we cannot show that the potential counter examples survive the spectral sequence; if they did survive this would show the telescope conjecture is false. One of my goals is to study the analogous problem at the prime $p=3$. In this case, the homotopy groups of tmf are computationally simpler than at $p=2$, while still having a large Hurewicz image. This makes the telescope conjecture at the prime 3 more approachable with this method.

The tmf-ASS can also be $K(2)$-localized to give a spectral sequence converging to the $K(2)$-local sphere $S^0_{K(2)}$. Techniques to compute the homotopy of $S^0_{K(2)}$ were developed by Goerss-Henn-Mahowald-Rezk (GHMR) in [19] and later shown in [7],[8], [9] to have an interpretation in terms of elliptic curves and modular forms. In particular, Behrens shows that for any prime ℓ there is a spectrum $Q(\ell)$, which is built out of isogenies of elliptic curves of degree ℓ^k, and shows at $p=3$ that there is a cofiber sequence

$$L_{K(2)}S \to Q(2) \to \Sigma D_{K(2)}Q(2). \tag{3.5}$$

The second map in this sequence is often referred to as the *middle map*. This is the height 2 analogue of the Adams-Baird resolution, which is a cofiber sequence

$$L_{K(1)}S^0 \longrightarrow KO \xrightarrow{\psi^3-1} KO.$$

A connective version of the Adams-Baird resolution can be located within the bo-Adams spectral sequence, and after $K(1)$-localization reduces to it. I am interested in a height 2 analogue of this fact.

Conjecture 3.6. There is a connective version of the resolution (3.5) which sits within the tmf-based ASS. By this, we mean that when we $K(2)$-localize, we recover (3.5).

Towards proving this, I plan to work on the following.

Goal 1. *Construct connective versions of the spectra $Q(\ell)$.*

A periodic version has already been constructed by Behrens. It would seem that the work of Lawson-Hill ([20]) will be relevant here to extend the construction over the cusps of the relevant moduli spaces. Provided that I can find such a connective version of (3.5), I am hoping to compute the middle map through the tmf-based ASS. This map appears as an unresolved d_1-differential in [19] and continues to be an obstruction for better understanding/computing v_2-periodic homotopy theory. For example, this unresolved d_1-differential has stymied efforts to give a completely satisfactory computation of $\pi_* S_{K(2)}$ at $p = 3$. Finding a version of the Behrens/GHMR resolution in the tmf-based ASS would also imply that the $K(2)$-localized tmf-based ASS has a horizontal vanishing line, which would be helpful in studying the height 2 telescope conjecture at the prime $p = 3$.

3.2. **Motivic and equivariant v_1-periodic homotopy theory.** Voevodsky and others have constructed a category of motivic spectra, which is built from schemes over a base field rather than topological spaces. It turns out that this category has a lot of similar properties as the classical stable homotopy category. In particular, in [14],[15], motivic analogs of the Morava K-theories $K(n)$ are constructed. In recent work with JD Quigley, I have been studying v_1-periodicity in the motivic setting. In particular, we have carried out Mahowald's program on bo-resolutions in the $\mathbb{C}$-motivic category, using the motivic analog of bo, denoted as kq ([1]). We have shown the following.

Theorem 3.7 (Culver-Quigley)**.** *In the kq-based ASS, the 0- and 1-lines of the E_∞-page can be explicitly computed and display a pattern similar to the image of J. Moreover, the E_∞-page has a vanishing line of slope 1/5, which implies all the v_1-periodic elements are detected in the 0- and 1-line.*

In motivic homotopy theory, the first Hopf invariant one element η is non-nilpotent, and an important question in the subject is to compute the η-inverted homotopy of the motivic sphere spectrum. Over $\mathbb{C}$, this has been computed by Andrews-Miller ([2]). Quigley and I have shown the following:

Theorem 3.8 (Culver-Quigley)**.** *We can recover $\eta^{-1}\pi_{**}S^{0,0}$ from the 0- and 1-line of the E_∞-page of the kq-based ASS.*

This suggests that the kq-based ASS might be a useful tool for studying $\eta^{-1}\pi_{**}S^{0,0}$. Quigley and I have begun to consider the analogous situation over other base fields. The spectra KQ and kq are defined over any field F whose characteristic is not 2, and hence the kq-resolution can be studied over such bases. We are especially interested in the case $F = \mathbb{Q}, \mathbb{R}, \mathbb{Q}_p$, and $\mathbb{F}_q$ for q odd. More specifically, we are interested in the following question.

Question 3.9. Can we use the kq-based ASS over $F = \mathbb{F}_q, \mathbb{Q}_p, \mathbb{R}$ to compute the v_1-periodic elements? Can we use it to compute some new 2-torsion in the motivic homotopy groups of spheres? Can we use it to compute $\eta^{-1}\pi_{**}S^0$, the stable homotopy groups of spheres where we have inverted η.

A natural starting point to tackle this question is the case when $F = \mathbb{R}$. It is now understood that there is a close connection between C_2-equivariant homotopy theory and $\mathbb{R}$-motivic homotopy theory. In the C_2-equivariant setting we have a bit more technology at our disposal, which may allow us to more directly generalize the classical story. I am working on the following.

Goal 2. *Use recent work of Hahn-Wilson to prove a decomposition of $bo_{C_2} \wedge bo_{C_2}$ analogous to Mahowald's. Use this to calculate the E_1-term of the bo_{C_2}-based Adams spectral sequence.*

This is the first step towards the following goal of Quigley and myself,

Goal 3. *Study the bo_{C_2}-based ASS much as in Culver-Quigley. See if the 0- and 1-line capture the equivariant image of J as calculated by Minami ([26], [27]).*

Having done this our next natural goal is the following.

Goal 4. *Use the C_2-equivariant and $\mathbb{C}$-motivic calculations and Quigley's squeeze lemma to obtain the $\mathbb{R}$-motivic calculation.*

3.3. **Invariants of commutative ring spectra.** With Gabe Angelini-Knoll, I have been computing the homotopy groups of certain spectral invariants of ring spectra. Given a classical ring R, one can associate many invariants such as the Hochschild homology $HH(R)$, the cyclic homology $HC(R)$, and its algebraic K-groups $K(R)$. The construction of these invariants all have analogues in spectra, and to a commutative ring spectrum R give rise to *topological Hochschild homology* $THH(R)$, *topological cyclic homology* $TC(R)$, and its *algebraic K-theory* $K(R)$.

Currently, a major question in the subject is the *chromatic redshift conjecture*. This conjecture asserts that if R is a v_n-periodic commutative ring spectrum, then $K(R)$ is a commutative ring spectrum which is v_{n+1}-periodic. The case when $n=0$ corresponds to the Lichtenbaum-Quillen conjecture and is intimately related to the computation of the image of the J-homomorphism. A standard way to compute K-theory of a commutative ring spectrum is via *trace methods*. In particular, if R is a ring spectrum, then there is a canonical map, called the *cyclotomic trace*,

$$\mathrm{trc} : K(R) \to TC(R)$$

which often exhibits $TC(R)$ as a very good approximation to K-theory. Currently, Angelini-Knoll and I are interested in studying the $n=2$ case of the chromatic red-shift conjecture. Our approach is to study, following Ausoni and Rognes, the THH and TC of the second truncated Brown-Peterson spectrum $BP\langle 2\rangle$. We are currently completing the following computation:

Goal 5. *Compute the homotopy groups of the topological Hochschild homology of $BP\langle 2\rangle$ with coefficients in connective Adams summand $\ell := BP\langle 2\rangle/v_1$.*

We have been approaching this problem using the THH-May spectral sequence constructed in [3] along with techniques used by McClure-Staffeldt [24] and Angletviet-Hill-Lawson [4]. After completing this computation, we plan to tackle the following question.

Conjecture 3.10. There exists a multiplicative filtration of $BP\langle 2\rangle$ in ℓ-modules.

Given this, we ought to be able to use a version of the THH-May spectral sequence, which should take the form

$$THH_*(BP\langle 2\rangle;\ell)\otimes \Lambda_{\mathbb{Z}_p}\sigma v_2 \implies THH_*(BP\langle 2\rangle),$$

and we expect this spectral sequence to be computable. The next step in our project is to compute $TC(BP\langle 2\rangle)$. Most likely we will have to smash with a type 3-complex. Armed with this computation, we hope to see if the red-shift conjecture holds in this case.

I am also interested in several questions related to this project. At the primes $p=2,3$, the spectrum $BP\langle 2\rangle$ admits an E_∞-model as $\mathrm{tmf}_1(3)$ when $p=3$ and as a topological automorphic forms when $p=3$. Currently, Bruner-Rognes are completing a computation of the THH of tmf. However, given any congruence subgroup $\Gamma \subseteq SL_2(\mathbb{Z})$, there is a spectrum $Tmf(\Gamma)$ which often admits a good connective cover $\mathrm{tmf}(\Gamma)$. I have begun to think about the following problem.

Goal 6. *Compute the topological Hochschild homology of* $\mathrm{tmf}(\Gamma)$ *(and/or its non-connective variants).*

The spectra $\mathrm{Tmf}(\Gamma)$ and $\mathrm{TMF}(\Gamma)$ can be regarded as the global sections of a derived sheaf on the moduli spaces $\mathscr{M}(\Gamma)$ of elliptic curves of level Γ. Recent work of Lurie allows us to think of this in terms of certain "derived moduli spaces," and in this context THH is the de Rham complex. This leads me to wonder about the following.

Question 3.11. Can we give computations of THH of $\mathrm{Tmf}(\Gamma)$ and $\mathrm{TMF}(\Gamma)$ in terms of invariants of the moduli stack $\mathscr{M}(\Gamma)$ of elliptic curves of level Γ?

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