RESEARCH STATEMENT

D. CULVER

1. Introduction

My research lies within the area of *stable homotopy theory*. This is the part of homotopy theory which studies algebraic invariants of spaces which don't change under suspension. My primary interests are concerned with calculating the stable homotopy groups of spheres. In particular, I work on understanding certain patterns in the stable homotopy groups of spheres which generalize well-known patterns such as the image of the J-homomorphism. This is known as *chromatic homotopy theory*. I am also working on analogues of these problems in motivic and equivariant stable homotopy theory. In all of these situations, I approach these problems by considering Adams type spectral sequences based on interesting multiplicative cohomology theories, such as *topological modular forms* or *Hermitian K-theory*. By doing so, I hope to describe these elaborate patterns in terms of invariants of elliptic curves, quadratic forms, and other well-known objects in mathematics.

2. Background

A natural tool for approaching the stable homotopy groups of spheres is the classical *Adams spectral sequences* (ASS). This is a spectral sequence of the form

$$\mathrm{Ext}^{s,t}_{A_*}(\mathbb{F}_p, \mathbb{F}_p) \Longrightarrow \pi_{t-s} S^0 \otimes \mathbb{Z}_p,$$

where p is a prime number. This spectral sequence is particularly good for computing the stable homotopy groups in a finite range, but obscures infinite patterns in the stable homotopy groups of spheres.

However, given any ring spectrum E[1], one can always construct an ASS based on E. Under suitable conditions, this spectral sequence takes the form

$$\mathrm{Ext}_{E_*E}(E_*, E_*) \Longrightarrow \pi_* L_E S^0.$$

Here $E_*E = \pi_*(E \wedge E)$ and $L_E S^0$ is a certain *localization* of the sphere. When E is $H\mathbb{F}_p$, the spectrum representing mod p homology, then we

[1] or multiplicative cohomology theory

recover the ASS. In the case when $E = MU$, the complex cobordism spectrum, we get the *Adams-Novikov spectral sequence* (ANSS). Usually one localizes at one prime at a time, and replaces MU with a summand called the Brown-Peterson spectrum BP. The ANSS takes the form

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{BP_*BP}(BP_*, BP_*) \implies \pi_{t-s}S^0 \otimes \mathbb{Z}_{(p)}.$$

A classical theorem of Quillen tells us that the ring

$$BP_* := \pi_* BP \cong \mathbb{Z}_{(p)}[v_1, v_2, v_3, \dots]$$

is the ring which carries the *universal p-typical formal group law* and that the pair (BP_*, BP_*BP) classifies p-typical formal group laws and isomorphisms between them. It is a deep insight of Quillen and Morava that this pair gives a presentation of the moduli stack $\mathcal{M}_{pFG}$ of p-typical formal groups and that the E_2-term of the ANSS can be described as the cohomology of this stack, i.e.

$$\mathrm{Ext}_{BP_*BP}(BP_*, BP_*) \cong H^*(\mathcal{M}_{pFG}).$$

There is a well-known filtration on $\mathcal{M}_{pFG}$ by an invariant called the *height*, which turns out to be controlled by the elements v_i in π_*BP. This filtration leads to the *algebraic chromatic spectral sequence* which converges to $\mathrm{Ext}_{BP_*BP}(BP_*, BP_*)$. The E_1-term of this spectral sequence consists of elements which *periodic* with respect to some v_n, and shows that each element of the ANSS E_2-term belongs to some v_n-periodic family. From this perspective, the stable homotopy groups have a lot of structure and contain many intricate, but systematic, patterns. Computations for $n = 1, 2, 3$ and for sufficiently large primes have been done in [27].

This led Doug Ravenel to ask if this algebraic structure has its origins in topology, leading him to formulate several conjectures ([28]). All but the now infamous *telescope conjecture* have been proven. In particular, it is now understood that there are certain homology theories $K(n)$, the nth *Morava K-theory*, so that the homotopy groups of the sphere spectrum S^0 are built out of the localizations $L_{K(n)}S^0$. The elements of $\pi_*L_{K(n)}S^0$ are periodic with respect to v_n. For example, the homotopy of $L_{K(1)}S^0$ corresponds to the image of J, which was intensely studied by Adams in the 60s. In this case, there is a periodicity arising from the Bott element in topological K-theory and v_1 corresponds to the Bott element. The higher v_n-periodic families are analogues of the image of J with periodicity arising from v_n. In recent years, the focus has moved to computing $L_{K(2)}X$ for various X and to settling the telescope conjecture. The latter says, roughly, that for appropriate X, we have an isomorphism $v_n^{-1}\pi_*X \cong \pi_*L_{K(n)}X$.

A majority of my research is concerned with better understanding these periodic families at height $n=2$ using the spectrum tmf of *topological modular forms* and studying analogous problems in motivic homotopy theory.

3. Current and Future Research

3.1. **Chromatic homotopy theory.** In [24] and [25], Mahowald intensely studied the image of J through the lens of the Adams spectral sequence based on the connective real K-theory spectrum, denoted bo. Using it he was able to prove the height 1 telescope conjecture and perform computations of the stable homotopy groups of spheres. In recent years, the spectrum tmf of topological modular forms has entered the scene, and is a height 2 version of bo. One of my main research problems is to carry out Mahowald's program for tmf. This leads to three subgoals for the tmf-based ASS:

(1) settle the height 2 telescope conjecture,
(2) compute homotopy groups, and
(3) study/compute $K(2)$-local homotopy groups.

The first step in approaching each of these problems is to study the homotopy groups of $\mathrm{tmf}\wedge\mathrm{tmf}$. The 2-primary version of this problem has been studied in [11] and [26]. In previous work, I studied a more approachable version of the problem at all primes. In particular, I computed the $BP\langle 2\rangle$ co-operations. Here

$$BP\langle 2\rangle_* = BP_*/(v_3, v_4, \ldots) = \mathbb{Z}_{(p)}[v_1, v_2]$$

Theorem 3.1 (Culver, [18], [17]). *There is an algorithm for computing v_2-torsion free component of $BP\langle 2\rangle\wedge BP\langle 2\rangle$ at all primes. Moreover, the v_2-torsion in $\pi_* BP\langle 2\rangle\wedge BP\langle 2\rangle$ is simple torsion and concentrated in Adams filtration 0.*

In the case when $p=2$, then $BP\langle 2\rangle$ is equivalent to $\mathrm{tmf}_1(3)$, a variant of tmf built out of elliptic curves with a chosen point of order three, and so this theorem can viewed as a calculation of $\mathrm{tmf}_1(3)\wedge\mathrm{tmf}_1(3)$. In a different direction, Paul VanKoughnett and I have computed the $K(1)$-local homotopy groups of $\mathrm{tmf}\wedge\mathrm{tmf}$ using Hopkins' original construction of $\mathrm{tmf}_{K(1)}$.

Theorem 3.2 (Culver-VanKoughnett, [19]). *For the primes $p=2,3$, the $K(1)$-local homotopy of $\mathrm{tmf}\wedge\mathrm{tmf}$ is given by*

$$\pi_* L_{K(1)}(\mathrm{tmf}\wedge\mathrm{tmf}) \cong \left(KO_*[j^{-1},\overline{j^{-1}}]\otimes\mathbb{T}(\lambda)/(\psi^p(\lambda)-\lambda-j^{-1}+\overline{j^{-1}})\right)^{\wedge}_p$$

where $\mathbb{T}$ refers to the free ϑ-algebra.

In joint work with Beaudry, Behrens, Bhattacharya, and Xu, [4], I have studied the bo-based ASS, and we have extended previous computations of Mahowald ([24]) and Mahowald-Lellmann ([25]). This project is a warm-up to computing the tmf-resolution. In particular, we have

Computation 1 (Beaudry-Behrens-Bhattachary-C.-Xu, [9]). *There are various spectral sequences which allow us to compute the E_2-term of the bo-based ASS through a range. Moreover, the bo-based ASS collapses at the E_2-term up through the* 40*-stem.*

We hope that the techniques we developed for the bo-ASS will extend to the tmf-ASS. We have already had success in applying them to the tmf-ASS for the Bhattacharya-Egger spectrum Z ([12]). In particular, we have showed the following.

Theorem 3.3 (Beaudry-Behrens-Bhattacharya-C.-Xu, [5]). *The* tmf*-ASS for Z is computable using similar techniques to those in* [4]*, and there are several possible counter examples to the telescope conjecture at the E_2-page. Furthermore, a previously unresolved potential d_3-differential in the $K(2)$-local ANSS for Z (*[13]*) does not occur, thereby determining the entire spectral sequence.*

At the moment we cannot show that the potential counter examples survive the spectral sequence; if they did survive this would show the telescope conjecture is false. One of my goals is to study the analogous problem at the prime $p=3$. In this case, the homotopy groups of tmf are computationally simpler than at $p=2$, while still having a large Hurewicz image. This leads to another of my current goals.

Goal 1. *Compute the homotopy groups of $tmf \wedge tmf$.*

The tmf-ASS can also be $K(2)$-localized to give a spectral sequence converging to the $K(2)$-local sphere $S^0_{K(2)}$. Techniques to compute the homotopy of $S^0_{K(2)}$ were developed by Goerss-Henn-Mahowald-Rezk (GHMR) in [21] and later shown in [6],[7], [8] to have an interpretation in terms of elliptic curves and modular forms. In particular, Behrens shows that for any prime ℓ there is a spectrum $Q(\ell)$, which is built out of isogenies of elliptic curves of degree ℓ^k, and shows at $p=3$ that there is a cofiber sequence

$$L_{K(2)}S \to Q(2) \to \Sigma D_{K(2)}Q(2). \tag{3.4}$$

The second map in this sequence is often referred to as the *middle map*. This is the height 2 analogue of the Adams-Baird resolution, which is a cofiber sequence

$$L_{K(1)}S^0 \longrightarrow KO \xrightarrow{\psi^3-1} KO.$$

A connective version of the Adams-Baird resolution can be located within the bo-Adams spectral sequence, and after $K(1)$-localization reduces to it. I am interested in a height 2 analogue of this fact.

Conjecture 3.5. There is a connective version of the resolution (3.4) which sits within the tmf-based ASS. By this, we mean that when we $K(2)$-localize, we recover (3.4).

Towards proving this, I plan to work on the following.

Goal 2. *Construct connective versions of the spectra $Q(\ell)$.*

A periodic version has already been constructed by Behrens. It would seem that the work of Lawson-Hill ([23]) will be relevant here to extend the construction over the cusps of the relevant moduli spaces. Provided that I can find such a connective version of (3.4), I am hoping to compute the middle map through the tmf-based ASS. This map appears as an unresolved d_1-differential in [21] and continues to be an obstruction for better understanding/computing v_2-periodic homotopy theory. For example, this unresolved d_1-differential has stymied efforts to give a completely satisfactory computation of $\pi_* S_{K(2)}$ at $p = 3$.

Behrens' result (3.4), as mentioned, provides a modular interpretation of the $K(2)$-local sphere. At higher heights, there are analogues of $Q(\ell)$ and tmf which are constructed using Shimura varieties and automorphic forms ([10]). On the other hand, obtaining explicit computations of the $K(n)$-local sphere $L_{K(n)}S$ for $n > 2$ by generators and relations seems implausible. Instead we should try to produce *automorphic* interpretations of $L_{K(n)}S$. As a first step in this direction, I am currently working on the following.

Goal 3 (joint with Behrens, Bobkova, and VanKoughnett)**.** *Provide a modular description of $L_{K(2)}S$ for primes $p \geq 5$.*

Interestingly, this turns out to be more difficult then the case of $p = 3$ because at larger primes there is an abundance of non-isomorphic supersingular elliptic curves (unlike when $p = 3$). Part of Behrens' proof of (3.4) relied on explicit computations of $\pi_*(L_{K(2)}Q(2))$ and $\pi_*(L_{K(2)}V(1))$ ([20]), however because of the abundance of non-isomorphic supersingular elliptic curves at primes $p > 5$ makes it impossible to give a *uniform* computation at all large primes. Thus, a more conceptual argument is required.

3.2. **Motivic homotopy theory.** Voevodsky and others have constructed a category of motivic spectra, which is built from schemes over a base field rather than topological spaces. It turns out that this category has a lot of similar properties as the classical stable homotopy category. In particular, in [14],[15], motivic analogs of the Morava K-theories $K(n)$ are constructed. In recent work with JD Quigley, I have been studying v_1-periodicity in the motivic setting. In particular, we have carried out Mahowald's program on bo-resolutions in the $\mathbb{C}$-motivic category, using the motivic analog of bo, denoted as kq ([1]). We have shown the following.

Theorem 3.6 (Culver-Quigley)**.** *In the kq-based ASS, the 0- and 1-lines of the E_∞-page can be explicitly computed and display a pattern similar to the image of J. Moreover, the E_∞-page has a vanishing line of slope 1/5, which implies all the v_1-periodic elements are detected in the 0- and 1-line.*

In motivic homotopy theory, the first Hopf invariant one element η is non-nilpotent, and an important question in the subject is to compute the η-inverted homotopy of the motivic sphere spectrum. Over $\mathbb{C}$, this has been computed by Andrews-Miller ([2]). There is ongoing research to compute $\pi_{**}\eta^{-1}S$ over $\mathbb{R}$. Quigley and I have shown the following:

Theorem 3.7 (Culver-Quigley)**.** *We can recover $\eta^{-1}\pi_{**}S^{0,0}$ over $\mathbb{C}$ from the 0- and 1-line of the E_∞-page of the kq-based ASS.*

This suggests that the kq-based ASS is a useful tool for studying $\eta^{-1}\pi_{**}S^{0,0}$. Quigley and I have begun to consider the analogous situation over other base fields. The spectra KQ and kq are defined over any field F whose characteristic is not 2, and hence the kq-resolution can be studied over such bases. We are especially interested in the case $F = \mathbb{Q}, \mathbb{R}, \mathbb{Q}_p$, and $\mathbb{F}_q$ for q odd. More specifically, we are interested in the following question.

Question 3.8. Can we use the kq-based ASS over $F = \mathbb{F}_q, \mathbb{Q}_p, \mathbb{R}$ to compute the v_1-periodic elements? Can we use it to compute some new 2-torsion in the motivic homotopy groups of spheres? Can we use it to compute $\eta^{-1}\pi_{**}S^0$, the stable homotopy groups of spheres where we have inverted η.

The first natural starting point would be to generalize Mahowald's splitting result. This leads to the following goal.

Goal 4. *Construct motivic Brown-Gitler spectra over general base fields and derive a motivic version of Mahowald's splitting for $kq \wedge kq$.*

There are several strategies for accomplishing this. Currently, Quigley and I are investigating how to generalize the original construction due to

Brown and Gitler to the motivic setting. This relies on knowing what the motivic Λ-algebra is. In joint work with William Balderrama and J.D. Quigley, we have given an explicit presentation of such an algebra.

Theorem 3.9 (Balderrama-C-Quigley)**.** *There is an explicit subcomplex of the cobar complex for the dual motivic Steenrod algebra, Λ^{mot}, whose cohomology is $\mathrm{Ext}_A(\mathbb{M}_p, \mathbb{M}_p)$, where $\mathbb{M}_p$ denotes the motivic cohomology of the base field with mod p coefficients.*

Classically, the Λ-algebra has been used to study low lines in the Adams spectral sequence, and notably leads to a quick proof of the Hopf invariant 1 problem (cf. [30]). We are currently exploring the analogous calculation in the motivic setting over various fields. We hope that this might settle the motivic Hopf invariant 1 problem.

The Λ-algebra also plays an interesting role in classical unstable homotopy theory. There are certain subalgebras $\Lambda(n)$ which are the E_2-term of an *unstable Adams spectral sequence* converging to the (unstable) homotopy groups $\pi_* S^n$. I have become interested in the following question.

Question 3.10. Can we produce a motivic unstable Adams spectral sequence which calculates $\pi_{**}S^{p,q}$ and can we identify its E_2-term with a certain subalgebra $\Lambda^{mot}(p,q)$ of Λ^{mot}?

In [31], the authors produced a simplicial EHP sequence, generalizing the important EHP sequence from classical homotopy theory. Classically, there is an algebraic analogue of this sequence involving the $\Lambda(n)$'s. At the moment, a "$\mathbb{P}^1$ EHP sequence" is unknown. A positive answer to the above question, or even just an identification of the $\Lambda^{mot}(p,q)$ might give us some clues as to what a $\mathbb{P}^1$-EHP sequence would be.

Furthermore, in establishing an unstable Adams spectral sequence classically, one has to consider the category of unstable modules over the Steenrod algebra A. An important fact that allows one to build the unstable Adams spectral sequence is that the mod 2 cohomology of the Eilenberg-MacLane spaces $K(\mathbb{Z}/2, n)$ gives the free unstable A-module generated by an element in degree n. This is accomplished using the Serre spectral sequence and Kudo's transgression theorem: this conditions under which $d_r(\mathrm{Sq}^n x) = \mathrm{Sq}^n d_r(x)$ in the Serre spectral sequence. For a long time, there wasn't a known analogue of the Serre spectral sequence in motivic homotopy theory, but recently one has been constructed in [3]. This leads to the following.

Question 3.11. Is there an analogue of Kudo's transgression theorem for the motivic Serre spectral sequence in [3]? Can this be used to compute the motivic cohomology of the motivic Eilenberg-MacLane spaces $K(\mathbb{Z}/2; p, q)$?

With Hana Kong and J.D. Quigley ([16]), we have produced an algebraic analogue of the motivic effective slice spectral sequence introduced by Voevodsky in [29]. Using this spectral sequence, we were able to prove a correspondence between the differentials in the effective slice spectral sequence and the ρ-BSS for certain motivic spectra. Using Betti realization, we were also able to reproduce differentials found by Hill-Hopkins-Ravenel ([22]). In our work, our methods only applied to the truncated motivic Brown-Peterson spectra $\mathrm{BP}\langle n\rangle$. I've begun thinking about the following question.

Question 3.12. Is there an analogue of the slice filtration in the derived category of comodules over the dual motivic Steenrod algebra? If so, can we calculate the algebraic slices of the homologies of spectra such as $\mathrm{BP}\langle n\rangle$ and recover the results of [16]? Can we identify the algebraic slices of the sphere? If the answer is yes, then can we make a precise relationship between differentials in the algebraic slice spectral sequence and differentials in the slice spectral sequence.

References

[1] Alexey Ananyevskiy, Oliver Röndigs, and Paul Arne Østvær, *On very effective hermitian K-theory*.

[2] Michael Andrews and Haynes Miller, *Inverting the Hopf map*, Journal of Topology **10** (2017), no. 4, 1145–1168, available at `https://londmathsoc.onlinelibrary.wiley.com/doi/pdf/10.1112/topo.12034`.

[3] Aravind Asok, Frédéric Déglise, and Jan Nagel, *The homotopy leray spectral sequence* (201812), available at `1812.09574`.

[4] Agnes Beaudry, Mark Behrens, Prasit Bhattacharya, Dominic Culver, and Zhouli Xu, *On the E2-term of the* bo*-Adams spectral sequence*.

[5] Agnès Beaudry, Mark Behrens, Prasit Bhattacharya, Dominic Culver, and Zhouli Xu, *On the tmf resolution for Z*. available at https://arxiv.org/pdf/1909.13379.pdf.

[6] Mark Behrens, *A modular description of the $K(2)$-local sphere at the prime 3*, Topology **45** (2006), 343–402.

[7] ______, *Buildings, elliptic curves, and the $K(2)$-local sphere*, American Journal of Mathematics **129** (2007), 1513–1563.

[8] ______, *Congruences between modular forms given by the divided beta family in homotopy theory*, Geometry and Topology **13** (2009), no. 319-357.

[9] Mark Behrens, Agnès Beaudry, Prasit Bhattacharya, Dominic Culver, and Zhouli Xu, *On the E_2-term of the* bo*-Adams spectral sequence*. available at https://arxiv.org/pdf/1702.00230.pdf.

[10] Mark Behrens and Tyler Lawson, *Topological automorphic forms*, Mem. Amer. Math. Soc. **204** (2010), no. 958, xxiv+141. MR2640996

[11] Mark Behrens, Kyle Ormsby, Nathaniel Stapleton, and Vesna Stojanoska, *On the ring of* tmf *cooperations at the prime 2*, Journal of Topology **12** (2019), 577–657.

[12] Prasit Bhattacharya and Philip Egger, *A class of 2-local finite spectra which admit a v_2^1-self-map*. available at https://arxiv.org/abs/1608.06250.

[13] ———, *Towards the $K(2)$-local homotopy groups of Z*. available at https://arxiv.org/abs/1706.06170.
[14] Simone Borghesi, *Algebraic Morava K-theories*, Inventiones mathematicae **151** (2003), no. 2, 381–413.
[15] ———, *Algebraic Morava K-theory spectra over perfect fields*, Annali della Scuola Normale Superiore di Pisa - Classe di Scienze **Ser. 5, 8** (2009), no. 2, 369–390 (en). MR2548251
[16] D. Culver, Hana Jia Kong, and J. Quigley, *Algebraic slice spectral sequences*, 2020. preprint.
[17] Dominic Culver, *The $BP\langle 2\rangle$-cooperations at odd primes*. to appear in the Journal of Pure and Applied Algebra.
[18] ———, *On $BP\langle 2\rangle \wedge BP\langle 2\rangle$-cooperations*, Algebr. Geom. Topol. **19** (2019), 807–862.
[19] Dominic Leon Culver and Paul VanKoughnett, *On the $K(1)$-local homotopy of* $\mathrm{tmf} \wedge \mathrm{tmf}$.
[20] Paul Goerss, Hans-Werner Henn, and Mark Mahowald, *The Homotopy of $L_2V(1)$ for the Prime 3*, Categorical Decomposition Techniques in Algebraic Topology (2003), 125–151.
[21] Paul Goerss, Hans-Werner Henn, Mark Mahowald, and Charles Rezk, *A resolution of the K(2)-local sphere at the prime 3*, Annals of Mathematics **162** (2005Sep), no. 2, 777–822.
[22] M. A. Hill, M. J. Hopkins, and D. C. Ravenel, *On the nonexistence of elements of Kervaire invariant one*, Ann. of Math. (2) **184** (2016), no. 1, 1–262. MR3505179
[23] Michael Hill and Tyler Lawson, *Topological modular forms with level structure*, Inventiones Mathematicae **203** (2016), no. 2.
[24] Mark Mahowald, *bo-resolutions.*, Pacific J. Math. **92** (1981), no. 2, 365–383.
[25] Mark Mahowald and Wolfgang Lellmann, *The bo-Adams spectral sequence*, Transactions of the American Mathematical Society **300** (1987), no. 2, 593–623.
[26] Mark Mahowald and Charles Rezk, *Topological modular forms of level 3*, Pure Appl. Math. Q. **5** (2009), no. 2, 853–872.
[27] Haynes Miller, Douglas Ravenel, and Stephen Wilson, *Perodic Phenomena in the Adams-Novikov Spectral Sequence*, Annals of Mathematics **106** (1977), no. 3, 469–516.
[28] Douglas C. Ravenel, *Localization with respect to certain periodic homology theories*, American Journal of Mathematics **106** (1984), no. 2, 351–414.
[29] Vladimir Voevodsky, *Open problems in the motivic stable homotopy theory. i; motives, polylogarithms and hodge theory, part i (irvine, ca, 1998)*, Int. Press Lect. Ser., vol. 3, Int. Press, Somerville, MA, 2002.
[30] John S. P. Wang, *On the cohomology of the mod-2 steenrod algebra and the non-existence of elements of hopf invariant one*, Illinois Journal of Mathematics **11** (1967sep), no. 3, 480–490.
[31] Kirsten Wickelgren and Ben Williams, *The simplicial EHP sequence in $\mathbb{A}^1$–algebraic topology*, Geometry & Topology **23** (2019Jun), no. 4, 1691–1777.

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