ON $BP\langle 2\rangle$-COOPERATIONS

D. CULVER

ABSTRACT. In this paper we develop techniques to compute the cooperations algebra for the second truncated Brown-Peterson spectrum $BP\langle 2\rangle$. We prove that the cooperations algebra $BP\langle 2\rangle_* BP\langle 2\rangle$ decomposes as a direct sum of a $\mathbb{F}_2$-vector space concentrated in Adams filtration 0 and a $\mathbb{F}_2[v_0, v_1, v_2]$-module which is concentrated in even degrees and is v_2-torsion free. We also develop a recursive method which produces a basis for the v_2-torsion free part.

CONTENTS

1. INTRODUCTION

The purpose of this paper is to give a description of the algebra of cooperations for the second *truncated Brown-Peterson spectrum*, denoted by $BP\langle 2\rangle$, at the prime 2. At chromatic height 1, the cooperations algebra of $BP\langle 1\rangle$ was computed by Adams in [1]. At the prime is 2, the spectrum $BP\langle 1\rangle$ is the 2-localization of the connective complex K-theory spectrum, denoted by bu, and when the prime is odd, $BP\langle 1\rangle$ is the Adams summand of connective complex K-theory. In his calculation, Adams observed that the E_2-page of the Adams spectral sequence for $BP\langle 1\rangle \wedge BP\langle 1\rangle$ has a non-canonical direct sum decomposition

$$\mathrm{Ext}_{E(1)_*}(\mathbb{F}_2, (A/\!\!/E(1))_*) \simeq V \oplus \mathcal{C}$$

where the subspace V is concentrated in Adams filtration 0 and $\mathcal{C}$ is v_1-torsion free. Adams also gave a complete description of $\mathcal{C}$ in terms of Adams covers of $\mathrm{Ext}_{E(1)_*}(\mathbb{F}_2, \mathbb{F}_2)$.

The interest in studying $BP\langle 2\rangle$-cooperations originates in Mark Mahowald's work on the Adams spectral sequence based on connective real K-theory, denoted as bo. Armed with his calculation of $\mathrm{bo}_*\mathrm{bo}$, Mahowald proved the 2-primary v_1-telescope conjecture in [6]. With Wolfgang Lellmann, he was able to compute the bo-based Adams spectral sequence for the sphere, and showed that it collapses in a large range (cf. [7]). These calcuations have been extended in [2]. At chromatic height 2 and $p = 2$, the role of bo is played by tmf and the role of bu is played by the spectrum $\mathrm{tmf}_1(3)$, in that it is a form of $BP\langle 2\rangle$ (cf. [5]). Partial calculations of $\mathrm{tmf}_*\mathrm{tmf}$ have been achieved in [4].

The goal of this work is to compute the cooperations algebra for $\mathrm{tmf}_1(3)$. This is motivated by the fact that one can descend from $\mathrm{tmf}_1(3)$ to tmf through the Bousfield-Kan spectral sequence of the cosimplicial resolution

$$\mathrm{tmf}^{\wedge 2} \longrightarrow \mathrm{tmf}_1(3)^{\wedge 2} \rightrightarrows (\mathrm{tmf}_1(3)^{\wedge_{\mathrm{tmf}} 2})^{\wedge 2} \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} \cdots .$$

Since the spectrum $\mathrm{tmf}_1(3)$ is a form of $BP\langle 2\rangle$, for the purposes of calculations, we can replace $\mathrm{tmf}_1(3)$ by $BP\langle 2\rangle$. Consequently, a natural choice for computing the cooperations algebra is the Adams spectral sequence

$$\mathrm{Ext}_{A_*}(\mathbb{F}_2, H_*(BP\langle 2\rangle \wedge BP\langle 2\rangle; \mathbb{F}_2)) \implies BP\langle 2\rangle_* BP\langle 2\rangle \otimes \mathbb{Z}_2^{\wedge}.$$

There are two main contributions of this paper. The first is a structural result regarding the algebra $BP\langle 2\rangle_*BP\langle 2\rangle$. In particular, we will show there is a direct sum decomposition into a vector space V concentrated in Adams filtration 0, and a v_2-torsion free component. The second is an inductive calculation of $BP\langle 2\rangle_*BP\langle 2\rangle$. This inductive calculation is similar to the one produced in [4]. Moreover, this decomposition of $BP\langle 2\rangle_*BP\langle 2\rangle$ implies that the methods developed in [2] to calculate the bo-ASS can be applied to the $BP\langle 2\rangle$-ASS. One of our goals for later work is to prove an analogous splitting for $\mathrm{tmf}_*\mathrm{tmf}$ and develop the tmf-resolution as a computational device.

Conventions. We will let A denote the mod 2 Steenrod algebra and A_* its dual. We will let ζ_k denote the conjugate of the the generator ξ_k in the dual Steenrod algebra A_*. Given a Hopf algebra B and a comodule M over B, we will often abbreviate $\mathrm{Ext}_B(\mathbb{F}_2, M)$ to $\mathrm{Ext}_B(M)$. Homology and cohomology are implicitly with mod 2 coefficients. All spectra are implicitly 2-complete.

Acknowledgements. I would like to thank Mark Behrens for suggesting this project and for innumerable helpful discussions along the way. Special thanks also go to Stephan Stolz for many enlightening exchanges and for carefully reading an earlier draft of this paper. I would also like to thank Prasit Bhattacharya and Nicolas Ricka for several helpful conversations. I would also like to thank Doug Ravenel for suggesting that I prove the topological version of Theorem 2.1. Finally, I thank Dylan Wilson and Paul VanKoughnett for pointing out an error in my original approach to proving Theorem 2.1.

2. The Adams spectral sequence for $BP\langle 2\rangle \wedge BP\langle 2\rangle$

In this section, we will prove $BP\langle 2\rangle_*BP\langle 2\rangle$ decomposes into a v_2-torsion and v_2-torsion free component. This will be accomplished through the Adams spectral sequence

$$\mathrm{Ext}_{A_*}(\mathbb{F}_2, H_*(BP\langle 2\rangle \wedge BP\langle 2\rangle)) \implies BP\langle 2\rangle_*BP\langle 2\rangle.$$

In particular, we will begin by determining the structure of the E_2-page. Recall that the mod 2 homology of $BP\langle 2\rangle$ is given by

$$H_*BP\langle 2\rangle \simeq (A /\!\!/ E(2))_*$$

where $E(2)$ denotes the subHopf algebra of the Steenrod algebra A generated by the first three Milnor primitives Q_0, Q_1, Q_2 (cf. [1]):

$$E(2) := E(Q_0, Q_1, Q_2).$$

As a subalgebra of the dual Steenrod algebra, the homology of $BP\langle 2\rangle$ is explicitly given as

$$(A /\!\!/ E(2))_* = \mathbb{F}_2[\zeta_1^2, \zeta_2^2, \zeta_3^2, \zeta_4, \zeta_5, \ldots],$$

see [10] for this calculation. By the Künneth theorem, we have

$$H_*(BP\langle 2\rangle \wedge BP\langle 2\rangle) \simeq H_*BP\langle 2\rangle \otimes H_*BP\langle 2\rangle$$

and hence, via a change of rings, we find that the E_2-term of this spectral sequence is

$$\mathrm{Ext}_{E(2)_*}(\mathbb{F}_2, (A /\!\!/ E(2))_*).$$

Here, the dual of $E(2)$ is given by

$$E(2)_* \simeq E(\zeta_1, \zeta_2, \zeta_3).$$

The $E(2)_*$-comodule structure of $(A /\!\!/ E(2))_*$ uniquely determines, and is uniquely determined by, a corresponding $E(2)$-module structure (given by the dual action of $E(2)$). Thus we may rewrite the E_2-page as

$$\mathrm{Ext}_{E(2)_*}(\mathbb{F}_2, (A /\!\!/ E(2))_*) = \mathrm{Ext}_{E(2)}(\mathbb{F}_2, (A /\!\!/ E(2))_*)$$

where the right hand side corresponds to Ext of modules. In order to calculate the Adams spectral sequence, ASS, we need to calculate this Ext group of modules over $E(2)$. Recall that the Adams spectral sequence for $BP\langle 2\rangle$ takes the form

$$\mathrm{Ext}_{E(2)}(\mathbb{F}_2, \mathbb{F}_2) \implies \pi_* BP\langle 2\rangle$$

and that the E_2-term is isomorphic to $\mathbb{F}_2[v_0, v_1, v_2]$. The main theorem of this section concerns the structure of $\mathrm{Ext}_{E(2)}(\mathbb{F}_2, (A /\!\!/ E(2))_*)$ as a module over this three variable polynomial algebra.

Theorem 2.1. *The E_2-page of the ASS for $BP\langle 2\rangle \wedge BP\langle 2\rangle$ admits a decomposition as modules over $\mathbb{F}_2[v_0, v_1, v_2]$ as $\mathcal{C} \oplus V$ where $\mathcal{C}$ is v_2-torsion free and is concentrated in even $(t-s)$-degree, and V is concentrated in Adams filtration 0.*

Corollary 2.2. *The ASS for $BP\langle 2\rangle \wedge BP\langle 2\rangle$ collapses at E_2.*

Proof. We will show how to prove this from the theorem. Suppose that E_r is the first page in which we have nontrivial differentials, and let

$$d_r x = y$$

be such a nontrivial differential. Since V is concentrated in Ext^0, it follows that y cannot be an element of V, as y necessarily has Adams filtration at least r. Since $\mathcal{C}$ is concentrated in even $(t-s)$-degree, it also follows that x cannot be an element of $\mathcal{C}$. Thus, we have $x \in V$ and $y \in \mathcal{C}$. Since $BP\langle 2\rangle \wedge BP\langle 2\rangle$ is a $BP\langle 2\rangle$-module, the differentials in the ASS are linear over $\mathrm{Ext}_{E(2)}(\mathbb{F}_2)$. So multiplying by v_2 on the differential gives

$$0 = d_r v_2 x = v_2 y.$$

As $y \in \mathcal{C}$, that $v_2 y = 0$ implies $y = 0$. And so there are no differentials on the E_r-page, contradiction. So this spectral sequence has no differentials. $\square$

Corollary 2.3. *The summand V satisfies*

$$v_0 V = v_1 V = v_2 V = 0.$$

Proof. Since V is concentrated in Adams filtration 0, and since the Adams filtration of v_0, v_1, and v_2 is 1, the corollary follows. $\square$

2.1. **The (co)module structure of** $(A/\!/E(2))_*$**.** We will now describe the structure of $(A/\!/E(2))_*$ as a module over $E(2)$, which is necessary in order to compute Ext. To do this, we will use first describe the $E(2)$-comodule structure. Recall that the coproduct on the dual Steenrod algebra A_* is given by

$$\psi : A_* \to A_* \otimes A_*; \zeta_n \mapsto \sum_{0 \le i \le n} \zeta_i \otimes \zeta_{n-i}^{2^i}.$$

This formula shows that, when restricted to $(A/\!/E(2))_*$, the coproduct satisfies

$$\psi : (A/\!/E(2))_* \to A_* \otimes (A/\!/E(2))_*,$$

making $(A/\!/E(2))_*$ into an A_*-comodule algebra. Moding out by $(A/\!/E(2))_*$ on the left gives a map

$$\alpha : (A/\!/E(2))_* \to E(2)_* \otimes (A/\!/E(2))_*.$$

Applying the formula for the coproduct on A_*, we find that

$$\alpha : \zeta_n \mapsto 1 \otimes \zeta_n + \zeta_1 \otimes \zeta_{n-1}^2 + \zeta_2 \otimes \zeta_{n-2}^4 + \zeta_3 \otimes \zeta_{n-3}^8, \;\; n > 3$$

and

$$\alpha : \zeta_n^2 \mapsto 1 \otimes \zeta_n^2, \;\; n = 0, 1, 2, 3.$$

As in [1], given a locally finite $E(2)_*$-comodule M, we may define a $E(2)$-module structure on M via the following formula: if $\alpha(x)$ is given by $\sum_i x_i' \otimes x_i''$ then define

$$Q_n x := \sum_i \langle Q_n, x_i' \rangle x_i''.$$

Thus

$$Q_i \zeta_n = \zeta_{n-i-1}^{2^{i+1}}.$$

Since the Q_i are primitive elements in A_*, we see that Q_i is a derivation on $(A/\!\!/E(2))_*$, and so we have completely determined the structure of $(A/\!\!/E(2))_*$ as a module over $E(2)$. Though the module structure on $(A/\!\!/E(2))_*$ is rather simple, $(A/\!\!/E(2))_*$ is very large, making calculating the Ext groups difficult. The following subsections will develop means of breaking up $(A/\!\!/E(2))_*$ into simpler pieces. This will rely heavily on the *Margolis homology* of $(A/\!\!/E(2))_*$. We will now briefly review Margolis homology.

Let P be a module over an exterior algebra $E(x)$ on one generator. Then the multiplication by x on P can be regarded as a differential, as it squares to 0, making P into a chain complex. The homology of this chain complex is called the *Margolis homology* of P, and is denoted by $M_*(P;x)$. In other words,

$$M_*(P;x) := \ker(x\cdot : P \to P) / \operatorname{im}(x\cdot : P \to P).$$

Modules P over the algebra $E(2)$ will have three different Margolis homology groups, namely the ones arising from restricting P to a module over the exterior algebras $E(Q_0), E(Q_1), E(Q_2)$. The following theorem demonstrates the importance of the Margolis homology groups.

Theorem 2.4 (Margolis, cf [1], [8]). *Let P be a module over E where E is an exterior algebra on a (possibly countably infinite) set of generators $x_1, x_2, \ldots$ so that their degrees satisfy $0 < |x_1| < |x_2| < \cdots$. If P is bounded below, then P is free if and only if all of its Margolis homology groups vanish.*

Recall that two modules P and Q are *stably equivalent* if there are free modules F and F' such that there is an isomorphism

$$P \oplus F \simeq Q \oplus F'.$$

Corollary 2.5. *If $f : P \to Q$ is a map of bounded below E-modules, then f is a stable equivalence if and only if f induces an isomorphism in all Margolis homology groups.*

For later subsections, we will now record the Margolis homology of $BP\langle 2\rangle$.

Proposition 2.6. *The Margolis homology of $BP\langle 2\rangle$ is given by*

$$M_*(BP\langle 2\rangle; Q_0) = \mathbb{F}_2[\zeta_1^2, \zeta_2^2]$$
$$M_*(BP\langle 2\rangle; Q_1) = \mathbb{F}_2[\zeta_1^2] \otimes E(\zeta_i^2 \mid i \geq 2)$$
$$M_*(BP\langle 2\rangle; Q_2) = \frac{\mathbb{F}_2[\zeta_i^2 \mid i \geq 1]}{(\zeta_i^8 \mid i \geq 1)}$$

Proof. This is an easy generalization of the proof given in [1] for the case of bu. □

2.2. **An $E(2)$-module splitting of $(A/\!\!/E(2))_*$.**

Definition 2.7. Let $m \in A_*$ be a monomial, say it is

$$m = \zeta_1^{i_1}\zeta_2^{i_2}\zeta_3^{i_3}\cdots.$$

Define the *length* of m to be the number of odd exponents in m:

$$\ell(m) := \#\{k \mid i_k \equiv 1(2)\}.$$

We will let $A_*^{(\ell)}$ denote the subspace of A_* spanned by monomials of length ℓ, and we will say elements of $A_*^{(\ell)}$ have *length* ℓ.

The following lemma states that the action by a Milnor primitive on $(A/\!\!/E(2))_*$ decreases length by exactly 1.

Lemma 2.8. *Given $m \in (A/\!\!/E(2))_*$ a monomial and $i = 0, 1, 2$, we have*

$$\ell(Q_i m) = \ell(m) - 1.$$

Proof. This follows from the formula for the action of Q_i on ζ_k and the fact that Q_i acts via derivations. □

Remark 2.9. The previous lemma allows us to put an extra grading on the Margolis homologies of $(A/\!\!/E(2))_*$. Namely, define $M^{(\ell)}$ to be the span of length ℓ monomials in $(A/\!\!/E(2))_*$. From the lemma, we can consider the chain complex

$$M_\bullet : \cdots \xrightarrow{\cdot Q_i} M^{(\ell+1)} \xrightarrow{\cdot Q_i} M^{(\ell)} \xrightarrow{\cdot Q_i} \cdots \xrightarrow{\cdot Q_i} M^{(0)}$$

and the homology of this chain complex is precisely the Margolis homology $M_*(BP\langle 2\rangle; Q_i)$. This puts a bigrading on the Margolis homology

$$M_*(BP\langle 2\rangle; Q_i) \cong \bigoplus_{\ell \geq 0} M_{*,\ell}(BP\langle 2\rangle; Q_i).$$

Consequently, Proposition 2.6 implies the following,

Corollary 2.10. *If $\ell > 0$ and $i = 0, 1, 2$, then*

$$M_{*,\ell}(BP\langle 2\rangle; Q_i) = 0.$$

From Adams' calculation of $\mathrm{bu}_*\mathrm{bu}$ we expect there to be infinitely many torsion classes concentrated in Adams filtration 0. The purpose in introducing the notion of length is to locate a large amount of the torsion inside $\mathrm{Ext}_{E(2)_*}((A /\!/ E(2))_*)$. Recall that the group $\mathrm{Ext}^0_{E(2)_*}(\mathbb{F}_2, M)$ is the group of primitive elements in the comodule M. Translating this into the language of modules, this corresponds to elements $y \in (A /\!/ E(2))_*$ for which Q_0y, Q_1y, and Q_2y are all zero. One source, then, for torsion elements in Ext^0 are the bottom cells of free submodules of $(A /\!/ E(2))_*$.

Suppose that $x \in (A /\!/ E(2))_*$ generates a free $E(2)$-module. This is equivalent to the element $Q_0Q_1Q_2x$ being nonzero. From Lemma 2.8, this implies that $\ell(x)$ is at least 3. Motivated by this observation, we define S to be the $E(2)$-submodule of $(A /\!/ E(2))_*$ generated by monomials of length at least 3,

$$S := E(2)\{m \in (A /\!/ E(2))_* \mid \ell(m) \geq 3\}.$$

Proposition 2.11. *The Margolis homology of S is trivial.*

Proof. Let $x \in S$ with $Q_ix = 0$. If $\ell(x) = 0$ then we can write

$$x = Q_2Q_1Q_0y$$

for some y of length 3, in which case x represents 0 in $M_*(S; Q_i)$. If $\ell(x) \geq 2$, then by Corollary 2.10, it follows that there is a y in $(A /\!/ E(2))_*$ with $Q_iy = x$. So $\ell(y) \geq 3$, and so $y \in S$. Thus x represents zero in $M_*(S; Q_i)$.

So the only interesting case is when $\ell(x) = 1$. For concreteness, suppose $Q_0x = 0$. As $\ell(x) = 1$, Corollary 2.10 implies there is a $y \in (A /\!/ E(2))_*$ with $Q_0y = x$. If $Q_1y = 0$, then since $\ell(y) = 2$, there is a $z \in (A /\!/ E(2))_*$ with $Q_1z = y$, again by Corollary 2.10. Note that $\ell(z) = 3$. Then $x = Q_0Q_1z$, which shows that x represents zero in $H_*(S; Q_0)$. This is similarly true if $Q_2y = 0$. So we will assume that $Q_1y \neq 0$ and $Q_2y \neq 0$.

Observe that if $\ell(x) = 1$, then there are x_1, x_2, x_3 of length 3 with

$$x = Q_0Q_1x_1 + Q_0Q_2x_2 + Q_1Q_2x_3$$

and so $Q_0x = 0$ if and only if $Q_0Q_1Q_2x_3 = 0$. So we may assume that x is of the form $x = Q_1Q_2x_3$ for some x_3 with $\ell(x_3) = 3$. This implies $Q_1x = Q_2x = 0$.

So assume that $Q_1 y \neq 0$ and $Q_2 y \neq 0$. We will modify y to produce an element y' with $Q_0 y' = x$ and $Q_1 y' = 0$. Define $y_0 := y$ and $x_0 := Q_1 y$. Since $Q_1 x = 0$, we see

$$Q_0 x_0 = Q_0 Q_1 y = Q_1 x = 0$$

Thus, there is a y_1 with $Q_0 y_1 = x_0$ by Corollary 2.10. Note that $\ell(y_1) = 2$ and note that $|y_1| = |y_0| - 3$. We now ask if $Q_1 y_1$ is zero. If it is, then we stop, otherwise we take $x_1 := Q_1 y_1$ and note again that $Q_0 x_1 = 0$. Thus we find y_2 with $Q_0 y_2 = x_1$. We continue this procedure, producing elements $y_0, y_1, \ldots$ of length two, and we stop once we reach n with $Q_1 y_n = 0$. Such an n will occur eventually because $|y_i| = |y_{i-1}| - 3$ and $(A /\!\!/ E(2))_*$ is a connective algebra.

Let the procedure stop at stage n. Then we have produced $y_0, y_1, \ldots, y_n$ and $x_0, \ldots, x_{n-1}$ with

- $Q_0 y_i = x_{i-1}$,
- $Q_1 y_i = x_i$.

In other words, we have produced an $E(2)_*$-module with generators $y_0, \ldots, y_n$. Since $Q_1 y_n = 0$ and $\ell(y_n) = 2$, there is a z_n with $Q_1 z_n = y_n$. So define $y'_{n-1} := Q_0 z_n + y_{n-1}$. Then $Q_0 y'_{n-1} = x_{n-2}$ and

$$Q_1(y'_{n-1}) = Q_1 Q_0 z_n + Q_1 y_{n-1} = Q_0 y_n + Q_1 y_{n-1} = x_n + x_n = 0.$$

Thus we can find z_{n-1} with $Q_1 z_{n-1} = y'_{n-1}$. Define $y'_{n-2} := y'_{n-1} + Q_0 z_{n-1}$. Then $Q_0 y'_{n-2} = x_{n-3}$ and $Q_1 y'_{n-2} = 0$. Keep performing this procedure to produce an element y'_1 with $Q_0 y'_1 = x_0$ and $Q_1 y'_1 = 0$. Then we can find z_1 with $Q_1 z_1 = y'_1$ and so we can define $y'_0 := Q_0 z_1 + y_0$. Then $Q_0 y'_0 = x$ and

$$Q_1 y'_0 = Q_0 y'_1 + Q_1 y_0 = x_0 + x_0 = 0.$$

Thus there is z_0 with $Q_1 z_0 = y'_0$. Since $\ell(z_0) = 3$, we have $z_0 \in S$ and so $x = Q_0 Q_1 z_0$, which shows x represents zero in $M_*(S; Q_0)$. Thus we have shown $M_*(S; Q_0) = 0$.

A similar proof for $M_*(S; Q_1)$ works with Q_1 replacing Q_0 and Q_2 replacing Q_1. However, the proof that $M_*(S; Q_2) = 0$ requires some adjustment. The problem is that if we were to perform an analogous procedure, say with Q_0 and Q_2, then the y_i's would not decrease degree, rather

$$|y_i| = |y_{i-1}| + 5$$

To rectify this, we may without loss of generality restrict to the case when x belongs to some particular weight as defined in [4] (we review this concept later in section 3.1). Since the action by Q_0, Q_1, Q_2 preserves the weight, all the y_i will have the same weight as x. The subcomodule $M_2(k) \subseteq (A /\!\!/ E(2))_*$ of weight $2k$ is finite dimensional. So the procedure we have described will terminate at a finite stage if we restrict to $x \in M_2(k)$ for some k. $\square$

Corollary 2.12. *The submodule S is a free $E(2)$-module.*

Proof. As S is bounded below, Theorem 2.4 implies that S is free. $\square$

Recall the following important theorem.

Theorem 2.13 ([8], pg. 245)**.** *Let B be a finite Hopf algebra over a field k, and let M be a module over B. Then the following are equivalent:*

- *M is free,*
- *M is projective,*
- *M is injective.*

In particular, we conclude that S is an injective $E(2)$-module. Considering the short exact sequence of $E(2)$-modules

$$0 \to S \to (A /\!\!/ E(2))_* \to Q \to 0.$$

Since S is injective, there is a splitting

$$(A /\!\!/ E(2))_* \simeq S \oplus Q.$$

Thus we get a corresponding decomposition for Ext. Our goal is to show that that $\mathrm{Ext}_{E(2)_*}(Q)$ is v_2-torsion free.

2.3. **The Bockstein spectral sequence for Q.** We begin with an overview of how to construct the Bockstein spectral sequence. Consider the short exact sequence of modules.

$$0 \to Q \to (E(2) /\!\!/ E(1))_* \otimes Q \to \Sigma^7 Q \to 0.$$

Recall that $(E(2) /\!\!/ E(1))_*$ is an exterior Hopf algebra $E(\zeta_3)$ with ζ_3 primitive. The first map comes from including Q into $1 \otimes Q$. The second map is the projection onto $\zeta_3 \otimes Q$. Applying $\mathrm{Ext}_{E(2)_*}$ gives an exact couple

$$\begin{tikzcd}
\mathrm{Ext}_{E(2)_*}(\Sigma^7 Q) \arrow[rr, "\partial"] & & \mathrm{Ext}_{E(2)_*}(Q) \arrow[dl] \\
& \mathrm{Ext}_{E(1)_*}(Q) \arrow[ul] &
\end{tikzcd}$$

Note that

$$\mathrm{Ext}^{s,t}_{E(2)_*}(\Sigma^7 Q) = \mathrm{Ext}^{s,t-7}_{E(2)_*}(Q)$$

which shows that the connecting homomorphism is of bidegree $(1,7)$. This is the correct degree for ∂ to be multiplication by v_2. In order to set up the BSS, we will want to show that the connecting map is indeed multiplication by v_2.

Proposition 2.14. *For the short exact sequence of $E(2)_*$-comodules*

$$0 \to \mathbb{F}_2 \to (E(2)/\!\!/E(1))_* \to \mathbb{F}_2\{\zeta_3\} \to 0,$$

the connecting homomorphism in $\mathrm{Ext}_{E(2)_*}$ induces multiplication by v_2.

Proof. This short exact sequence of comodules induces a short exact sequence of cobar complexes

$$0 \to C^\bullet_{E(2)_*}(\mathbb{F}_2) \to C^\bullet_{E(2)_*}((E(2)/\!\!/E(1))_*) \to C^\bullet_{E(2)_*}(\mathbb{F}_2\{\zeta_3\}) \to 0.$$

To calculate the boundary map, let

$$z = \sum_i [a_{1i} \mid \cdots \mid a_{si}]\zeta_3$$

be a cycle in the cobar complex for $\mathbb{F}_2\{\zeta_3\}$. This means that

$$\begin{aligned}
dz &= \sum_i d\left([a_{1i} \mid \cdots \mid a_{si}]1 \otimes \zeta_3\right) \\
&= \sum_i \sum_{j=0}^{s} [a_{1i} \mid \cdots \mid \psi(a_{ji}) \mid \cdots \mid a_{si}]1 \otimes \zeta_3 \\
&= 0
\end{aligned}$$

A lift of z to $C^\bullet_{E(2)_*}((E(2)/\!\!/E(1))_*)$ is

$$\overline{z} = \sum_i [a_{1i} \mid \cdots \mid a_{si}]\zeta_3$$

In the cobar complex for $(E(2)/\!\!/E(1))_*$,

$$\begin{aligned} d\,\overline{z} &= \sum_i d\left([a_{1i} \mid \cdots \mid a_{si}]\zeta_3\right) \\ &= \sum_i \sum_{j=0}^{s} [a_{1i} \mid \cdots \mid \psi(a_{ji}) \mid \cdots \mid a_{si}]\zeta_3 + \sum_i [a_{1i} \mid \cdots \mid a_{si} \mid \zeta_3]. \end{aligned}$$

Since z was a cycle, the first term is zero. So

$$d\,\overline{z} = \sum_i [a_{1i} \mid \cdots \mid a_{si} \mid \zeta_3].$$

Since ζ_3 represents v_2 in the cobar complex, and multiplication is induced by concatenation, it follows that the boundary map is indeed multiplication by v_2. $\square$

The upshot of this proposition is that we have the following distinguished triangle in the derived category $\mathcal{D}_{E(2)_*}$ of $E(2)_*$-comodules,

$$\Sigma^7\mathbb{F}_2[-1] \xrightarrow{\cdot v_2} \mathbb{F}_2 \longrightarrow (E(2)/\!\!/E(1))_* \longrightarrow \Sigma^7\mathbb{F}_2.$$

Tensoring with Q gives a distinguished triangle

$$\Sigma^7 Q[-1] \xrightarrow{\cdot v_2} Q \longrightarrow (E(2)/\!\!/E(1))_* \otimes Q \longrightarrow \Sigma^7 Q.$$

This allows us to consider the unrolled exact couple

$$\begin{tikzcd} \mathrm{Ext}^{s,t}_{E(2)}(Q) \arrow[d] & \mathrm{Ext}^{s-1,t-7}_{E(2)}(Q) \arrow[l, "\cdot v_2"'] \arrow[d] & \mathrm{Ext}^{s-2,t-14}_{E(2)}(Q) \arrow[l, "\cdot v_2"'] \arrow[d] & \cdots \arrow[l, "\cdot v_2"'] \\ \mathrm{Ext}^{s,t}_{E(1)}(Q) \arrow[ur, dotted] \arrow[d, equal] & \mathrm{Ext}^{s-1,t-7}_{E(1)}(Q) \arrow[ur, dotted] \arrow[d, equal] & \mathrm{Ext}^{s-2,t-14}_{E(1)}(Q) \arrow[ur, dotted] \arrow[d, equal] & \\ E_1^{s,t,0} & E_1^{s-1,t-7,1} & E_1^{s-2,t-14,2} & \end{tikzcd}$$

This results in the Bockstein spectral sequence, which is of the form

$$E_1^{***} = \mathrm{Ext}^{**}_{E(1)}(Q) \otimes \mathbb{F}_2[v_2] \implies \mathrm{Ext}^{**}_{E(2)}(Q).$$

This spectral sequence is trigraded, where

$$E_1^{s,t,r} = \mathrm{Ext}^{s,t}_{E(1)}(Q)\{v_2^r\},$$

and in this spectral sequence, $E_1^{s,t,r}$ converges to $\mathrm{Ext}_{E(2)}^{s+r,t+7r}(Q)$. The d_k-differential has trigrading

$$d_k : E_k^{s,t,r} \to E_k^{s-k+1,t-7k,r+k}$$

and $E_\infty^{s,t,r}$ converges to $\mathrm{Ext}_{E(2)}^{s+r,t+7r}(Q)$. So in Adams indexing, all the differentials look like d_1-differentials.

The utility of this spectral sequence is to prove that $\mathrm{Ext}_{E(2)}(Q)$ is v_2-torsion free. Towards this end, consider the short exact sequence

$$0 \to S' \to Q \to \overline{Q} \to 0$$

where S' is the $E(1)$-submodule of Q generated by length 2 monomials (or rather images of monomials).

Proposition 2.15. *The submodule S' has trivial Q_0 and Q_1-Margolis homology. Thus S' is a free $E(1)$-submodule.*

Proof. Suppose that $x \in S'$ is such that $Q_i x = 0$. So x represents an element in $M_*(S', Q_i)$. We are tasked with showing that x represents the zero element. If $\ell(x) = 0$, then as $x \in S'$ there must be a $y \in Q$ with $\ell(y) = 2$ such that $Q_0 Q_1 y = x$. So x is zero in $M_*(S'; Q_i)$.

So suppose that $\ell(x) = 1$. Since the Margolis homology of Q is isomorphic to that of $(A /\!\!/ E(2))_*$, Corollary 2.10 implies that there is a $y \in Q$ with $Q_i y = x$. Since $\ell(x) = 1$, then $\ell(y) = 2$, and so $y \in S'$. Thus showing that x is zero in the Margolis homology group $M_*(S'; Q_i)$. Finally, if $\ell(x) = 2$ and $Q_i x = 0$, then there is a $z \in Q$ with $Q_i z = x$. But as Q was the quotient of $(A /\!\!/ E(2))_*$ by S, any element of Q of length 3 is necessarily zero. Thus, $x = 0$ in Q.

This shows that the Margolis homology groups of S' are both zero, and so by Theorem 2.4, the module S' must be free over $E(1)$. □

Corollary 2.16. *There is a splitting of Q as an $E(1)$-module*

$$Q \simeq S' \oplus \overline{Q}$$

and thus we get a splitting

$$\mathrm{Ext}_{E(1)_*}(Q) \simeq \mathrm{Ext}_{E(1)_*}(S') \oplus \mathrm{Ext}_{E(1)_*}(\overline{Q}).$$

To prove that $\mathrm{Ext}_{E(2)_*}(Q)$ is v_2-torsion free, we will show that the v_2-BSS collapses at E_1. Since all the differentials look like d_1-differentials in Adams indexing, this will follow if we can prove $\mathrm{Ext}_{E(1)_*}(Q)$ is concentrated in even $(t-s)$-degree.

Observation 2.17. *Note that if m is a monomial of length 0, then by the definitions, the degree of m is even. Indeed, m being length zero means*

$$m = \zeta_1^{2i_1}\zeta_2^{2i_2}\zeta_3^{2i_3}\zeta_4^{2i_4}\cdots$$

where the exponents are all of the form $2i_k$. So

$$|m| = \sum_k 2|\zeta_k| \equiv 0 \mod 2.$$

Corollary 2.18. *The groups $\mathrm{Ext}_{E(1)_*}(S')$ are concentrated in even $(t-s)$-degree.*

Proof. Observe that if $x \in (A /\!/ E(2))_*$ is a length 2 monomial, then $x = m\zeta_i\zeta_j$ for some monomial m of length 0 and $i \neq j$. Observe that every length 0 monomial is in even degree. The degree of ζ_k is $2^k - 1$. So the degree of $\zeta_i\zeta_j$ is

$$|\zeta_i\zeta_j| = 2^i - 1 + 2^j - 1$$

which is even. Thus length 2 elements of $(A /\!/ E(2))_*$ are concentrated in even degree, and this remains true when projecting to Q. If $x \in Q$ generates a free $E(1)$-module M, then the unique nonzero element in $\mathrm{Ext}_{E(1)_*}(M)$ lives in degree $|x| - 4$. Thus if x is in even degree, it determines an element in $\mathrm{Ext}^{0,*}_{E(1)_*}(S')$ in even degree. Combining all these observations shows that $\mathrm{Ext}_{E(1)_*}(S')$ is concentrated in even degree. □

Proposition 2.19. *The Ext-groups of $\overline{Q}$ are concentrated in even $(t-s)$-degree.*

Proof. Since S' is free, the Q_0 and Q_1-Margolis homology groups of Q and $\overline{Q}$ are isomorphic. Consequently, the Margolis homology groups of $\overline{Q}$ are concentrated in length 0, and so in even degree.

Observe that for $x \in Q$ to generate a free $E(1)$-module, that the length of x must be 2. So $\overline{Q}$ has no free summands. By the classification theorem of modules over $E(1)$ (cf. [1], pg. 345), it follows that $\overline{Q}$ is a direct sum of finite lightning flash modules, a consequence of which is that $\mathrm{Ext}_{E(1)}(\overline{Q})$ is a direct sum of Adams covers of $\mathrm{Ext}_{E(1)_*}(\mathbb{F}_2)$.

The Adams cover associated to a lightning flash in $\overline{Q}$ is uniquely determined by the two unique nonzero elements of its Margolis homology. In the case of $\overline{Q}$, the Margolis homology is in even $(t-$

s)-degree, and this forces the Adams covers to be in even $(t-s)$-degree. $\square$

Corollary 2.20. *The BSS for Q collapses at E_1. Thus* $\mathrm{Ext}_{E(2)_*}(Q)$ *is v_2-torsion free.*

We can now prove Theorem 2.1.

Proof of Theorem 2.1. We have shown that there is a splitting of $E(2)$-modules
$$(A/\!\!/E(2))_* \simeq S \oplus Q$$
and so applying $\mathrm{Ext}_{E(2)}$ gives a decomposition
$$\mathrm{Ext}_{E(2)_*}((A/\!\!/E(2))_*) \simeq \mathrm{Ext}_{E(2)_*}(S) \oplus \mathrm{Ext}_{E(2)_*}(Q).$$
We have shown that S is free, and so $\mathrm{Ext}_{E(2)_*}(S)$ is concentrated in Ext^0. We have also just shown that $\mathrm{Ext}_{E(2)_*}(Q)$ is v_2-torsion free. So we define
$$V := \mathrm{Ext}_{E(2)_*}(S),$$
$$\mathcal{C} := \mathrm{Ext}_{E(2)_*}(Q).$$
The previous two propositions show that $\mathcal{C}$ is concentrated in even $(t-s)$-degree. This completes the proof of Theorem 2.1. $\square$

Remark 2.21. Note that the v_2-BSS for Q has many hidden extensions.

Remark 2.22. A consequence of the discussion thus far is that $\mathrm{Ext}_{E(2)_*}(Q)$ is generated as a module over $\mathbb{F}_2[v_0, v_1, v_2]$ by elements in $\mathrm{Ext}^{0,*}_{E(2)_*}$.

Remark 2.23. One could attempt to generalize the above arguments to the spectra $BP\langle n\rangle \wedge BP\langle m\rangle$ when $m \geq n$. In this case, the homology of $BP\langle n\rangle$ is $(A/\!\!/E(n))_*$. One would like to show that $\mathrm{Ext}_{E(n)_*}((A/\!\!/E(m))_*)$ splits into a v_n-torsion summand concentrated in Adams filtration 0, and a v_n-torsion free summand. Towards this end, one could define
$$S(m,n) \subseteq (A/\!\!/E(m))_*$$
to be the $E(n)$-submodule generated by monomials of length at least $n+1$, in analogy with S above. If it could be shown that $S(m,n)$ is a free module over $E(n)$, then we would have a splitting of $E(n)$-modules
$$(A/\!\!/E(m))_* = S(m,n) \oplus Q(m,n).$$
An inductive argument with the v_n-BSS would then allow one to show that $\mathrm{Ext}_{E(n)_*}(Q(m,n))$ is v_n-torsion free. In the case $n = m =$

2, $S(2,2)$ coincides with the submodule S defined above. The problem is that the arguments presented hitherto to show that $M_*(S;Q_i)$ is trivial do not seem to generalize to other values of m and n.

2.4. **Topological splitting.** In the previous subsection, we established a decomposition

$$\pi_*(BP\langle 2\rangle \wedge BP\langle 2\rangle) = V \oplus \mathcal{C}$$

where V is the $\mathbb{F}_2$-vector space of v_2-torsion elements and $\mathcal{C}$ is v_2-torsion free. In this section, we will establish that this splitting of homotopy groups in fact lifts to the stable homotopy category. That is, we will show that there is an equivalence of spectra

$$BP\langle 2\rangle \wedge BP\langle 2\rangle \simeq HV \vee C$$

with HV the Eilenberg-MacLane spectrum with $\pi_*(HV) = V$ and $\pi_*C = \mathcal{C}$.

Let X denote $BP\langle 2\rangle \wedge BP\langle 2\rangle$. We will establish this spectrum level splitting by showing there is a map

$$X \to HV$$

and defining C to be the fibre. This will produce a fibre sequence

$$C \to X \to HV.$$

We will show that there is a section to the map $X \to HV$.

In the previous section, we established a splitting

$$(A/\!\!/E(2))_* = S \oplus Q$$

with S a free $E(2)$-module. Dualizing gives a decomposition

$$A/\!\!/E(2) = S^* \oplus Q^* \tag{2.24}$$

and S^* is free as an $E(2)$-module.

In applying the change of rings theorem for an A-module M, one has to use the sheering isomorphism

$$A/\!\!/E(2) \otimes_{\mathbb{F}_2} M \simeq A \otimes_{E(2)} M.$$

where the left hand side is endowed with the diagonal action. In the case of H^*X, we have the isomorphism

$$A/\!\!/E(2) \otimes A/\!\!/E(2) \simeq A \otimes_{E(2)} (A/\!\!/E(2)).$$

Coupled with the decomposition 2.24, we see that as a module over A, the cohomology H^*X decomposes as

$$H^*X \simeq (A \otimes_{E(2)} S^*) \oplus (A \otimes_{E(2)} Q^*).$$

As S^* is free as an $E(2)$-module, the first factor is free as an A-module. Let us denote this free factor by F. Note that $H^*(HV)$ is precisely F. The idea is to show that the maps

$$F \to H^*X \to F$$

in the splitting of H^*X lift to maps of spectra via the Adams spectral sequence.

Consider the Adams spectral sequence

$$\mathrm{Ext}_A(F, H^*X) \implies [X, HV]_*.$$

Since F is free as an A-module, the E_2-page is concentrated in Adams filtration 0, and so it collapses at E_2. Note that

$$\mathrm{Ext}^0_A(F, H^*X) = \hom_A(F, H^*X),$$

so the inclusion of F into H^*X determines a map of spectra

$$X \to HV.$$

For the map in the other direction, we shall use the Adams spectral sequence again,

$$\mathrm{Ext}_A(H^*X, F) \implies [HV, X]_*.$$

For this spectral sequence, we can apply the change-of-rings isomorphism on the E_2-term,

$$\mathrm{Ext}_A(H^*X, F) \simeq \mathrm{Ext}_{E(2)}(A /\!/ E(2), F).$$

By Theorem 4.4 in [9], A is free over $E(2)$. Since S^* is also locally finite, it follows that $F = A \otimes_{E(2)} S^*$ is locally finite. Thus F is a locally finite free $E(2)$-module. If $\{b_\alpha\}$ is an $E(2)$-basis, then

$$F = \bigoplus_\alpha E(2)\{b_\alpha\} = \prod_\alpha E(2)\{b_\alpha\}$$

since F is locally finite. Thus

$$\mathrm{Ext}^{s,*}_{E(2)}((A /\!/ E(2))_*, F) \simeq \prod_\alpha \mathrm{Ext}^{s,*}_{E(2)}((A /\!/ E(2))_*, E(2)\{b_\alpha\}).$$

Since $E(2)$ is self-injective, it follows that each component group on the right-hand side is zero when $s > 0$. Thus the Ext^s groups are trivial for $s > 0$. So the E_2-page of the ASS is concentrated in Adams filtration 0, and hence collapses. Therefore we have the desired map of spectra. Thus we get the section of the cofibre sequence

$$C \to X \to HV$$

and hence the desired splitting

$$X \simeq C \vee HV.$$

Remark 2.25. If we could prove the analogous splitting for

$$\mathrm{Ext}_{E(n)_*}((A/\!\!/E(m))_*)$$

when $m \geq n$, then the above argument could be used to show that $BP\langle n\rangle \wedge BP\langle m\rangle$ splits as a spectrum into an analogous wedge $C(n,m) \vee HV(n,m)$.

3. Calculations

In this section we develop techniques to provide an inductive calculation of

$$\mathrm{Ext}_{E(2)_*}((A/\!\!/E(2))_*).$$

The first step in accomplishing this is to introduce a notion of *weight* analogous to the one found in [4]. This allows us to define *Brown-Gitler sub-comodules* of $(A/\!\!/E(2))_*$. We will show that there is a decomposition

$$(A/\!\!/E(2))_* \cong_{E(2)_*} \bigoplus_{j\geq 0} \Sigma^{2j}\underline{\mathrm{bu}}_{\lfloor j/2\rfloor}$$

We will inductively compute the groups $\mathrm{Ext}_{E(2)_*}(\Sigma^{2j}\underline{\mathrm{bu}}_{2j})$ modulo torsion by first developing a short exact sequence (3.14) and a 4-term exact sequence (3.32), which leads to spectral sequences. The v_0-inverted spectral sequences collapse immediately, and we develop inductive methods for identifying the generators in the v_0-inverted Ext groups. After this we determine some multiplicative extensions.

3.1. **Brown-Gitler (co)modules.** The majority of this and the following three sections are adapt the techniques of [3] to our setting.

Let $E(n)$ denote the sub-Hopf algebra of the Steenrod algebra which is generated by the first $n+1$ Milnor primitives, $Q_0, \ldots, Q_n$. Let $E(n)_*$ denote the dual of this algebra. This will be a quotient of the dual Steenrod algebra and it is given by

$$E(n)_* = E(\zeta_1, \ldots, \zeta_{n+1})$$

where the ζ_i's are the images of ζ_i in A_*. In $E(n)_*$, these elements are primitive, and so $E(n)$ is a self-dual Hopf algebra.

Following [4] (cf. pg 7), we define a *weight filtration* on A_* which induces a filtration on the A_*-subcomodule

$$(A/\!\!/E(n))_* = A_* \square_{E(n)_*} \mathbb{F}_2 \cong \mathbb{F}_2[\zeta_1^2, ..., \zeta_{n+1}^2, \zeta_{n+2}, \ldots].$$

We define the *weight* of the generators ζ_k by

$$\mathrm{wt}(\zeta_k) := 2^{k-1}$$

and extend multiplicatively by

$$\mathrm{wt}(xy) := \mathrm{wt}(x) + \mathrm{wt}(y).$$

The *Brown-Gitler comodule* $N_i(j)$ is the subspace of $(A/\!\!/E(i))_*$ spanned by elements of weight less than or equal to $2j$. From the coproduct formula for the dual Steenrod algebra:

$$\psi(\zeta_k) = \sum_{i+j=k} \zeta_i \otimes \zeta_j^{2^i}, \tag{3.1}$$

we see that $N_i(j)$ is an A_*-subcomodule of $(A/\!\!/E(i))_*$. The algebra $(A/\!\!/E(i))_*$ can also be regarded as a comodule over $E(i)_*$, in fact it is a comodule algebra. For consistency with the notation for Brown-Gitler spectra and their associated subcomodules (cf. [4]), we shall write

$$\underline{BP\langle i\rangle}_j := N_i(j).$$

For $i = 1$, we shall write

$$\underline{\mathrm{bu}}_j := N_1(j),$$

and for $i = 0$ we write

$$\underline{\mathrm{H}\mathbb{Z}}_j := N_0(j)$$

As in [3], we can define a map of ungraded rings

$$\varphi_i : (A/\!\!/E(i))_* \to (A/\!\!/E(i-1))_*$$

which is defined on generators by

$$\varphi_i : \zeta_k^{2^\ell} \mapsto \begin{cases} \zeta_{k-1}^{2^\ell} & k > 1 \\ 1 & k = 1 \end{cases}$$

and extended multiplicatively. So, for example,

$$\varphi_2(\zeta_4\zeta_5^2) = \zeta_3\zeta_4^2.$$

Lemma 3.2. *The map φ_i is a map of ungraded $E(i)_*$-comodules.*

Proof. Since $(A/\!\!/E(i))_*$ is generated by $\{\zeta_1^2, \ldots, \zeta_{i+1}^2, \zeta_{i+2}, \ldots\}$, it is enough to check that φ_i commutes with coaction on these generators. This follows immediately from the coproduct formula (3.1) and the fact that $E(i)_*$ is exterior. $\square$

Let $M_i(j)$ denote the subspace of $(A/\!\!/E(i))_*$ spanned by the monomials of weight exactly $2j$. Observe that the coaction on $(A/\!\!/E(i))_*$ (as a $E(i)_*$-comodule) preserves the weight. Thus the subspaces $M_i(j)$ are $E(i)_*$-subcomodules. In particular we have shown

Proposition 3.3. *There is a splitting of $E(i)_*$-comodules*

$$(A/\!\!/E(i))_* \cong \bigoplus_{j\geq 0} M_i(j)$$

Lemma 3.4. *For $i > 0$, the map φ_i maps the subspace $M_i(j)$ isomorphically onto $N_{i-1}(\lfloor j/2 \rfloor)$.*

Proof. Given a monomial in $M_i(j)$, it can be written as $\zeta_1^{2\ell}x$ where x is a monomial which is a product of ζ_k^i for $k \geq 2$. In this case, the weight of x is $2j - 2\ell$. Observe that

$$\varphi_i(\zeta_1^{2\ell}x) = \varphi_i(x)$$

and the weight of $\varphi_i(x)$ is $j - \ell$. Write x as

$$x = \zeta_2^{i_2}\zeta_3^{i_3}\zeta_4^{i_4}\cdots.$$

Then the weight of x is

$$\mathrm{wt}(x) = 2i_2 + 4i_3 + 8i_4 + \cdots + 2^{n-1}i_n + \cdots.$$

Since $i > 0$, it follows that i_2 is even, and hence $\mathrm{wt}(x)$ is divisible by 4. It follows that $2j - 2\ell$ is divisible by 4, whence $j - \ell$ is divisible by 2. So $\varphi_i(x)$ belongs to $M_{i-1}\left(\frac{j-\ell}{2}\right)$. This shows that φ_i maps the subspace spanned by monomials of the form $\zeta_1^{2\ell}x$ isomorphically onto the subspace $M_{i-1}\left(\frac{j-\ell}{2}\right)$. Letting ℓ vary, we see that the image of φ_i restricted to $M_i(j)$ maps isomorphically onto $N_{i-1}(\lfloor j/2 \rfloor)$. $\square$

Remark 3.5. The inverse to the isomorphism in the previous lemma is given by

$$\varphi_i^{-1} : N_{i-1}(\lfloor j/2 \rfloor) \to M_i(j);\ \zeta_1^{i_1}\zeta_2^{i_2}\cdots \mapsto \zeta_1^{a}\zeta_2^{i_1}\zeta_3^{i_2}\cdots$$

where $a = 2j - \mathrm{wt}(\zeta_2^{i_1}\zeta_3^{i_2}\cdots)$.

Corollary 3.6. *There is an isomorphism of graded $E(i)_*$-comodules*
$$M_i(j) \cong \Sigma^{2j} N_{i-1}(\lfloor j/2 \rfloor).$$

Corollary 3.7. *There is an isomorphism of $E(i)_*$-comodules*
$$M_i(2j) \cong \Sigma^2 M_i(2j+1)$$
which is given by multiplication by ζ_1^2.

In light of Corollary 3.6, we will always make the identification
$$M_2(2j) \cong \Sigma^{4j} N_1(j)$$
in the rest of this paper.

3.2. **Exact sequences.** Inspired by [3], we construct exact sequences relating the Brown-Gitler comodules $N_1(j)$ and $M_2(j)$. Recall that
$$(E(2)/\!\!/E(1))_* \cong E(\zeta_3).$$
Consider the $\mathbb{F}_2$-linear map
$$\tau : (A/\!\!/E(1))_* \to (A/\!\!/E(2))_* \otimes (E(2)/\!\!/E(1))_*$$
defined on the monomial basis by
$$\zeta_1^{2i_1}\zeta_2^{2i_2}\zeta_3^{2i_3+\epsilon}\zeta_4^{i_4}\cdots \mapsto \zeta_1^{2i_1}\zeta_2^{2i_2}\zeta_3^{2i_3}\zeta_4^{i_4}\cdots \otimes \zeta_3^{\epsilon}.$$
Note this is not a map of $E(2)_*$-comodules when the target is endowed with the diagonal coaction. For example, in $(A/\!\!/E(1))_*$, there is the coaction
$$\alpha(\zeta_3) = 1 \otimes \zeta_3 + \zeta_1 \otimes \zeta_2^2 + \zeta_2 \otimes \zeta_1^4 + \zeta_3 \otimes 1$$
whereas in $(A/\!\!/E(2))_* \otimes (E(2)/\!\!/E(1))_*$, we have
$$\alpha(1 \otimes \zeta_3) = 1 \otimes 1 \otimes \zeta_3 + \zeta_3 \otimes 1 \otimes 1.$$
However, we do have that τ is an isomorphism of $\mathbb{F}_2$-vector spaces.

Following [3], we will put a decreasing filtration on $(A/\!\!/E(1))_*$. Define
$$F^j(A/\!\!/E(1))_* := \tau^{-1}\left(\left(\bigoplus_{k\geq j} M_2(k)\right) \otimes (E(2)/\!\!/E(1))_*\right)$$
which gives the decreasing filtration
$$(A/\!\!/E(1))_* = F^0(A/\!\!/E(1))_* \supset F^1(A/\!\!/E(1))_* \supset F^2(A/\!\!/E(1))_* \supset \cdots.$$

The following observation will be useful in later arguments,

Observation 3.8. *Let x be a monomial in $(A/\!\!/E(1))_*$. If $x \in F^j((A/\!\!/E(1))_*)$, then $\mathrm{wt}(x)$ is bounded below by $2j$. If the power of ζ_3 in x is odd, then its weight is bounded below by $2j+4$.*

The coproduct formula (3.1) shows that this is a filtration by $E(2)_*$-subcomodules. Passing to filtration quotients gives a map on the associated graded comodule algebra

$$E^0\tau : E^0(A/\!\!/E(1))_* \to E^0((A/\!\!/E(2))_* \otimes (E(2)/\!\!/E(1))_*) \tag{3.9}$$

which we will show is a map of $E(2)_*$-comodules. Here, the filtration on the target of τ is given by

$$D^j := \left(\bigoplus_{k \geq j} M_2(k) \right) \otimes (E(2)/\!\!/E(1))_*$$

so that $F^j = \tau^{-1}D^j$. However, from Proposition 3.3, it follows that

$$E^0((A/\!\!/E(2))_* \otimes (E(2)/\!\!/E(1))_*) \cong_{E(2)_*} (A/\!\!/E(2))_* \otimes (E(2)/\!\!/E(1))_*,$$

and so $E^0\tau$ is a map

$$E^0\tau : E^0(A/\!\!/E(1))_* \to (A/\!\!/E(2))_* \otimes (E(2)/\!\!/E(1))_*$$

Here as an example illustrating why $E^0\tau$ is a map of comodules over $E(2)_*$. In the coaction, $\alpha(\zeta_3)$, of ζ_3 above, the terms which prevented the map τ from being a comodule map were $\zeta_1 \otimes \zeta_2^2$ and $\zeta_2 \otimes \zeta_1^4$. Note that $\zeta_3 \in F^0$, whereas ζ_2^2 and ζ_1^4 are both in F^2. In general, the coproduct formula shows that the coaction of an element x of $(A/\!\!/E(1))_*$ is the same as the coaction on $\tau(x)$ modulo elements of higher filtration. From now on we will write τ interchangeably for τ and $E^0\tau$.

Proposition 3.10. *The map* (3.9) *is an isomorphism of $E(2)_*$-comodules.*

Proof. To check that this is a map of $E(2)_*$-comodules, it is enough to check that it is a map of $E(2)$-modules. So let x be a monomial in $(A/\!\!/E(1))_*$, say

$$x = \zeta_1^{2i_1}\zeta_2^{2i_2}\zeta_3^{2i_3+\epsilon}\zeta_4^{i_4}\cdots .$$

Then as Q_i acts via a derivation, we have that

$$Q_i x = \zeta_1^{2i_1}\zeta_2^{2i_2}Q_i(\zeta_3^{2i_3+\epsilon})\zeta_4^{i_4}\cdots + \sum_{j>3} \zeta_1^{2i_1}\zeta_2^{2i_2}\cdots\zeta_{j-1}^{i_{j-1}}(Q_i\zeta_j^{i_j})\zeta_{j+1}^{i_{j+1}}\cdots . \tag{3.11}$$

When $\epsilon = 0$, the action by Q_i commutes with the action of $E(2)$ on $(A/\!\!/E(2))_* \otimes (E(2)/\!\!/E(1))_*$, since the action by Q_i preserves the weight.

Suppose then that $\epsilon = 1$. Then

$$Q_i\tau(x) = Q_i(\zeta_1^{2i_1}\zeta_2^{2i_2}\zeta_3^{2i_3}\zeta_4^{i_4}\cdots \otimes \zeta_3).$$

Since Q_i is a derivation, and since the target of τ is endowed with the diagonal $E(2)$-action, we can rewrite this as

$$\begin{aligned} Q_i(\tau(x)) = \zeta_1^{2i_1}\zeta_2^{2i_2}\zeta_3^{2i_3}\zeta_4^{i_4}\cdots \otimes Q_i\zeta_3 + \\ \sum_{j>3} \zeta_1^{2i_1}\zeta_2^{2i_2}\zeta_3^{2i_3}\cdots\zeta_{j-1}^{i_j}(Q_i\zeta_j^{i_j})\zeta_{j+1}^{i_{j+1}}\cdots \otimes \zeta_3. \end{aligned} \tag{3.12}$$

Note that τ carries the sum in (3.11) to the sum in (3.12). When $i = 0, 1$, then $Q_i\zeta_3 = 0$ in $(E(2)/\!\!/E(1))_*$, and so the first term in (3.12) vanishes. In $(A/\!\!/E(1))_*$, $Q_i\zeta_3 \neq 0$, but the first term in (3.11) does land in strictly higher filtration than x, and so the first term in (3.11) vanishes in the associated graded $E^0(A/\!\!/E(1))_*$. When $i = 2$, then in both $E^0(A/\!\!/E(1))_*$ and $(E(2)/\!\!/E(1))_*$, $Q_2\zeta_3 = 1$. In this case, τ carries the first term in (3.11) to the first term in (3.12). This shows that τ is a comodule map in the associated graded comodule. To see that this map is an isomorphism, just note that τ is an $\mathbb{F}_2$-isomorphism and induces a linear isomorphism on the associated graded. So τ is an isomorphism of comodules. □

Define quotients

$$Q^j(A/\!\!/E(1))_* := (A/\!\!/E(1))_*/F^{j+1}(A/\!\!/E(1))_*,$$

then this is an $E(2)_*$-comodule and it inherits a filtration from $(A/\!\!/E(1))_*$. The map τ induces an isomorphism of $\mathbb{F}_2$-vector spaces

$$\tau : Q^j(A/\!\!/E(1))_* \to \underline{BP\langle 2\rangle}_j \otimes (E(2)/\!\!/E(1))_*$$

which induces an isomorphism of associated graded $E(2)_*$-comodules,

$$\tau : E^0Q^j(A/\!\!/E(1))_* \to \underline{BP\langle 2\rangle}_j \otimes (E(2)/\!\!/E(1))_*.$$

Lemma 3.13. *There is a short exact sequence of $E(2)_*$-comodules*

$$0 \to \Sigma^{4j}\underline{\mathrm{bu}}_j \otimes \underline{\mathrm{bu}}_1 \to \underline{\mathrm{bu}}_{2j+1} \to Q^{2j-1}(A/\!\!/E(1))_* \to 0 \tag{3.14}$$

Proof. Observe there is a commutative diagram

$$\begin{tikzcd} \underline{BP\langle 2\rangle}_{2j-1} \otimes E(\zeta_3) \arrow[r, hook] \arrow[drr] & (A/\!/E(2))_* \otimes E(\zeta_3) \arrow[r, "\tau^{-1}"] & (A/\!/E(1))_* \\ & & \underline{\mathrm{bu}}_{2j+1} \arrow[u, hook] \end{tikzcd}.$$

This factorization arises because if $x \in \underline{BP\langle 2\rangle}_{2j-1}$, then the image of $x \otimes \zeta_3^{\epsilon}$ in $(A/\!/E(1))_*$ has weight bounded by

$$\mathrm{wt}(\tau^{-1}(x \otimes \zeta_3^{\epsilon})) \leq 4j - 2 + 4 = 4j + 2,$$

and hence must lie in $\underline{\mathrm{bu}}_{2j+1}$. The projection of $(A/\!/E(1))_*$ onto $Q^{2j-1}(A/\!/E(1))_*$ restricts to give a surjection

$$\rho : \underline{\mathrm{bu}}_{2j+1} \twoheadrightarrow Q^{2j-1}(A/\!/E(1))_*.$$

Note that the kernel of ρ is the intersection of $F^{2j}(A/\!/E(1))_*$ with $\underline{\mathrm{bu}}_{2j+1}$. So let x be a nonzero element of this intersection. Note that, since both $\underline{\mathrm{bu}}_{2j+1}$ and F^{2j} are defined in terms of the monomial basis of $(A/\!/E(1))_*$, it follows that their intersection is as well. Thus, we may suppose that x is in fact a monomial, say it is the monomial $\zeta_1^{2i_1}\zeta_2^{2i_2}\zeta_3^{2i_3+\epsilon}\zeta_4^{i_4}\cdots$. Since x is an element of $\underline{\mathrm{bu}}_{2j+1}$, its weight is bounded above by $4j+2$. Since $x \in F^{2j}(A/\!/E(1))_*$, Observation 3.8 implies that $\mathrm{wt}(x)$ is at least $4j$. Observation 3.8 also implies that if $\epsilon = 1$, then the weight of x is bounded below by $4j+4$, which is a contradiction. Thus $\epsilon = 0$. All of this implies that the kernel of ρ is

$$\ker \rho = M_2(2j) \otimes \underline{\mathrm{bu}}_1$$

where $\underline{\mathrm{bu}}_1$ is the subcomodule $\mathbb{F}_2\{1, \zeta_1^2\}$ of $(A/\!/E(1))_*$. By 3.6, we get the desired short exact sequence. ☐

Lemma 3.15. *There is an exact sequence of $E(2)_*$-comodules*

$$0 \to \Sigma^{4j}\underline{\mathrm{bu}}_j \to \underline{\mathrm{bu}}_{2j} \to Q^{2j-1}(A/\!/E(1))_* \to \Sigma^{4j+5}\underline{\mathrm{bu}}_{j-1} \to 0 \tag{3.16}$$

Proof. As an $\mathbb{F}_2$-vector space, the image of $\underline{\mathrm{bu}}_{2j}$ in $(A/\!/E(2))_* \otimes E(\zeta_3)$ under τ is

$$\tau(\underline{\mathrm{bu}}_{2j}) = (\underline{BP\langle 2\rangle}_{2j-2} \otimes E(\zeta_3)) \oplus (M_2(2j-1)\otimes \mathbb{F}_2\{1\}) \oplus (M_2(2j) \otimes \mathbb{F}_2\{1\})$$

which gives the following exact sequence on the level of $\mathbb{F}_2$-vector spaces

$$0 \longrightarrow M_2(2j) \xrightarrow{\varphi} \underline{\mathrm{bu}}_{2j} \xrightarrow{\psi} Q^{2j-1}(A/\!\!/E(1))_* \xrightarrow{\omega}$$

$$\xrightarrow{\omega} M_2(2j-1) \otimes \mathbb{F}_2\{\zeta_3\} \longrightarrow 0.$$

The map ψ is defined by the following commutative diagram

$$\begin{tikzcd}
\underline{\mathrm{bu}}_{2j} \arrow[r, "\psi"] \arrow[d, "="] & Q^{2j-1}(A/\!\!/E(1))_* \arrow[d, "\tau"] \\
\underline{\mathrm{bu}}_{2j} \arrow[r, "\widetilde{\psi}"] & \underline{BP\langle 2\rangle}_{2j-1} \otimes E(\zeta_3)
\end{tikzcd}$$

where $\widetilde{\psi}$ is defined on monomials by

$$\widetilde{\psi} : \zeta_1^{2i_1}\zeta_2^{2i_2}\zeta_3^{2i_3+\epsilon}\zeta_4^{i_4}\cdots \mapsto \begin{cases} \zeta_1^{2i_1}\zeta_2^{2i_2}\zeta_3^{2i_3}\zeta_4^{i_4}\cdots \otimes \zeta_3^{\epsilon} & \mathrm{wt}(\zeta_1^{2i_1}\zeta_2^{2i_2}\zeta_3^{2i_3}\zeta_4^{i_4}\cdots) \leq 4j-2 \\ 0 & \text{otherwise} \end{cases}.$$

We need to check that the maps are maps of $E(2)_*$-comodules. Let us start with ω. Note the commutative diagram

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
\tau^{-1}\ker\pi \arrow[r] \arrow[d] & \ker\pi \arrow[d] \\
Q^{2j-1}(A/\!\!/E(1))_* \arrow[r, "\tau"] \arrow[d, "\omega"] & \underline{BP\langle 2\rangle}_{2j-1} \otimes E(\zeta_3) \arrow[d, "\pi"] \\
M_2(2j-1) \otimes \mathbb{F}_2\{\zeta_3\} \arrow[r, "="] \arrow[d] & M_2(2j-1) \otimes \mathbb{F}_2\{\zeta_3\} \arrow[d] \\
0 & 0
\end{tikzcd}$$

where π is the projection. So in order to check that ω is a comodule map, we just need to check that $\tau^{-1}\ker\pi$ is a subcomodule of $Q^{2j-1}(A/\!\!/E(1))_*$. That ω is a comodule map follows then follows from the following lemma.

Lemma 3.17. $\tau^{-1}\ker\pi$ *is a subcomodule of* $Q^{2j-1}(A/\!\!/E(1))_*$.

Proof. First note that

$$\ker \pi = (\underline{BP\langle 2\rangle}_{2j-2} \otimes E(\zeta_3)) \oplus (M_2(2j-1) \otimes \mathbb{F}_2\{1\}).$$

So let $x \in \tau^{-1}\ker\pi$. Since $\underline{BP\langle 2\rangle}_{2j-1}$ and $M_2(2j-1)$ have monomial bases, and since τ is defined on the monomial basis, it follows that $\tau^{-1}\ker\pi$ also has a monomial basis. So we may suppose that x is of the form

$$x = \zeta_1^{2i_1}\zeta_2^{2i_2}\zeta_3^{2i_3+\epsilon}\zeta_4^{i_4}\cdots$$

in $\tau^{-1}\ker\pi$. Thus,

$$\tau(x) = \underbrace{\zeta_1^{2i_1}\zeta_2^{2i_2}\zeta_3^{2i_3}\zeta_4^{i_4}\cdots}_{\text{define } y \text{ to be this}} \otimes \zeta_3^{\epsilon}$$

In particular, the weight of y is bounded above by $4j-2$, and when the weight of y is $4j-2$, we must have $\epsilon = 0$. Moreover $x = y\zeta_3^{\epsilon}$. Applying any of the Milnor primitives Q_i to x gives

$$Q_i x = Q_i(y)\zeta_3^{\epsilon} + yQ_i\zeta_3^{\epsilon}.$$

Applying τ to this expression gives

$$\tau(Q_i x) = Q_i y \otimes \zeta_3^{\epsilon} + (yQ_i\zeta_3^{\epsilon}) \otimes 1. \tag{3.18}$$

It needs to be checked that both terms are in the kernel of π.

First suppose that $\epsilon = 0$. Then $x = y$ is an element of $\underline{BP\langle 2\rangle}_{2j-1}$ and in this case the action of the Milnor primitives preserves the weight, and so $\tau(Q_i y) = (Q_i y) \otimes 1$ is an element of $\ker\pi$. Suppose then that $\epsilon = 1$, in which case $\mathrm{wt}(y) \leq 4j-4$. First consider the case when $\mathrm{wt}(y) < 4j-4$. Then

$$\mathrm{wt}(yQ_i\zeta_3) < 4j$$

for $i = 0, 1, 2$. This implies that $(yQ_i\zeta_3) \otimes 1 \in \underline{BP\langle 2\rangle}_{2j-1} \otimes \mathbb{F}_2\{1\}$. Since the action by the Q_i's preserves the weight of y, we find then that both terms on the right hand side of (3.18) lie in $\ker\pi$.

We are thus left with checking the case when $\mathrm{wt}(y) = 4j-4$. When $i = 0$ or $i = 1$, then $yQ_i\zeta_3$ has weight $4j$, and so is zero in $Q^{2j-1}(A/\!\!/E(1))_*$. Thus, for $i = 0, 1$,

$$Q_i x = (Q_i y)\zeta_3 \xmapsto{\tau} Q_i y \otimes \zeta_3,$$

and since the weight of $Q_i y$ is still $4j-4$, this is an element of $\ker\pi$. Finally, we have

$$Q_2(x) = Q_2(y\zeta_3) = Q_2(y)\zeta_3 + y$$

which is mapped under τ to

$$Q_2 y \otimes \zeta_3 + y \otimes 1.$$

The both terms lie in $\underline{BP\langle 2\rangle}_{2j-2} \otimes E(\zeta_3)$, and so belong to $\ker \pi$. This completes the proof that $\tau^{-1}\ker\pi$ is a $E(2)_*$-subcomodule. □

It is clear that the map φ is a $E(2)_*$-comodule map. Indeed, we can regard $M_2(j)$ as a subspace of $\underline{\mathrm{bu}}_{2j}$. If $x \in M_2(j)$ is a monomial, then an odd power of ζ_3 cannot occur in x. So the action of Q_0, Q_1, Q_2 on x preserves the weight, and consequently lies in $M_2(j)$. Thus $M_2(j)$ is a subcomodule of $\underline{\mathrm{bu}}_{2j}$.

Finally, we check that ψ is a map of comodules. Let K denote the cokernel of φ. Then we have an induced morphism of short exact sequences of $E(2)_*$-comodules

$$\begin{tikzcd}
0 \arrow[r] & M_2(2j) \otimes \mathbb{F}_2\{1\} \arrow[r, "\varphi"] \arrow[d, hook] & \underline{\mathrm{bu}}_{2j} \arrow[r, "\psi_1"] \arrow[d] & K \arrow[r] \arrow[d, dashed, "\psi_2"] & 0 \\
0 \arrow[r] & M_2(2j) \otimes E(\zeta_3) \arrow[r] & Q^{2j}(A/\!\!/E(1))_* \arrow[r] & Q^{2j-1}(A/\!\!/E(1))_* \arrow[r] & 0
\end{tikzcd}$$

Since ψ is the composite $\psi_2\psi_1$, we may conclude that ψ is a comodule map. □

Remark 3.19. The quotients $Q^j(A/\!\!/E(1))_*$ have finite filtrations projected from the filtration on $(A/\!\!/E(1))_*$. Applying $\mathrm{Ext}_{E(2)_*}$ to this filtration produces a spectral sequence

$$E_1 = \mathrm{Ext}_{E(2)_*}(E^0 Q^j(A/\!\!/E(1))_*) \implies \mathrm{Ext}_{E(2)_*}(Q^j(A/\!\!/E(1))_*).$$

Since $E^0Q^j(A/\!\!/E(1))_*$ is isomorphic to $\underline{BP\langle 2\rangle}_j \otimes (E(2)/\!\!/E(1))_*$ as an $E(2)_*$-comodule, we have that the E_1-page is $\mathrm{Ext}_{E(1)}(\underline{BP\langle 2\rangle}_j)$ which we know consists of v_1-torsion elements on the 0-line and a v_1-torsion free component concentrated in even $(t-s)$-degree. The spectral sequence is a linear over $\mathrm{Ext}_{E(2)_*}(\mathbb{F}_2)$, which implies that this spectral sequence collapses. Consequently, for the purposes of the inductive calculations in the following section, we will regard $Q^j(A/\!\!/E(1))_*$ as $\underline{BP\langle 2\rangle}_j \otimes (E(2)/\!\!/E(1))_*$.

3.3. **Inductive calculations.** In the last section, we produced the exact sequences of comodules (3.14) and (3.16). As in [4], we regard these as providing spectral sequences converging to $\mathrm{Ext}_{E(2)}(\underline{\mathrm{bu}}_{2j+1})$ and $\mathrm{Ext}_{E(2)}(\underline{\mathrm{bu}}_{2j})$ respectively. Following [4], we write

$$\bigoplus M_i[k_i] \implies M$$

to denote the existence of a spectral sequence

$$\bigoplus \mathrm{Ext}^{s-k_i,t+k_i}_{E(2)_*}(M_i) \implies \mathrm{Ext}^{s,t}_{E(2)_*}(M).$$

We shall abbreviate $M_i[0]$ by M_i. Below, we shall always be identifying $\underline{\mathrm{bu}}_j$ with $M_2(2j)$ via the isomorphism

$$\varphi_2 : M_2(2j) \xrightarrow{\cong} \Sigma^{4j}\underline{\mathrm{bu}}_j.$$

The exact sequence (3.14) gives a spectral sequence

$$\Sigma^{8j+4}Q^{2j-1}(A/\!\!/E(1))_* \oplus \left(\Sigma^{12j+4}\underline{\mathrm{bu}}_j \otimes \underline{\mathrm{bu}}_1\right) \implies \Sigma^{8j+4}\underline{\mathrm{bu}}_{2j+1} \tag{3.20}$$

and (3.16) gives a spectral sequence

$$\Sigma^{12j}\underline{\mathrm{bu}}_j \oplus Q^{2j-1}(A/\!\!/E(1))_* \oplus \Sigma^{12j+5}\underline{\mathrm{bu}}_{j-1}[1] \implies \Sigma^{8j}\underline{\mathrm{bu}}_{2j} \tag{3.21}$$

In the exact sequences, there were the comodules $Q^{2j-1}(A/\!\!/E(1))_*$, and in Remark 3.19, it was pointed out that

$$\begin{aligned}\mathrm{Ext}_{E(2)_*}(Q^{2j-1}(A/\!\!/E(1))_*) &\cong \mathrm{Ext}_{E(2)_*}((E(2)/\!\!/E(1))_* \otimes \underline{BP\langle 2\rangle}_{2j-1}) \\ &\cong \mathrm{Ext}_{E(1)}(\underline{BP\langle 2\rangle}_{2j-1}).\end{aligned}$$

In order to carry out the computation, we must then calculate $\mathrm{Ext}_{E(1)_*}$ of $\underline{BP\langle 2\rangle}_{2j-1}$.

Lemma 3.22. *For any j, there are isomorphisms*

$$\begin{aligned}\underline{BP\langle 2\rangle}_j \cong_{E(2)_*} \bigoplus_{0\le k\le j} M_2(k) &\cong_{E(2)_*} \bigoplus_{0\le k\le j} \Sigma^{2k}\underline{\mathrm{bu}}_{\lfloor k/2\rfloor} \\ &\cong_{E(1)_*} \bigoplus_{k=0}^{j}\bigoplus_{\ell=0}^{\lfloor k/2\rfloor} \Sigma^{2k+2\ell}\underline{H\mathbb{Z}}_{\lfloor \ell/2\rfloor}\end{aligned}$$

Proof. The first isomorphism just follows from Proposition 3.3. From an application of Corollary 3.6 we obtain

$$M_2(k) \cong_{E(2)_*} \Sigma^{2k}\underline{\mathrm{bu}}_{\lfloor k/2\rfloor}$$

which gives the second isomorphism. Applying Proposition 3.3 and Corollary 3.6 again, but with $i = 1$ gives the third isomorphism.

$\square$

Example 3.23. We have

$$M_2(4) \cong_{E(2)_*} \Sigma^8 \underline{\mathrm{bu}}_2 \cong_{E(1)_*} \Sigma^8(\underline{\mathrm{H}\mathbb{Z}}_0 \oplus \Sigma^2 \underline{\mathrm{H}\mathbb{Z}}_0 \oplus \Sigma^4 \underline{\mathrm{H}\mathbb{Z}}_1).$$

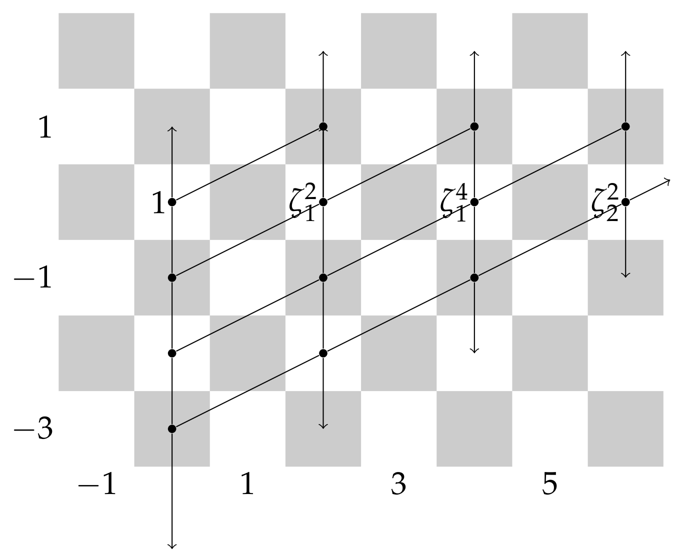
It follows from ([1], part III, §16-17) that there is an isomorphism

$$\mathrm{Ext}_{E(1)}(\underline{\mathrm{H}\mathbb{Z}}_k)/v_0\text{-torsion} \cong \mathrm{Ext}_{E(1)}(\mathbb{F}_2)^{\langle 2k-\alpha(k)\rangle}$$

where $\mathrm{Ext}(M)^{\langle n\rangle}$ denotes the nth Adams cover of $\mathrm{Ext}(M)$ and $\alpha(k)$ denotes the number of 1's in the dyadic expansion of k. In [1], this is a consequence of the fact that the Margolis homology groups $M_*(\underline{\mathrm{H}\mathbb{Z}}_k; Q_0)$ and $M_*(\underline{\mathrm{H}\mathbb{Z}}_k; Q_1)$ are both one dimensional. Observe that when v_0 is inverted then

$$\begin{aligned} v_0^{-1}\,\mathrm{Ext}_{E(1)}(\underline{\mathrm{H}\mathbb{Z}}_k) &\cong v_0^{-1}\,\mathrm{Ext}_{E(1)}(\mathbb{F}_2)^{\langle 2k-\alpha(k)\rangle} \\ &= \mathbb{F}_2[v_0^{\pm 1}, v_1] \otimes_{\mathbb{F}_2} M_*(\underline{\mathrm{H}\mathbb{Z}}_k; Q_0) \end{aligned} \tag{3.24}$$

Example 3.25. Below, we illustrate this isomorphism for $\underline{\mathrm{H}\mathbb{Z}}_2$.



In the chart, elements below the 0-line are obtained by inverting v_0 and downward pointing arrows indicate v_0^{-1}-towers. Note that as a module over $\mathbb{F}_2[v_0^{\pm 1}, v_1]$ that this Ext-group is generated by the element in $(0, -3)$. Since v_0 is inverted, this module is also generated by the element in $(0, 0)$, which is precisely the nonzero element of $M_*(\underline{\mathrm{H}\mathbb{Z}}_2; Q_0)$.

As a first step to determining $\mathrm{Ext}_{E(1)}(\underline{BP\langle 2\rangle}_{2j-1})$, we will compute v_0-inverted Ext groups first. This will allow us to locate the starting points of the Adams covers within the integral Ext groups.

Proposition 3.26 ([4]). *We have the isomorphism*

$$v_0^{-1}\,\mathrm{Ext}_{E(1)}(\underline{BP\langle 2\rangle}_j) \cong \mathbb{F}_2[v_0^{\pm 1}, v_1]\{\zeta_1^{2i_1}\zeta_2^{2i_2} \mid i_1 + 2i_2 \leq j\}$$

Proof. Given an A-comodule M, there is an isomorphism

$$v_0^{-1}\,\mathrm{Ext}_{A_*}(M) \cong v_0^{-1}\,\mathrm{Ext}_{A(0)_*}(M).$$

This is an algebraic analogue of Serre's theorem that rational stable homotopy is the same as rational homology. Consider the following sequence of isomorphisms

$$\begin{aligned} v_0^{-1}\,\mathrm{Ext}_{E(1)_*}(\underline{BP\langle 2\rangle}_j) &\cong v_0^{-1}\,\mathrm{Ext}_{A_*}((A/\!\!/E(1))_* \otimes \underline{BP\langle 2\rangle}_j) \\ &\cong v_0^{-1}\,\mathrm{Ext}_{A(0)_*}((A/\!\!/E(1))_* \otimes \underline{BP\langle 2\rangle}_j) \end{aligned}$$

When v_0 is inverted, the functor $v_0^{-1}\,\mathrm{Ext}_{A(0)_*}(-)$ has a Künneth formula. Hence the last Ext group is isomorphic to

$$v_0^{-1}\,\mathrm{Ext}_{A(0)_*}((A/\!\!/E(1))_*) \otimes_{\mathbb{F}_2[v_0^{\pm 1}]} v_0^{-1}\,\mathrm{Ext}_{A(0)_*}(\underline{BP\langle 2\rangle}_j).$$

Calculating $v_0^{-1}\,\mathrm{Ext}_{A(0)_*}$ is extremely simple, it is the free $\mathbb{F}_2[v_0^{\pm 1}]$-module generated by the Q_0-Margolis homology of M,

$$v_0^{-1}\,\mathrm{Ext}_{A(0)_*}(M) = \mathbb{F}_2[v_0^{\pm 1}] \otimes_{\mathbb{F}_2} M_*(M; Q_0).$$

From Part 3 of [1], the Q_0-Margolis homology of $(A/\!\!/E(1))_*$ is

$$M_*((A/\!\!/E(1))_*; Q_0) = \mathbb{F}_2[\zeta_1^2].$$

Since the action by Q_0 preserves the weight on $(A/\!\!/E(2))_*$, there is an associated weight filtration on

$$M_*(BP\langle 2\rangle; Q_0) = \mathbb{F}_2[\zeta_1^2, \zeta_2^2]$$

which implies that

$$M_*(\underline{BP\langle 2\rangle}_j; Q_0) = \mathbb{F}_2\{\zeta_1^{2i_1}\zeta_2^{2i_2} \mid i_1 + 2i_2 \leq j\}.$$

In the v_0-inverted Adams spectral sequence

$$v_0^{-1}\,\mathrm{Ext}_{A(0)_*}((A/\!\!/E(1))_*) \implies H\mathbb{Q}_{2*}\mathrm{bu} = \mathbb{Q}_2[v_1]$$

ζ_1^2 is detecting v_1. Thus we get the desired isomorphism. $\square$

Remark 3.27. Each of the monomials in $v_0^{-1}\operatorname{Ext}_{E(1)_*}(\underline{BP\langle 2\rangle}_j)$ determines an Adams cover of $\operatorname{Ext}_{E(1)_*}(\mathbb{F}_2)$. Here is an algorithm for determining the the Adams covers associated to a monomial $\zeta_1^{2i_1}\zeta_2^{2i_2}$:

(1) If $2i_2 \geq 4$, then the next element in Adams cover is $\zeta_1^{2i_1}\zeta_2^{2i_2-4}\zeta_3^2$, and there is the relation
$$v_1\zeta_1^{2i_1}\zeta_2^{2i_2} = v_0\zeta_1^{2i_1}\zeta_2^{2i_2-4}\zeta_3^2.$$
If $2i_2 - 4 < 4$, then the process terminates.

(2) If $2i_2 - 4 \geq 4$, then repeat the previous step. Continue this until the exponent of ζ_2 is 0 or 2.

(3) Perform the previous steps on ζ_3 until the exponent on ζ_3 is 0 or 2. Then continue onto $\zeta_4, \zeta_5, \ldots$ and so on until the process terminates.

Observe that in the spectral sequences (3.21) and (3.20) that upon inverting v_0, all the terms in the E_1-page are in even degree. So these spectral sequences collapse after v_0-localization. We also know from our general structural results that there can be no differential originating from a torsion class and hitting a v_2-torsion free class. Thus we have

Proposition 3.28. *In the spectral sequences* (3.21) *and* (3.20)*, the only nontrivial differentials must be between torsion classes. Consequently, the spectral sequences* (3.20) *collapse immediately.*

Proof. Since the v_2-torsion free component is concentrated in even $(t-s)$-degree, there are no differentials between v_2-torsion free classes. Recall that we had the decomposition
$$(A /\!\!/ E(2))_* = S \oplus Q$$
and that the BSS
$$\operatorname{Ext}_{E(1)_*}(Q) \otimes \mathbb{F}_2[v_2] \implies \operatorname{Ext}_{E(2)_*}(Q)$$
collapses. The latter implies that $\operatorname{Ext}_{E(2)}(Q)$ is generated by the elements in $\operatorname{Ext}^{0,*}$ as a module over $\mathbb{F}_2[v_0, v_1, v_2]$. Thus $\operatorname{Ext}_{E(2)}(\underline{\mathrm{bu}}_j)$ is generated as a module over $\mathbb{F}_2[v_0, v_1, v_2]$ by elements in $\operatorname{Ext}^{0,*}_{E(2)_*}$.

Now consider one of the spectral sequences (3.20) or (3.21). Note that the differentials are linear over $\mathbb{F}_2[v_0, v_1, v_2]$. So if there were a differential $d(x) = y$ where x is a torsion class and y is v_2-torsion free, then $v_2 y$ would be a permanent cycle in the spectral sequence. But this would contradict that $\mathrm{Ext}_{E(2)}(\underline{\mathrm{bu}}_j)$ is generated over $\mathbb{F}_2[v_0, v_1, v_2]$ by elements in $\mathrm{Ext}^{0,*}$. A similar argument shows that there cannot be a differential $d(y) = x$. $\square$

A consequence of this is that in the inductive calculations, we can essentially ignore the torsion classes on the E_1-page.

We will now carry out the inductive calculations. We begin with some remarks on the v_0-inverted calculations, starting with developing analogues of Lemmas 4.19 and 4.20 of [4].

Since the v_0-inverted versions of (3.16) and (3.14) collapse at E_2, we get summands

$$v_0^{-1}\,\mathrm{Ext}_{E(1)_*}(\Sigma^{8k}\underline{BP\langle 2\rangle}_{2k-1}) \subseteq v_0^{-1}\,\mathrm{Ext}_{E(2)_*}(\Sigma^{8k}\underline{\mathrm{bu}}_{2k}) \tag{3.29}$$

$$v_0^{-1}\,\mathrm{Ext}_{E(1)_*}(\Sigma^{8k+4}\underline{BP\langle 2\rangle}_{2k-1}) \subseteq v_0^{-1}\,\mathrm{Ext}_{E(2)_*}(\Sigma^{8k+4}\underline{\mathrm{bu}}_{2k+1}) \tag{3.30}$$

We will identify the generators of these summands. This is accomplished by contemplating the following portion of the $8k$-fold suspension of the exact sequence (3.16):

$$\begin{tikzcd}
\Sigma^{8k}\underline{\mathrm{bu}}_{2k} \arrow[r] \arrow[d, "\cong"] & \Sigma^{8k}(E(2)/\!\!/E(1))_* \otimes \underline{BP\langle 2\rangle}_{2k-1}. \\
M_2(4k) &
\end{tikzcd}$$

Let $\zeta_1^{2i_1}\zeta_2^{2i_2} \in v_0^{-1}\,\mathrm{Ext}_{E(1)_*}(\underline{BP\langle 2\rangle}_{2k-1})$, then in the diagram above we have

$$\begin{tikzcd}
\zeta_1^{2i_1}\zeta_2^{2i_2} \arrow[r, mapsto] \arrow[d, mapsto] & \zeta_1^{2i_1}\zeta_2^{2i_2} \\
\zeta_1^{a}\zeta_2^{2i_1}\zeta_3^{2i_2} &
\end{tikzcd}$$

where

$$a := 8k - 4i_1 - 8i_2.$$

Similarly, we could contemplate the diagrams below coming from (3.14):

$$\begin{tikzcd}
\Sigma^{8k+4}\underline{\mathrm{bu}}_{2k+1} \arrow[r] \arrow[d, "\cong"] & \Sigma^{8k+4}(E(2)/\!\!/E(1))_* \otimes \underline{BP\langle 2\rangle}_{2j-1} \arrow[r] & 0 \\
M_2(4k+2) & &
\end{tikzcd}$$

This results in

Lemma 3.31. *The summands* (3.29) *and* (3.30) *are generated as modules over* $\mathbb{F}_2[v_0^{\pm 1}, v_1]$ *by elements*

$$\zeta_1^a \zeta_2^{2i_2} \zeta_3^{2i_3}$$

with $i_2 + 2i_3 \leq 2j - 1$ *and* $a = 8j - 4i_2 - 8i_3$ *(resp.* $a = 8j + 4 - 4i_2 - 8i_3$*).*

Next we need to determine the generators arising from the terms $\Sigma^{4j}\underline{\mathrm{bu}}_j$ in the case of (3.16) and $\Sigma^{4j}\underline{\mathrm{bu}}_j \otimes \underline{\mathrm{bu}}_1$ in the case of (3.14). Because of Proposition 3.28, we obtain summands

$$\mathrm{Ext}_{E(2)}(\Sigma^{12j}\underline{\mathrm{bu}}_j)/v_0{-}tors \subseteq \mathrm{Ext}_{E(2)}(\Sigma^{8j}\underline{\mathrm{bu}}_{2j}) \tag{3.32}$$

$$\mathrm{Ext}_{E(2)}(\Sigma^{12j+4}\underline{\mathrm{bu}}_j \otimes \underline{\mathrm{bu}}_1)/v_o{-}tors \subseteq \mathrm{Ext}_{E(2)}(\Sigma^{8j}\underline{\mathrm{bu}}_{2j+1}) \tag{3.33}$$

Proposition 3.34. *Assume inductively that that the* $\mathrm{Ext}_{E(2)}(\Sigma^{4j}\underline{\mathrm{bu}}_j)$ *has generators of the form* $\{\zeta_1^{i_1}\zeta_2^{i_2}\cdots\}$*. Then the summand of* (3.32) *has generators of the form* $\{\zeta_2^{i_1}\zeta_3^{i_2}\cdots\}$ *and the summand* (3.33) *has generators*

$$\{\zeta_2^{i_1}\zeta_3^{i_2}\cdots\} \cdot \{\zeta_1^4, \zeta_2^2\}.$$

The proof of this proposition follows by considering the diagrams

$$\begin{tikzcd}
0 \arrow[r] & \Sigma^{12k}\underline{\mathrm{bu}}_k \arrow[r] & \Sigma^{8k}\underline{\mathrm{bu}}_{2k} \arrow[d, "\cong"] \\
 & \Sigma^{4k}M_2(2k) \arrow[u, "\cong"] & M_2(4k)
\end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[r] & \Sigma^{12k+4}\underline{\mathrm{bu}}_k \otimes \underline{\mathrm{bu}}_1 \arrow[r] & \Sigma^{8k+4}\underline{\mathrm{bu}}_{2k+1} \arrow[d, "\cong"] \\
 & \Sigma^{4k}M_2(2k) \otimes \underline{\mathrm{bu}}_1 \arrow[u, "\cong"] & M_2(4k+2)
\end{tikzcd}$$

For example, in the first diagram, given a monomial $\zeta_1^{i_1}\zeta_2^{i_2}\cdots$ in $\mathrm{Ext}_{E(2)}(\Sigma^{4k}\underline{\mathrm{bu}}_k)$, we would obtain

$$\begin{tikzcd} \zeta_1^{i_2}\zeta_3^{i_2}\cdots \arrow[r, mapsto] & \zeta_1^{a}\zeta_2^{i_2}\zeta_3^{i_3}\cdots \arrow[d, mapsto] \\ \zeta_1^{i_1}\zeta_2^{i_2}\cdots \arrow[u, mapsto] & \zeta_2^{i_1}\zeta_3^{i_2}\cdots \end{tikzcd}$$

the right hand vertical arrow follows from the fact that

$$a = 4k - \mathrm{wt}(\zeta_2^{i_2}\zeta_3^{i_3}\cdots) = i_1.$$

There remains two questions regarding these inductive calculations: What is the role of the summand $\mathrm{Ext}_{E(2)}(\Sigma^{12k+5}\underline{\mathrm{bu}}_{j-1})$, and how does one determine the v_2-extensions in the spectral sequences on summands (3.29) and (3.30)? The following lemma will be useful.

Lemma 3.35. *The composite*

$$M_2(2j-1)\otimes E(\zeta_3) \xrightarrow{\tau^{-1}} (A/\!\!/E(2))_* \longrightarrow Q^{2j-1}(A/\!\!/E(1))_*$$

is a map of $E(2)_*$-comodules.

Proof. Denote the composite by χ. Consider an element $y\otimes\zeta_3^{\epsilon}$ in $M_2(2j-1)\otimes E(\zeta_3)$. Then we need to check that

$$Q_i y\zeta_3^{\epsilon} = \chi(Q_i(y\otimes\zeta_3^{\epsilon})).$$

Note that

$$Q_i(y\otimes\zeta_3^{\epsilon}) = (Q_i y)\otimes\zeta_3^{\epsilon} + y\otimes Q_i\zeta_3^{\epsilon}$$

and under χ this is

$$\chi(Q_i(y\otimes\zeta_3^{\epsilon})) = (Q_i y)\zeta_3^{\epsilon} + yQ_i\zeta_3^{\epsilon}.$$

If $i = 0, 1$, then $yQ_i\zeta_3^{\epsilon}$ lies in $F^{2j}(A/\!\!/E(1))_*$, and so is zero in $Q^{2j-1}(A/\!\!/E(1))_*$. Moreover, in $M_2(2j-1)\otimes E(\zeta_3)$,

$$Q_i(y\otimes\zeta_3^{\epsilon}) = (Q_i y)\otimes\zeta_3^{\epsilon}.$$

And so χ commutes with Q_0 and Q_1. For the case of Q_2, if $\epsilon = 0$, there is nothing to check. So let $\epsilon = 1$. Then

$$Q_2(y\otimes\zeta_3) = (Q_2 y)\otimes\zeta_3 + y\otimes 1$$

which is sent under χ to

$$\chi(Q_2(y\otimes\zeta_3)) = (Q_2 y)\zeta_3 + y.$$

In $Q^{2j-1}(A/\!/E(1))_*$, one has

$$Q_2(y\zeta_3) = (Q_2 y)\zeta_3 + y$$

as Q_2 acts as a derivation and $Q_2\zeta_3 = 1$. Thus Q_2 also commutes with χ. □

Corollary 3.36. *Let $j \geq 1$ and let $M = M_2(2j-1)$. Then we have the following diagram, where the rows are exact, in the category of $E(2)_*$-comodules,*

$$\begin{array}{ccccccccc}
0 \to & \Sigma^{4j}\underline{\mathrm{bu}}_j & \to & \underline{\mathrm{bu}}_{2j} & \to & Q^{2j-1}(A/\!/E(1))_* & \to & \Sigma^{4j+5}\underline{\mathrm{bu}}_{j-1} & \to 0 \\
 & & & \uparrow & & \uparrow & & \uparrow & \\
0 \to & 0 & \to & M & \to & M \otimes E(\zeta_3) & \to & M \otimes \mathbb{F}_2\{\zeta_3\} & \to 0
\end{array}$$

From this we deduce the analogue of Lemma 4.22 in [4].

Corollary 3.37. *Consider the summand*

$$\begin{aligned} v_0^{-1}\operatorname{Ext}_{E(1)_*}(\Sigma^{12j-2}\underline{\mathrm{bu}}_{j-1}) &\subseteq v_0^{-1}\operatorname{Ext}_{E(1)_*}(\Sigma^{8j}\underline{BP\langle 2\rangle}_{2j-1}) \\ &\subseteq v_0^{-1}\operatorname{Ext}_{E(2)_*}(\Sigma^{8j}\underline{\mathrm{bu}}_{2j}), \end{aligned}$$

generated over $\mathbb{F}_2[v_0^{\pm 1}, v_1]$ by the generators

$$\zeta_1^4\zeta_2^{2i_2}\zeta_3^{2i_3} \in (A/\!/E(2))_*$$

where $i_1 + 2i_2 = 2j-1$. In the summand

$$v_0^{-1}\operatorname{Ext}_{E(2)_*}(\Sigma^{12j+5}\underline{\mathrm{bu}}_{j-1}[1]) \subseteq v_0^{-1}\operatorname{Ext}_{E(2)_*}(\Sigma^{8j}\underline{\mathrm{bu}}_{2j}) \tag{3.38}$$

let x_i for $(1 \leq i \leq j-1)$ denote the generator of (3.38) *corresponding to ζ_1^{2i}. Then in the E_∞-page of the spectral sequence* (3.21), *we have*

$$v_2\zeta_1^4\zeta_2^{2i_2}\zeta_3^{2i_3} = x_{i_3}$$

We will now discuss the v_2-extensions concerning the other generators in the summand in (3.21),

$$v_0^{-1}\operatorname{Ext}_{E(1)_*}(\Sigma^{8j}\underline{BP\langle 2\rangle}_{2j-1}) \subseteq v_0^{-1}\operatorname{Ext}_{E(2)}(\Sigma^{8j}\underline{\mathrm{bu}}_{2j}),$$

and the generators of the summand

$$v_0^{-1}\operatorname{Ext}_{E(1)_*}(\Sigma^{8j+4}\underline{BP\langle 2\rangle}_{2j-1}) \subseteq v_0^{-1}\operatorname{Ext}_{E(2)_*}(\Sigma^{8j+4}\underline{\mathrm{bu}}_{2j+1})$$

in (3.20). That is the monomials of the form

$$\zeta_1^a\zeta_2^{2i_2}\zeta_3^{2i_3} \in (A/\!/E(2))_*$$

where $i_2 + 2i_3 \leq 2j - 1$ and $a = 8j - 4i_2 - 8i_3$ and $a > 4$ (resp. $a = 8j + 4 - 4i_2 - 8i_3$).

The notion of length discussed in section 2 induces an increasing filtration of $(A /\!\!/ E(2))_*$ by $E(2)_*$-comodules,

$$G^\ell(A /\!\!/ E(2))_* := E(2)\{m \in (A /\!\!/ E(2))_* \mid \ell(m) \leq \ell\}.$$

Since the action by Q_i lowers length by exactly one, it follows that the filtration quotients are trivial $E(2)_*$-comodules

$$G^\ell / G^{\ell-1} = \mathbb{F}_2\{m \mid \ell(m) = \ell\}.$$

Applying $\mathrm{Ext}_{E(2)_*}$ gives a spectral sequence converging to $\mathrm{Ext}_{E(2)}((A /\!\!/ E(2))_*)$. This spectral sequence is of the form

$$E_1^{s,t,\ell} = E_0(A /\!\!/ E(2))_* \otimes \mathbb{F}_2[v_0, v_1, v_2] \implies \mathrm{Ext}_{E(2)_*}((A /\!\!/ E(2))_*)$$

and we call it the *length spectral sequence*. By examining the induced short exact sequences in cobar complexes, one easily derives that the d_1-differential is

$$d_1(x) = v_0(Q_0 x) + v_1(Q_1 x) + v_2(Q_2 x)$$

which implies that $\mathrm{Ext}_{E(2)_*}((A /\!\!/ E(2))_*)$ has the following class of relations,

$$v_2(Q_2 x) = v_0(Q_0 x) + v_1(Q_1 x).$$

In particular, if m is a length 0 monomial, then we have the following relations in Ext:

$$v_2(\zeta_1^8 m) = v_1(\zeta_2^4 m) + v_0(\zeta_3^2 m). \tag{3.39}$$

This suggests that given a monomial m of length 0 in

$$\mathrm{Ext}_{E(2)_*}(\Sigma^{8j+4\epsilon}\underline{\mathrm{bu}}_{2j+\epsilon}),$$

we should have the relations

$$v_2(m) = v_0(\zeta_1^{-8}\zeta_3^2 m) + v_1(\zeta_1^{-8}\zeta_2^4 m) \tag{3.40}$$

Of course, under the hypothesis of Corollary 3.37, this does not make sense, but the following simple observation shows that this is a possibility for all other monomials in the summands

$$v_0^{-1}\,\mathrm{Ext}_{E(2)_*}(\Sigma^{8j+4\epsilon}\underline{BP\langle 2\rangle}_{2j-1}).$$

Lemma 3.41. *For the generators*

$$\zeta_1^a\zeta_2^{2i_2}\zeta_3^{2i_3} \in v_0^{-1}\operatorname{Ext}_{E(1)}(\Sigma^{8j}BP\langle 2\rangle_{2j-1}) \subseteq v_0^{-1}\operatorname{Ext}_{E(2)}(\Sigma^{8j}\mathrm{bu}_{2j})$$

the exponent a is always divisible by 4 and $a \geq 4$. For the generators

$$\zeta_1^a\zeta_2^{2i_2}\zeta_3^{2i_3} \in v_0^{-1}\operatorname{Ext}_{E(1)}(\Sigma^{8j+4}BP\langle 2\rangle_{2j-1}) \subseteq v_0^{-1}\operatorname{Ext}_{E(2)}(\Sigma^{8j+4}\mathrm{bu}_{2j+1})$$

the exponent a is always divisible by 4 and $a \geq 8$.

In the next several propositions, we prove that when $m = \zeta_1^a\zeta_2^{2i_2}\zeta_3^{2i_3}$ is a generator of $v_0^{-1}\operatorname{Ext}_{E(1)_*}(\Sigma^{8j+4\epsilon}\underline{BP\langle 2\rangle}_{2j-1})$, then the other monomials occuring in (3.40) are also generators of $v_0^{-1}\operatorname{Ext}_{E(2)_*}(\Sigma^{8j+4\epsilon}\underline{\mathrm{bu}}_{2j+\epsilon})$. The proofs are fairly direct; one simply breaks into several cases and makes sure certain inequalities hold.

Proposition 3.42. *Consider a monomial generator $\zeta_1^a\zeta_2^{2i_2}\zeta_3^{2i_3}$ in*

$$v_0^{-1}\operatorname{Ext}_{E(1)}(\Sigma^{8j}\underline{BP\langle 2\rangle}_{2j-1})$$

and suppose that $a > 8$. Then the monomials

$$\zeta_1^{a-8}\zeta_2^{2i_2}\zeta_3^{2i_3+2}, \zeta_1^{a-8}\zeta_2^{2i_2+4}\zeta_3^{2i_3} \in v_0^{-1}\operatorname{Ext}_{E(1)}(\Sigma^{8j}\mathrm{bu}_{2j})$$

are generators.

Proof. Under the hypothesis of the proposition, we have

$$i_2 + 2i_3 \leq 2j - 1$$

and

$$a = 8j - 4i_2 - 8i_3$$

Since the weights of the proposed monomials are still $8j$, all that needs to be checked is that

$$i_2 + 2i_3 + 2 \leq 2j - 1$$

Since

$$a = 8j - 4i_2 - 8i_3 > 8$$

we have

$$2j - i_2 - 2i_3 > 2$$

and hence

$$i_2 + 2i_3 < 2j - 2.$$

Therefore,

$$i_2 + 2i_3 + 2 < 2j$$

which proves the proposition. ∎

Proposition 3.43. *Consider a monomial generator $\zeta_1^8\zeta_2^{2i_2}\zeta_3^{2i_3}$ in the summand $v_0^{-1}\operatorname{Ext}_{E(1)}(\Sigma^{8j}\underline{BP\langle 2\rangle}_{2j-1})$. Then, the monomials $\zeta_2^{2i_2}\zeta_3^{2i_3+2}, \zeta_2^{2i_2+4}\zeta_3^{2i_3}$ are generators in the summand $v_0^{-1}\operatorname{Ext}_{E(2)}(\Sigma^{12j}\underline{\mathrm{bu}}_j)$.*

Proof. It needs to be checked that the monomials $\zeta_1^{2i_2+4}\zeta_2^{2i_3}$ and $\zeta_1^{2i_2}\zeta_3^{2i_3+2}$ are generators of $v_0^{-1}\operatorname{Ext}_{E(2)_*}(\Sigma^{4j}\mathrm{bu}_j)$. The proof will be broken up into several different cases. We will begin with the case in which $i_2 \neq 0$.

So assume that $i_2 \neq 0$, we will consider two further sub-cases. Let $k = \lfloor j/2 \rfloor$. Suppose first that $2k = j$. In this case, there is the exact sequence

$$0 \to \Sigma^{12k}\underline{\mathrm{bu}}_k \to \Sigma^{8k}\underline{\mathrm{bu}}_j \to \Sigma^{8k}(E(2)/\!\!/E(1))_* \otimes \underline{BP\langle 2\rangle}_{2k-1} \to \Sigma^{4k+5}\underline{\mathrm{bu}}_{k-1} \to 0.$$

Then we need to show the following

(1) $2i_2 + 4i_3 + 4 = 8k = 4j$
(2) $i_3 + 1 \leq 2k - 1 = j - 1$.

Since $\zeta_1^8\zeta_2^{2i_2}\zeta_3^{2i_3}$ is a generator of the summand

$$v_0^{-1}\operatorname{Ext}_{E(1)_*}(\Sigma^{8j}\underline{BP\langle 2\rangle}_{2j-1}),$$

we know that

$$8 + 4i_2 + 8i_3 = 8j$$

which shows the first condition. Observe that this implies i_2 is even. From this equality, we can write

$$8i_3 + 8 = 8j - 4i_2$$

which dividing by 8 and using the fact that i_2 is even shows

$$i_3 + 1 = j - \frac{i_2}{2} \leq j - 1,$$

showing the second condition.

So consider the case when $2k = j - 1$. Then we have a sequence

$$0 \to \Sigma^{12k}\underline{\mathrm{bu}}_k \otimes \underline{\mathrm{bu}}_j \to \Sigma^{4j}\underline{\mathrm{bu}}_j \to (E(2)/\!\!/E(1))_* \otimes \underline{BP\langle 2\rangle}_{2k-1} \to 0.$$

We will consider the monomials $\zeta_1^{2i_2+4}\zeta_2^{2i_3}$ and $\zeta_1^{2i_2}\zeta_2^{2i_3+2}$ separately. Consider first $\zeta_1^{2i_2+4}\zeta_2^{2i_3}$. For this to be a generator of $v_0^{-1}\operatorname{Ext}_{E(2)_*}\Sigma^{4j}\underline{\mathrm{bu}}_j$, it would have to be a generator originating in the summand

$$v_0^{-1}\operatorname{Ext}_{E(1)_*}(\Sigma^{4j}\underline{BP\langle 2\rangle}_{2k-1}).$$

So we need to show

(1) $2i_2 + 4 + 4i_3 = 8k + 4 = 4j$
(2) $i_3 \leq 2k - 1 = j - 2$

The first condition follows as before. We can again write

$$8i_3 = 8j - 4i_2 - 8$$

which dividing by 8 gives

$$i_3 = j - 1 - \frac{i_2}{2} \leq j - 2$$

since $i_2 \neq 0$ and i_2 is even.

So consider now the monomial $\zeta_1^{2i_2}\zeta_2^{2i_3+2}$. There are two further sub-cases to consider for this monomial. Suppose first that $i_2 > 2$. Then if this monomial is to be a generator of $v_0^{-1}\operatorname{Ext}_{E(2)_*}(\Sigma^{4j}\underline{\mathrm{bu}}_j)$, it would have to originate from the summand

$$v_0^{-1}\operatorname{Ext}_{E(1)_*}(\Sigma^{4j}\underline{BP\langle 2\rangle}_{2k-1}).$$

We thus need to check that

(1) $2i_2 + 4i_3 + 4 = 8k + 4 = 4j$
(2) $i_3 + 1 \leq 2k - 1 = j - 2$.

The first condition follows as before. For the second condition, note

$$8i_3 = 8j - 4i_2 - 8$$

gives

$$i_3 = j - 1 - \frac{i_2}{2}.$$

Since we are assuming $i_2 > 2$, then $i_2/2 > 1$, which implies

$$i_3 \leq j - 3$$

as desired. So consider then the case when $i_2 = 2$. In this case, for the monomial to be a generator of $v_0^{-1}\operatorname{Ext}_{E(2)}(\Sigma^{4j}\underline{\mathrm{bu}}_j)$, it would have to originate from the summand $v_0^{-1}\operatorname{Ext}_{E(2)_*}(\Sigma^{12jk}\underline{\mathrm{bu}}_k \otimes \underline{\mathrm{bu}}_1)$. Thus we need to check that $\zeta_1^{2i_3+2}$ is a generator of $v_0^{-1}\operatorname{Ext}_{E(2)_*}(\Sigma^{4k}\underline{\mathrm{bu}}_k)$. For this to be true, it needs to be the case that

$$i_3 + 1 = 2k = j - 1.$$

Writing

$$8i_3 = 8j - 4i_2 - 8 = 8j - 16$$

and dividing by 8 shows that indeed $i_3 = j-2$. This shows that $\zeta_1^{2i_3+2}$ is indeed a generator of $v_0^{-1}\operatorname{Ext}_{E(2)_*}(\Sigma^{4k}\underline{\mathrm{bu}}_k)$. This completes the case when $i_2 \neq 0$.

So consider the final case when $i_2 = 0$. Then we need to show that the monomials $\zeta_1^{1^4}\zeta_2^{2j-2}$ and ζ_2^{2j} are generators of $v_0^{-1}\operatorname{Ext}_{E(2)_*}(\Sigma^{4j}\underline{\mathrm{bu}}_j)$. Let $k := \lfloor j/2 \rfloor$. We again need to separate into the subcases when $2k = j$ and $2k = j-1$.

Suppose first that $2k = j$. Then we have the sequence

$$0 \to \Sigma^{12k}\underline{\mathrm{bu}}_k \to \Sigma^{8k}\underline{\mathrm{bu}}_j \to \Sigma^{8k}(E(2)/\!\!/E(1))_* \otimes \underline{BP\langle 2\rangle}_{2k-1} \\ \to \Sigma^{4k+5}\underline{\mathrm{bu}}_{k-1} \to 0.$$

In this case, the monomial $\zeta_1^4\zeta_2^{2j-2}$ would have to be a generator originating from the summand $v_0^{-1}\operatorname{Ext}_{E(1)_*}(\Sigma^{4j}\underline{BP\langle 2\rangle}_{2k-1})$. In order to check that it is indeed a generator, it just needs to be observed that

$$4 + 4j - 4 = 4j$$

and that

$$j - 1 = 2k - 1.$$

For the monomial ζ_2^{2j} to be a generator, it would have to be a generator originating from $v_0^{-1}\operatorname{Ext}_{E(2)_*}(\Sigma^{4k}\underline{\mathrm{bu}}_k)$, i.e. we need to check that ζ_1^{2j} is a generator for $v_0^{-1}\operatorname{Ext}_{E(2)_*}(\Sigma^{4k}\underline{\mathrm{bu}}_k)$. This is immediate since $2j = 4k$.

So finally consider the subcase when $2k = j-1$. Then we have an exact sequence

$$0 \to \Sigma^{12k}\underline{\mathrm{bu}}_k \otimes \underline{\mathrm{bu}}_j \to \Sigma^{4j}\underline{\mathrm{bu}}_j \to (E(2)/\!\!/E(1))_* \otimes \underline{BP\langle 2\rangle}_{2k-1} \to 0.$$

In this case, we would have to show that $\zeta_1^4\zeta_2^{2j-2}$ and ζ_2^{2j} are generators originating from the summand $v_0^{-1}\operatorname{Ext}_{E(2)_*}(\Sigma^{8k}\underline{\mathrm{bu}}_k \otimes \underline{\mathrm{bu}}_1)$. This will follow once we show that ζ_1^{2j-2} is a generator for

$$v_0^{-1}\operatorname{Ext}_{E(2)_*}(\Sigma^{4k}\underline{\mathrm{bu}}_k).$$

This is immediate since

$$2j - 2 = 4k,$$

which completes the proof in the case $i_2 = 0$. $\square$

We will now discuss the hidden v_2-extension in the spectral sequences (3.20). Many of the arguments are similar to those in the proof of the previous proposition.

Proposition 3.44. *Suppose $\zeta_1^a\zeta_2^{2i_2}\zeta_3^{2i_3}$ is a generator for*
$$v_0^{-1}\operatorname{Ext}_{E(1)}(\Sigma^{8j+4}BP\langle 2\rangle_{2j-1})$$
with $a > 12$. Then the monomials
$$\zeta_1^{a-8}\zeta_2^{2i_2}\zeta_3^{2i_3+2}, \zeta_1^{a-8}\zeta_2^{2i_2+4}\zeta_3^{2i_3}$$
are also generators for
$$v_0^{-1}\operatorname{Ext}_{E(1)}(\Sigma^{8j+4}BP\langle 2\rangle_{2j-1}).$$

Proof. It needs to be shown that
$$i_2 + 2i_3 + 2 \leq 2j - 1$$
Since $a > 12$ we have
$$a = 8j + 4 - 4i_2 - 8i_3 > 12$$
which implies
$$i_2 + 2i_3 < 2j - 2$$
and hence
$$i_2 + 2i_3 + 2 < 2j$$
which proves the proposition. □

Proposition 3.45. *Suppose $\zeta_1^{12}\zeta_2^{2i_2}\zeta_3^{2i_3}$ is a generator for*
$$v_0^{-1}\operatorname{Ext}_{E(1)}(\Sigma^{8j+4}BP\langle 2\rangle_{2j-1}).$$
Then the monomials
$$\zeta_1^{4}\zeta_2^{2i_2}\zeta_3^{2i_3+2}, \zeta_1^{4}\zeta_2^{2i_2+4}\zeta_3^{2i_3}$$
are generators for the summand $v_0^{-1}\operatorname{Ext}_{E(2)}(\Sigma^{12j+4}\mathrm{bu}_j \otimes \mathrm{bu}_1)$.

Proof. To prove the proposition, it needs to be checked that the monomials $\zeta_1^{2i_2}\zeta_2^{2i_3+2}$ and $\zeta_1^{2i_2+4}\zeta_2^{2i_3}$ are generators of $v_0^{-1}\operatorname{Ext}_{E(2)_*}(\Sigma^{4j}\underline{\mathrm{bu}}_j)$. The proof breaks down as in the proof of Proposition 3.43, *mutatis mutandis*. □

The following proposition deals with the last cases of v_2-extensions in the rational inductive calculations.

Proposition 3.46. *Let $\zeta_1^8\zeta_2^{2i_2}\zeta_3^{2i_3}$ be a generator of $v_0^{-1}\operatorname{Ext}_{E(2)}(\Sigma^{8j+4}\mathrm{bu}_{2j+1})$, then $\zeta_2^{2i_2+4}\zeta_3^{2i_3}$ and $\zeta_2^{2i_2}\zeta_3^{2i_3+2}$ are generators of $v_0^{-1}\operatorname{Ext}_{E(2)}(\Sigma^{8j+4}\mathrm{bu}_{2j+1})$.*

Proof. We have the short exact sequence

$$0 \to \Sigma^{12j+4}\underline{\mathrm{bu}}_j \otimes \underline{\mathrm{bu}}_1 \to \Sigma^{8j+4}\underline{\mathrm{bu}}_{2j+1} \\ \to \Sigma^{8j+4}(E(2) /\!\!/ E(1))_* \otimes \underline{BP\langle 2\rangle}_{2j-1} \to 0.$$

Note that we have

$$i_2 + 2i_3 = 2j - 1$$

and so i_2 must be an odd natural number. To prove the proposition, we need to show that $\zeta_1^{2i_2+2}\zeta_2^{2i_3}$ and $\zeta_1^{2i_2-2}\zeta_2^{2i_3+2}$ are generators of $v_0^{-1}\operatorname{Ext}_{E(2)}(\Sigma^{4j}\mathrm{bu}_j)$. Let $k = \lfloor j/2 \rfloor$, so that $j = 2k + \epsilon$. Consider first the case when $\epsilon = 0$. Then we have the sequence

$$0 \to \Sigma^{12k}\underline{\mathrm{bu}}_k \to \Sigma^{8k}\underline{\mathrm{bu}}_j \\ \to \Sigma^{8k}(E(2) /\!\!/ E(1))_* \otimes \underline{BP\langle 2\rangle}_{2k-1} \to \Sigma^{4k+5}\underline{\mathrm{bu}}_{k-1} \to 0.$$

To show that $\zeta_1^{2i_2+2}\zeta_2^{2i_3}$ and $\zeta_1^{2i_2-2}\zeta_2^{2i_3+2}$ are generators of $v_0^{-1}\operatorname{Ext}_{E(2)_*}(\Sigma^{4j}\underline{\mathrm{bu}}_j)$, we will consider several subcases. Consider the case when $i_2 \geq 3$. In this case, for these monomials to be generators, they would have to be generators of the summand

$$v_0^{-1}\operatorname{Ext}_{E(1)_*}(\Sigma^{8k}\underline{BP\langle 2\rangle}_{2k-1}) \subseteq v_0^{-1}\operatorname{Ext}_{E(2)_*}(\Sigma^{4j}\underline{\mathrm{bu}}_j).$$

To show they are indeed generators in this summand, it just needs to be checked that

$$i_3 \leq 2k - 1$$

for the first monomial and

$$i_3 + 1 \leq 2k - 1$$

for the second. Since $i_2 \geq 3$, it follows that

$$2i_3 \leq 2j - 4$$

and hence

$$i_3 \leq j - 2 = 2k - 2,$$

which shows that both monomials are generators in this case. So consider the sub-case when $i_2 = 1$, then

$$2i_3 = 2j - 2$$

and so

$$i_3 = j - 1 = 2k - 1$$

which shows that $\zeta_1^4\zeta_2^{2k-1}$ is a generator of the summand

$$v_0^{-1}\operatorname{Ext}_{E(1)_*}(\Sigma^{4j}BP\langle\rangle 2k-1).$$

For ζ_2^{2j} to be a generator, it would have to be a generator for the summand $v_0^{-1}\operatorname{Ext}_{E(2)}(\Sigma^{4j+4k}\mathrm{bu}_k)$. This follows since ζ_1^{2j} is a generator for $v_0^{-1}\operatorname{Ext}_{E(2)}(\Sigma^{4k}\mathrm{bu}_k)$.

Now suppose that $\epsilon = 1$, so that we have an exact sequence

$$0 \to \Sigma^{4k+4j}\underline{\mathrm{bu}}_k \otimes \underline{\mathrm{bu}}_1 \to \Sigma^{4j}\underline{\mathrm{bu}}_j \to \Sigma^{4j}(E(2) /\!\!/ E(1))_* \otimes \underline{BP\langle 2\rangle}_{2k-1} \to 0$$

We have to consider three cases: $i_2 > 3, i_2 = 3, i_2 = 1$. First suppose $i_2 > 3$. Then we need to show the two monomials are generators of $v_0^{-1}\operatorname{Ext}_{E(1)}(\Sigma^{4j}\underline{BP\langle 2\rangle}_{2k-1})$. Thus we need to check that $i_3 + 1 \leq 2k-1$. Since $i_2 \geq 3$, we get

$$2i_3 < 2j - 1 - 3 = 2j - 4$$

which shows that

$$i_3 < j - 2 = j - 2 = (2k+1) - 2 = 2k - 1$$

and this proves the case $i_2 > 3$.

So suppose now that $i_2 = 3$. Then the proof for the first monomial is the same as above. The second monomial becomes $\zeta_1^4\zeta_2^{2i_3+2}$. So we need to show that $\zeta_1^{2i_3}$ is a generator for $v_0^{-1}\operatorname{Ext}_{E(2)}(\Sigma^{4k}\mathrm{bu}_k)$. This follows from the fact that, in this case,

$$2i_3 + 2 = 2j - 1 - 3 = 2j - 4 = 2(2k+1) - 4 + 2 = 4k$$

and this finishes the case when $i_2 = 3$.

Finally, suppose that $i_2 = 1$. For both monomials, we need to show that $\zeta_1^{2i_3}$ is a generator of $v_0^{-1}\operatorname{Ext}_{E(2)}(\Sigma^{4k}\mathrm{bu}_k)$. This follows from the fact that in this case

$$2i_3 = 2j - 1 - 1 = 2j - 2 = 4k$$

This proves the case when $i_2 = 1$ and finishes the proposition. $\square$

We briefly mention how to infer the integral calculations from the rational calculations done above.

Remark 3.47. By Theorem 2.1, we know that there is an injection

$$\operatorname{Ext}_{E(2)_*}((A /\!\!/ E(2))_*)/v_2\text{-tors} \hookrightarrow v_0^{-1}\operatorname{Ext}_{E(2)_*}((A /\!\!/ E(2))_*)$$

and

$$\operatorname{Ext}_{E(2)_*}(\Sigma^{8j+4\epsilon}\underline{\mathrm{bu}}_{2j+\epsilon})/v_2\text{-tors} \hookrightarrow v_0^{-1}\operatorname{Ext}_{E(2)_*}(\Sigma^{8j+4\epsilon})$$

Recall that, integrally, $\mathrm{Ext}_{E(1)_*}(\underline{BP\langle 2\rangle}_{2j-1})$ decomposed into a sum of suspensions of Adams covers of $\mathrm{Ext}_{E(1)_*}(\mathbb{F}_2)$, and that inverting v_0 on each cover reduces it to a copy of $v_0^{-1}\,\mathrm{Ext}_{E(1)_*}(\mathbb{F}_2)$. To recover the Adams covers one simply uses the algorithm described in Remark 3.27. By Proposition 3.28, we can conclude that the rational generators produced above along with their associated Adams covers gives a basis of $\mathrm{Ext}_{E(2)_*}(\Sigma^{8j+4\epsilon}\underline{\mathrm{bu}}_{2j+\epsilon})/v_0$-tors as a module over $\mathbb{F}_2[v_0]$.

3.4. **Low degree computations.** In this section, we will provide examples of low degree computations using the inductive methods developed in the previous section. We tabulate the generators of the spectral sequences for low dimensional cases of $\underline{\mathrm{bu}}_j$. In the tables below, the summands of the form $(E(2)/\!\!/E(1))_*\otimes -$ are understood as being generators over $\mathbb{F}_2[v_0^{\pm 1}, v_1]$, while all other summands are generators over $\mathbb{F}_2[v_0^{\pm 1}, v_1, v_2]$. In the table below, generators having a hidden v_2-extension are indicated in red.

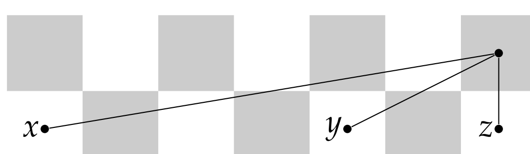
$\underline{\mathrm{bu}}_0$:	$\mathbb{F}_2$:	1
$\Sigma^4\underline{\mathrm{bu}}_1$:	$\Sigma^4\underline{\mathrm{bu}}_1$:	ζ_1^4, ζ_2^2
$\Sigma^8\underline{\mathrm{bu}}_2$:	$\Sigma^8(E(2)/\!\!/E(1))_*\otimes\underline{BP\langle 2\rangle}_1$:	$\zeta_1^8, \zeta_1^4\zeta_2^2$
	$\Sigma^{12}\underline{\mathrm{bu}}_1$:	ζ_2^4, ζ_3^2
	$\Sigma^{17}\underline{\mathrm{bu}}_0[1]$:	$v_2\zeta_1^4\zeta_2^2+\cdots$
$\Sigma^{12}\underline{\mathrm{bu}}_3$:	$\Sigma^{12}(E(2)/\!\!/E(1))_*\otimes\underline{BP\langle 2\rangle}_1$:	$\zeta_1^{12}, \zeta_1^8\zeta_2^2$
	$\Sigma^{16}\underline{\mathrm{bu}}_1\otimes\underline{\mathrm{bu}}_1$:	$\{\zeta_2^4, \zeta_3^2\}\cdot\{\zeta_1^4, \zeta_2^2\}$
$\Sigma^{16}\underline{\mathrm{bu}}_4$:	$\Sigma^{16}(E(2)/\!\!/E(1))_*\otimes\underline{BP\langle 2\rangle}_3$:	$\zeta_1^{16}, \zeta_1^{12}\zeta_2^2, \zeta_1^8\zeta_2^4, \zeta_1^8\zeta_3^2, \zeta_1^4\zeta_2^6, \zeta_1^4\zeta_2^2\zeta_3^2$
	$\Sigma^{24}(E(2)/\!\!/E(1))_*\otimes\underline{BP\langle 2\rangle}_1$:	$\zeta_2^8, \zeta_2^4\zeta_3^2$
	$\Sigma^{28}\underline{\mathrm{bu}}_1$:	ζ_3^4, ζ_4^2
	$\Sigma^{33}\underline{\mathrm{bu}}_0[1]$	$v_2\zeta_2^4\zeta_3^2+\cdots$
	$\Sigma^{29}\underline{\mathrm{bu}}_1[1]$:	$v_2\zeta_1^4\zeta_2^6+\cdots, v_2\zeta_1^4\zeta_2^2\zeta_3^2+\cdots$
$\Sigma^{20}\underline{\mathrm{bu}}_5$:	$\Sigma^{20}(E(2)/\!\!/E(1))_*\otimes\underline{BP\langle 2\rangle}_3$:	$\zeta_1^{20}, \zeta_1^{16}\zeta_2^2, \zeta_1^{12}\zeta_2^4, \zeta_1^8\zeta_2^6, \zeta_1^8\zeta_2^2\zeta_3^2$
	$\Sigma^{28}(E(2)/\!\!/E(1))_*\otimes\underline{BP\langle 2\rangle}_1\otimes\underline{\mathrm{bu}}_1$:	$\{\zeta_2^8, \zeta_2^4\zeta_3^2\}\cdot\{\zeta_1^4, \zeta_2^2\}$
	$\Sigma^{32}\underline{\mathrm{bu}}_1\otimes\underline{\mathrm{bu}}_1$:	$\{\zeta_3^4, \zeta_4^2\}\cdot\{\zeta_1^4, \zeta_2^2\}$

$$
\begin{array}{lrl}
 & \Sigma^{27}\underline{\mathrm{bu}}_1[1]: & \{v_2\zeta_2^4\zeta_3^2+\cdots\}\cdot\{\zeta_1^4,\zeta_2^2\} \\
\Sigma^{24}\underline{\mathrm{bu}}_6: & \Sigma^{24}(E(2)/\!\!/E(1))_*\otimes\underline{BP\langle 2\rangle}_5: & \zeta_1^{24},\zeta_1^{20}\zeta_2^2,\zeta_1^{16}\zeta_2^4,\zeta_1^{16}\zeta_3^2,\zeta_1^{12}\zeta_2^6, \\
 & & \zeta_1^{12}\zeta_2^2\zeta_3^2,\zeta_1^8\zeta_2^8,\zeta_1^8\zeta_2^4\zeta_3^2,\zeta_1^8\zeta_3^4, \\
 & & \zeta_1^4\zeta_2^{10},\zeta_1^4\zeta_2^6\zeta_3^2,\zeta_1^4\zeta_2^2\zeta_3^4 \\
 & \Sigma^{36}(E(2)/\!\!/E(1))_*\otimes\underline{BP\langle 2\rangle}_3: & \zeta_2^{12},\zeta_2^8\zeta_3^2 \\
 & \Sigma^{40}\underline{\mathrm{bu}}_1\otimes\underline{\mathrm{bu}}_1: & \{\zeta_3^4,\zeta_4^2\}\cdot\{\zeta_2^4,\zeta_3^2\} \\
 & \Sigma^{41}\underline{\mathrm{bu}}_2[1]: & v_2\zeta_1^4\zeta_2^{10}+\cdots,v_2\zeta_1^4\zeta_2^6\zeta_3^2+\cdots \\
 & & v_2\zeta_1^4\zeta_2^2\zeta_3^4+\cdots \\
\Sigma^{28}\underline{\mathrm{bu}}_7: & \Sigma^{28}(E(2)/\!\!/E(1))_*\otimes\underline{BP\langle 2\rangle}_5: & \zeta_1^{28},\zeta_1^{24}\zeta_2^2,\zeta_1^{20}\zeta_2^4,\zeta_1^{20}\zeta_3^2,\zeta_1^{16}\zeta_2^6, \\
 & & \zeta_1^{16}\zeta_2^2\zeta_3^2,\zeta_1^{12}\zeta_2^8,\zeta_1^{12}\zeta_2^4\zeta_3^2,\zeta_1^{12}\zeta_3^4, \\
 & & \zeta_1^8\zeta_2^{10},\zeta_1^8\zeta_2^6\zeta_3^2,\zeta_1^8\zeta_2^2\zeta_3^4 \\
 & \Sigma^{40}(E(2)/\!\!/E(1))_*\otimes\underline{BP\langle 2\rangle}_1\otimes\underline{\mathrm{bu}}_1: & \{\zeta_2^{12},\zeta_2^8\zeta_3^2\}\cdot\{\zeta_1^4,\zeta_2^2\} \\
 & \Sigma^{44}\underline{\mathrm{bu}}_1^{\otimes 3}: & \{\zeta_3^4,\zeta_4^2\}\cdot\{\zeta_2^4,\zeta_3^2\}\cdot\{\zeta_1^4,\zeta_2^2\}
\end{array}
$$

Below are charts for the spectral sequences (3.21) and (3.20). In the charts below, we will use the following key.

Symbol	Ring
$\circ$	$\mathbb{F}_2[v_0^{\pm 1}, v_1]$
$\triangle$	$\mathbb{F}_2[v_0^{\pm 1}, v_1, v_2]$

In the charts below, the following pattern



will denote a relation of the form

$$v_2x = v_1y + v_0z.$$

In particular, lines of slope $1/6$ denote multiplication by v_2, lines of slope $1/2$ denotes multiplication by v_1, and vertical lines denote multiplication by v_0.

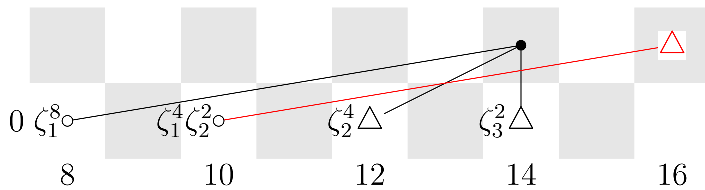


FIGURE 3.1. $v_0^{-1}\,\mathrm{Ext}_{E(2)_*}(\Sigma^8\underline{\mathrm{bu}}_2)$

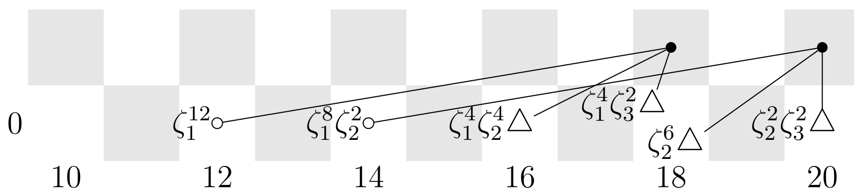


FIGURE 3.2. $v_0^{-1}\,\mathrm{Ext}_{E(2)_*}(\Sigma^{12}\underline{\mathrm{bu}}_3)$

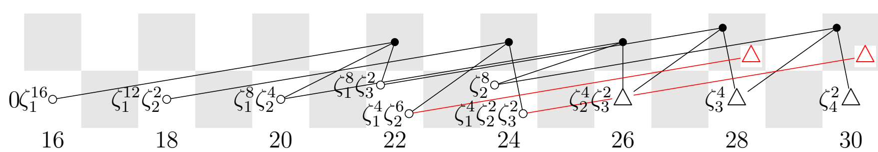
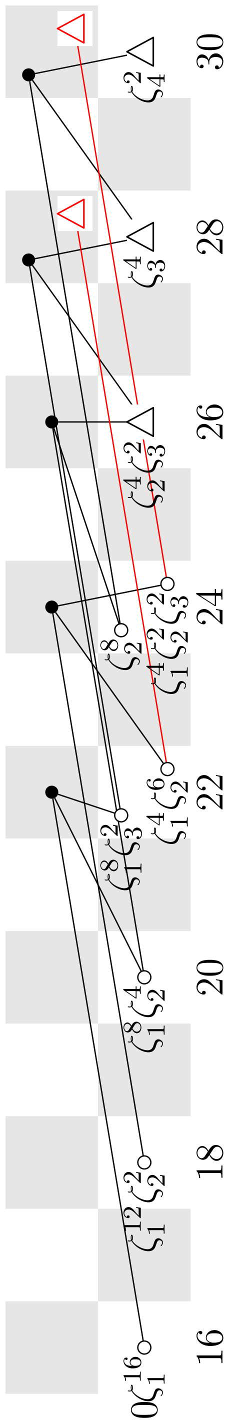
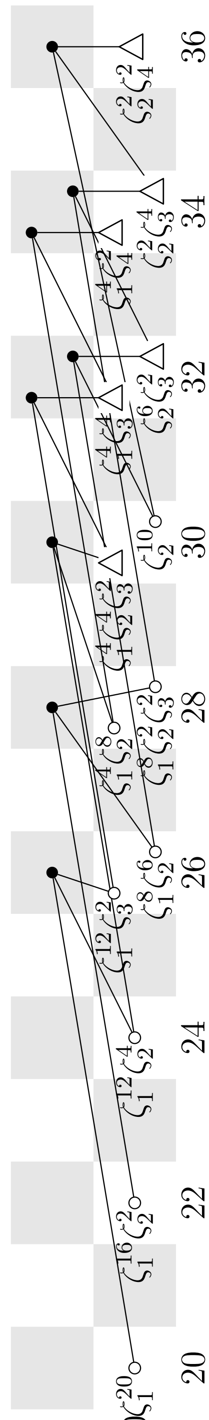


FIGURE 3.3. $v_0^{-1}\,\mathrm{Ext}_{E(2)_*}(\Sigma^{16}\underline{\mathrm{bu}}_4)$

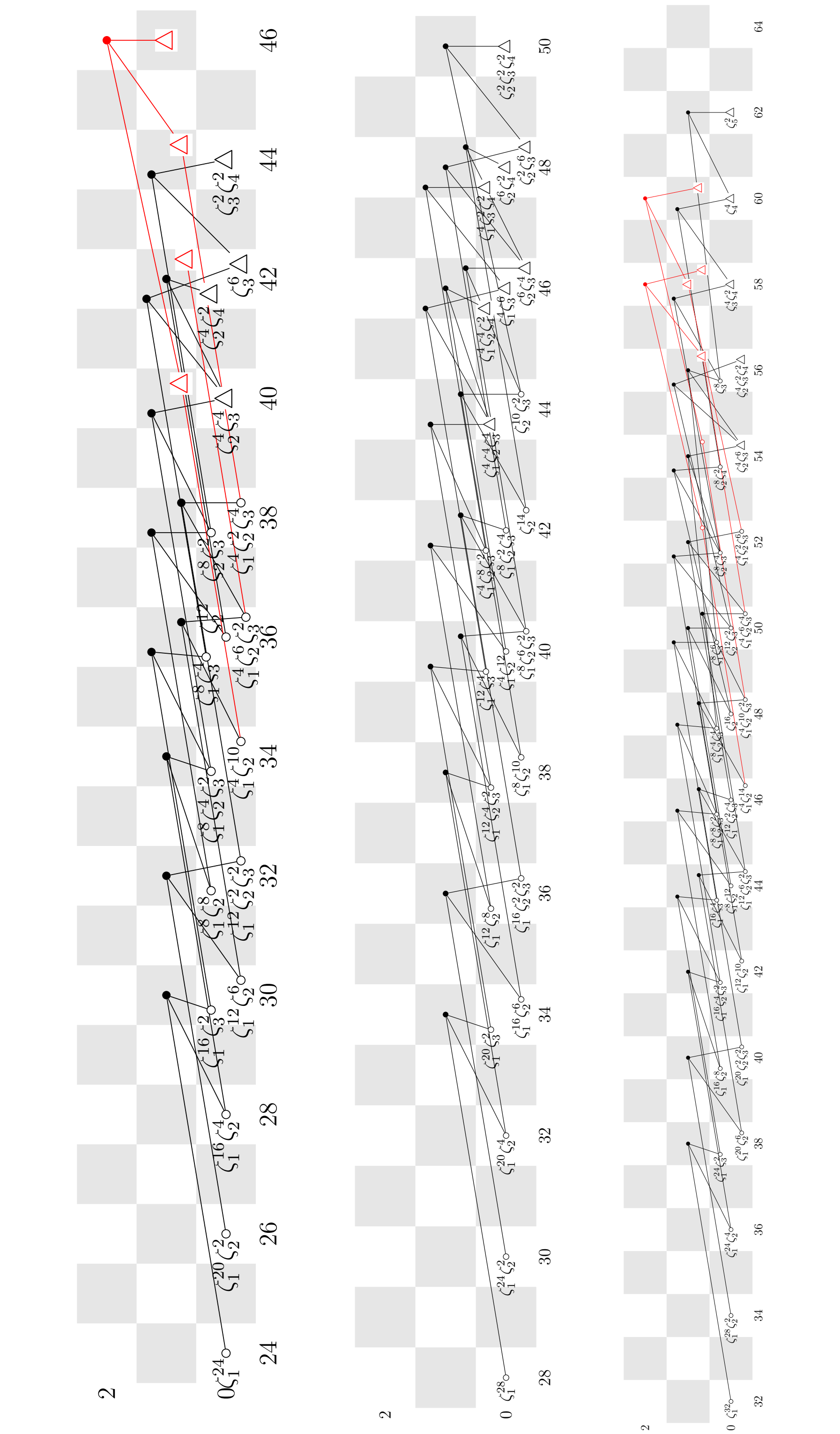


(A) $v_0^{-1}\operatorname{Ext}_{E(2)_*}(\Sigma^{16}\underline{\mathrm{bu}}_4)$



(B) $v_0^{-1}\operatorname{Ext}_{E(2)_*}(\Sigma^{20}\underline{\mathrm{bu}}_5)$

FIGURE 3.4



(A) $v_0^{-1}\,\mathrm{Ext}_{E(2)_*}(\Sigma^{24}\underline{\mathrm{bu}}_6)$

(B) $v_0^{-1}\,\mathrm{Ext}_{E(2)_*}(\Sigma^{28}\underline{\mathrm{bu}}_7)$

(C) $v_0^{-1}\,\mathrm{Ext}_{E(2)_*}(\Sigma^{32}\underline{\mathrm{bu}}_8)$

FIGURE 3.5

References

[1] J.F. Adams, *Stable homotopy and generalised homology*, University of Chicago Press, 1974.

[2] Agnes Beaudry, Mark Behrens, Prasit Bhattacharya, Dominic Culver, and Zhouli Xu, *On the E_2-term of the* bo-*Adams spectral sequence* (201702), available at `arXiv:1702.00230`.

[3] Mark Behrens, Mike Hill, Mike Hopkins, and Mark Mahowald, *On the existence of a v_2^{32} self map on $M(1,4)$ at the prime 2*, Homology, Homotopy, and Applications **10** (2008), 45–84.

[4] Mark Behrens, Kyle Ormsby, Nathaniel Stapleton, and Vesna Stojanoska, *On the ring of cooperations for 2-primary connective topological modular forms* (201501), available at `arXiv:1501.01050`.

[5] Tyler Lawson and Niko Naumann, *Commutativity conditions for truncated Brown-Peterson spectra of height 2*, Journal of Topology **5** (2012), no. 1, 137–168.

[6] Mark Mahowald, *bo-resolutions.*, Pacific J. Math. **92** (1981), no. 2, 365–383.

[7] Mark Mahowald and Wolfgang Lellmann, *The bo-Adams spectral sequence*, Transactions of the American Mathematical Society **300** (1987), no. 2, 593–623.

[8] Harvey R. Margolis, *Spectra and the Steenrod algebra*, North-Holland Mathematical Library, North-Holland, 1983.

[9] John W. Milnor and John C. Moore, *On the structure of Hopf algebras*, Annals of Mathematics **2** (1965), no. 81, 211–264.

[10] W. Stephen Wilson, *The Ω-Spectrum for Brown-Peterson Cohomology Part 2*, American Journal of Mathematics **97** (1975), no. 1, 101–123.

University of Illinois, Urbana-Champaign

E-mail address: `dculver@illinois.edu`