# THE ADAMS SPECTRAL SEQUENCE FOR 3-LOCAL tmf

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ABSTRACT. The purpose of this short article is to record the computation for 3-local tmf via the Adams spectral sequence.

## CONTENTS

## 1. INTRODUCTION

In this paper, we will carry out a computation of the homotopy groups of tmf. The homotopy groups of tmf have been known for quite awhile now. For example, the computation of $\pi_*$tmf was explicitly written up in [1], though it was known even earlier to Hopkins and his collaborators (cf. [5]). The usual approach to calculating $\pi_*$tmf is via the Adams-Novikov spectral sequence (also referred to as the descent spectral sequence in this context). One advantage of this approach is that the Adams-Novikov $E_2$-term can be computed using the theory of elliptic curves.

However, there are occasions where one wants to know the Adams spectral sequence for computing the homotopy groups of a spectrum. That is,

one may want to know the Adams $E_2$-term, all differentials, and all hidden extensions. The purpose of this paper is to record the Adams spectral sequence for 3-local topological modular forms.

The reader may wonder why one would be interested in knowing the Adams spectral sequence for tmf, especially since the Adams-Novikov spectral sequence is far more efficient for this purpose. We should mention that the analogous calculation at the prime 2 is being carried out by Rognes and Bruner. It is the author's understanding that their interest in that spectral sequence stemmed from their work on the topological Hochschild homology of tmf. We speculate knowing the Adams spectral sequence at the prime 3 might be useful for similar reasons.

1.1. **Outline of the paper.** Recall that the Adams spectral sequence is a convergent spectral sequence of the form

$$\mathrm{Ext}_{A_*}(\mathbb{F}_3, H_*\mathrm{tmf}) \implies \pi_*\mathrm{tmf}_3^\wedge.$$

Thus, a necessary input is $H_*\mathrm{tmf}$. This was determined, for example, in [8], where Rezk shows there is a short exact sequence of comodules

$$0 \to \Sigma^8 \mathscr{B} \to H_*\mathrm{tmf} \to \mathscr{B} \to 0$$

where $\mathscr{B}$ is a certain subalgebra of the dual Steenrod algebra $A_*$. This is the starting point of our calculation. We view this short exact sequence as giving a multiplicative filtration of $H_*\mathrm{tmf}$ by comodules, yielding an algebraic spectral sequence

$$E_1^{*,*,*} = \mathrm{Ext}_{A_*}(E_0 H_*\mathrm{tmf}) \cong \mathrm{Ext}_{A_*}(\mathscr{B}) \otimes E(b_4) \implies \mathrm{Ext}_{A_*}(H_*\mathrm{tmf}).$$

In §2 we recall these details and establish a change-of-rings formula for $\mathrm{Ext}_{A_*}(\mathscr{B})$. In §3 we use a Cartan-Eilenberg spectral sequence to compute $\mathrm{Ext}_{A_*}(\mathscr{B})$. An expert in these affairs can safely ignore this section; we include it for those who are trying to learn more computational methods in homotopy theory. In §4, we determine the Adams $E_2$-term. Finally, in §5, we establish the Adams differentials and derive $\pi_*\mathrm{tmf}$.

*Conventions.* In this article, we will implicitly assume that all spectra are 3-complete. Thus tmf refers, from here on out, to the 3-completion of the spectrum of topological modular forms. We will always denote mod 3 Eilenberg-MacLane spectrum by $H$. Given a Hopf algebra $\Gamma$ and a comodule $C$ over $\Gamma$, we will abbreviate $\mathrm{Ext}_\Gamma(\mathbb{F}_3, C)$ by $\mathrm{Ext}_\Gamma(C)$. In the case when $\Gamma = A_*$, we will write $\mathrm{Ext}(C)$. We will always employ Adams indexing unless specifically stated otherwise. We use the symbol $\cong$ to denote isomorphism and $\simeq$ to denote (weak) homotopy equivalence. We let $\zeta_n$ denote $\chi\xi_n$ and $\overline{\tau}_n$ denote $\chi\tau_n$ in the dual Steenrod algebra. All spectral sequences use Adams indexing.

## 2. THE MOD 3 HOMOLOGY OF tmf

In this section we recall necessary facts about the mod 3 homology of tmf. In [8], it is shown that, as an algebra, the homology of tmf is given by

$$H_* \mathrm{tmf} \cong E(b_4) \otimes \mathscr{B}$$

where $|b_4| = 8$ and

$$\mathscr{B} := \mathbb{F}_3[\zeta_1^3, \zeta_n \mid n \geq 2] \otimes E(\overline{\tau}_n \mid n \geq 3).$$

One can easily check that $\mathscr{B}$ is a comodule algebra over $A_*$. Furthermore, Rezk shows that there is nontrivial extension of comodules

$$(2.1) \qquad 0 \longrightarrow \Sigma^8 \mathscr{B} \longrightarrow H_* \mathrm{tmf} \longrightarrow \mathscr{B} \rightarrow 0.$$

Applying $\mathrm{Ext}_{A_*}(-)$ to this short exact sequence of comodules yields a long exact sequence in Ext. We regard this as a convergent spectral sequence

$$\mathrm{Ext}(\Sigma^8 \mathscr{B}) \oplus \mathrm{Ext}(\mathscr{B}) \implies \mathrm{Ext}(H_* \mathrm{tmf}).$$

The fact that 2.1 is a nontrivial extension implies that this spectral sequence does not immediately collapse. Determining the differentials in this spectral sequence is the subject of section 4. Thus, it is apparent that we need to compute the Ext groups of $\mathscr{B}$. We will simplify this by establishing a change-of-rings formula.

**Definition 2.2.** Let $\Gamma$ be the Hopf algebra

$$\Gamma := \mathbb{F}_3[\zeta_1]/(\zeta_1^3) \otimes E(\overline{\tau}_0, \overline{\tau}_1, \overline{\tau}_2)$$

with the induced coproduct from the dual Steenrod algebra.

**Example 2.3.** In the dual Steenrod algebra, the coproduct on $\overline{\tau}_2$ is given by

$$\psi(\overline{\tau}_2) = \overline{\tau}_2 \otimes 1 + \overline{\tau}_0 \otimes \zeta_2 + \overline{\tau}_1 \otimes \zeta_1^3 + 1 \otimes \overline{\tau}_2.$$

Thus, in $\Gamma$, $\overline{\tau}_2$ is a Hopf algebra primitive. On the other hand,

$$\psi(\overline{\tau}_1) = \overline{\tau}_1 \otimes 1 + \overline{\tau}_0 \otimes \zeta_1 + 1 \otimes \overline{\tau}_1.$$

Thus this Hopf algebra is not primitively generated..

The proof of the following proposition is standard.

**Proposition 2.4.** *There is an isomorphism*

$$\mathscr{B} \cong A_* \,\square_\Gamma\, \mathbb{F}_3.$$

**Corollary 2.5.** *There is a change-of-rings isomorphism*

$$\mathrm{Ext}(\mathscr{B}) \cong \mathrm{Ext}_\Gamma(\mathbb{F}_3).$$

Thus we must compute the cohomology of the Hopf algebra $\Gamma$. This is done in the next section.

## 3. COMPUTING THE COHOMOLOGY OF $\Gamma$

In the last section we showed that the Ext groups of $\mathscr{B}$ are $\mathrm{Ext}_\Gamma(\mathbb{F}_3)$. Since $\Gamma$ is a finite Hopf algebra, there is hope of computing its cohomology. Recall that $A(1)$ is the subalgebra of the Steenrod algebra generated by the Bockstein $\beta$ and $\mathscr{P}^1$. Its dual is

$$A(1)_* \cong \mathbb{F}_3[\zeta_1]/(\zeta_1^3) \otimes E(\overline{\tau}_0, \overline{\tau}_1).$$

In paticular, $A(1)_*$ is a sub-Hopf algebra of $\Gamma$. The following proposition relies on the first appendix in [7]. We recommend the reader look at Definition A1.1.15.

**Lemma 3.1.** *The following*

$$A(1)_* \to \Gamma \to E(\overline{\tau}_2)$$

*is a cocentral extension of Hopf algebras over $\mathbb{F}_3$.*

When one has a cocentral extension of Hopf algebras, one can consider the Cartan-Eilenberg spectral sequence. In general, if

$$(D, \Phi) \to (A, \Gamma) \to (A, \Sigma)$$

is an extension of Hopf algebroids, $N$ is a left comodule over $\Gamma$, then there is a natural convergent spectral sequence of the form

$$E_2^{f,s,t} = \mathrm{Ext}_\Phi^{f,t}(D, \mathrm{Ext}_\Sigma^s(A, N)) \implies \mathrm{Ext}_\Gamma^{f+s,t}(A, N).$$

Here, $f$ denotes the filtration degree and the differentials are of the form

$$d_r : E_r^{f,s,t} \to E_r^{f+r,s-r+1,t}.$$

See A1.3.14 and A1.3.15 of [7] for details on this spectral sequence. Applied to our extension of Hopf algebras, this spectral sequence takes on the form

$$(3.2) \qquad E_2^{f,s,t} = \mathrm{Ext}_{A(1)_*}^{f,t}(\mathbb{F}_3, \mathrm{Ext}_{E(\overline{\tau}_2)}^s(\mathbb{F}_3, \mathbb{F}_3)) \implies \mathrm{Ext}_\Gamma^{f+s,t}(\mathbb{F}_3).$$

Since $E(\overline{\tau}_2)$ is a primitively generated exterior Hopf algebra, we have that

$$\mathrm{Ext}_{E(\overline{\tau}_2)}(\mathbb{F}_3) \cong \mathbb{F}_3[v_2]$$

where the $(s, t)$-bidegree of $v_2$ is $(1, 17)$. Note that since $\mathbb{F}_3$ is a comodule algebra, the Cartan-Eilenberg spectral sequence is multiplicative.

In order to determine the $E_2$-page of this spectral sequence, we need to understand the coaction of $A(1)_*$ on $\mathbb{F}_3[v_2]$. As $\mathbb{F}_3[v_2]$ is a comodule algebra over $A(1)_*$, it is enough to determine the coaction on $v_2$.

**Lemma 3.3.** *Under the canonical $A(1)_*$-coaction on $\mathrm{Ext}_{E(\bar{\tau}_2)}(\mathbb{F}_3)$, the element $v_2$ is a comodule primitive.*

*Proof.* Observe that the largest degree element of $A(1)_*$ is $\zeta_1^2 \bar{\tau}_0 \bar{\tau}_1$, which has degree 13. Since

$$\mathrm{Ext}_{E(\bar{\tau}_2)}(\mathbb{F}_3) \cong \mathbb{F}_3[v_2], \quad |v_2| = (1, 17)$$

the coaction on $v_2$ must be $1 \otimes v_2$ for degree reasons. $\square$

**Corollary 3.4.** *The $E_2$-term of the Cartan-Eilenberg SS is given by*

$$E_2 \cong \mathrm{Ext}_{A(1)_*}(\mathbb{F}_3) \otimes \mathbb{F}_3[v_2].$$

*and the filtration degree of $v_2$ is 0.*

Thus we must determine the cohomology of $A(1)_*$. An elementary exercise in using the May spectral sequence (cf. [7]) will yield
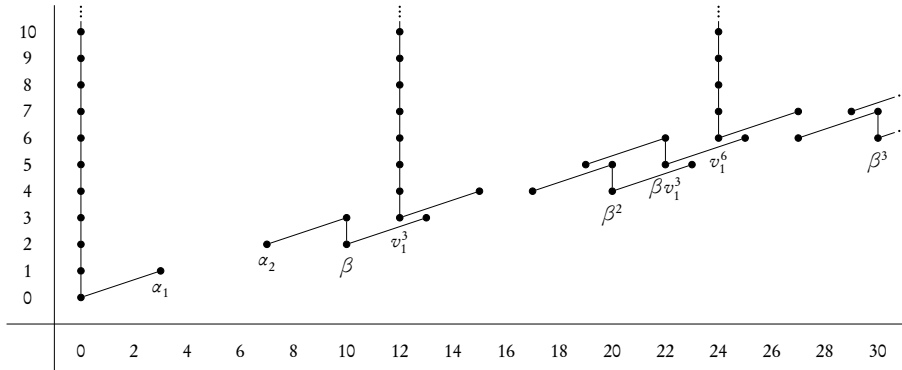
**Proposition 3.5.** *The algebra $\mathrm{Ext}_{A(1)_*}(\mathbb{F}_3)$ is given by*

$$\mathbb{F}_3[v_0, v_1^3, \beta] \otimes E(\alpha_1, \alpha_2)/(v_0 \alpha_1, v_0 \alpha_2, \alpha_1 \alpha_2 - v_0 \beta)$$

*where the $(s, t)$-bidegrees of the generators are given by*

- $|\alpha_1| = (1, 4)$
- $|\beta| = (2, 12),$
- $|\alpha_2| = (2, 9)$

A chart for this Ext group is given below.

From now on, we will write $c_6$ for $v_1^3$. For degree reasons, this spectral sequence collapses. Thus we can infer the following.

**Corollary 3.6.** *The cohomology of the Hopf algebra $\Gamma$ is given by*

$$\mathbb{F}_3[v_0, c_6, v_2, \beta] \otimes E(\alpha_1, \alpha_2)/(v_0\alpha_1, v_0\alpha_2, h_0\alpha_2 - v_0\beta).$$

## 4. Determining the Adams $E_2$-term

In this section we will determine the $E_2$-term of the Adams spectral sequence converging to $\pi_*\mathrm{tmf}$. The way this will be achieved is by applying the functor $\mathrm{Ext}(-)$ to the short exact sequence (2.1) to obtain a long exact sequence. Regarding this as a spectral sequence provides us with

$$E_1 = \mathrm{Ext}(\Sigma^8\mathscr{B}) \oplus \mathrm{Ext}(\mathscr{B}) \implies \mathrm{Ext}(\mathrm{tmf}).$$

For the purposes of this paper, we will refer to this spectral sequence as the *algebraic spectral sequence*.

4.1. **algebraic differentials.** The results of the previous section show that, as an algebra, the $E_1$-term of the algebraic spectral sequence is given by

$$E_1 = \mathrm{Ext}_\Gamma(\mathbb{F}_3) \otimes E(b_4).$$

Since this is a spectral sequence derived from a long exact sequence, there can only be $d_1$-differentials. Note that in Adams indexing, the $d_1$-differential looks like an Adams $d_1$-differential. Results of the previous sections imply the following.

**Lemma 4.1.** *The short exact sequence (2.1) gives a multiplicative filtration of $H_*\mathrm{tmf}$ by $A_*$-comodules. Thus the algebraic spectral sequence is a multiplicative spectral sequence. Moreover, there is an isomorphism of $A_*$-comodule algebras*

$$E_0 H_*\mathrm{tmf} \cong \mathscr{B} \otimes E(b_4)$$

*with $b_4$ a comodule primitive and has filtration degree 1. Thus*

$$E_1 \cong \mathrm{Ext}_\Gamma(\mathbb{F}_3) \otimes E(b_4)$$

*as a graded ring.*

Below is a chart for the $E_1$-page of the algebraic spectral sequence. The classes in blue are those in the coset for $b_4$ in the $E_1$-page. In other words, they have filtration 1 with respect to the filtration on $H_*\mathrm{tmf}$. Moreover, because the filtration on $H_*\mathrm{tmf}$ only changes when going from the $0$th filtration to the first filtration, this spectral sequence only has a $d^1$-differential. This differential is just the connecting homomorphism arising from the long exact sequence. In particular, this means that all differentials originate from a black class and target a blue class.

label classes
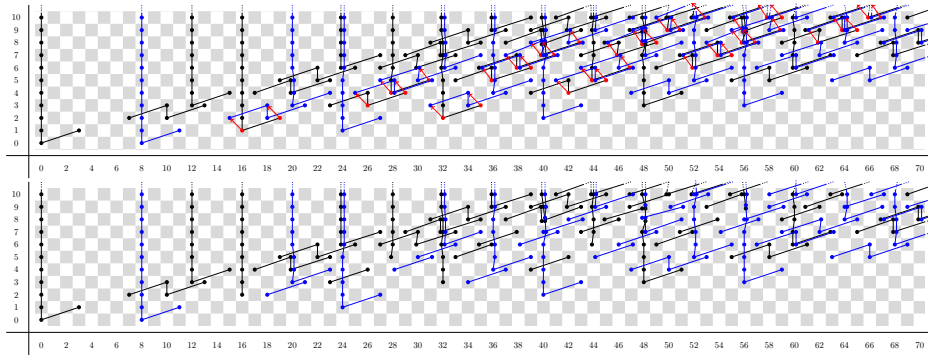
FIGURE 4.1. The $E_1$ and $E_2$-page of the algebraic spectral sequence.

We will now determine the differentials in the algebraic spectral sequence. First, we make the following simple observation.

**Lemma 4.2.** *For degree reasons, the classes $\alpha_1, \alpha_2, b_4, \beta$, and $c_6$ are permanent cycles of the algebraic spectral sequence.*

This observation and the multiplicativity of the spectral sequence eliminate many possible differentials.

From the known computation of $\pi_* \mathrm{tmf}$ (cf. [1]), we see that $\pi_{15}\mathrm{tmf} = 0$. In the $E_1$-term of the algebraic spectral sequence, there are two classes in stem 15; the class $b_4\alpha_2$ and the class $c_6\alpha_1$. Both of these classes must die, but the multiplicativity of the spectral sequence implies that a differential must originate on $v_2$. The structure of the spectral sequence thus forces the following differential[1]

$$d_1(v_2) \doteq b_4\alpha_2.$$

Multiplicativity of the spectral sequence and the previous lemma yields the following result.

**Proposition 4.3.** *The algebraic spectral sequence has the following $d_1$-differentials*

$$d_1(v_2^i v_0^j c_6^k \beta^\ell \alpha_1^\epsilon) \doteq v_2^{i-1} v_0^j c_6^k \beta^\ell \alpha_1^\epsilon b_4 \alpha_2 \qquad i \not\equiv 0 \mod 3$$

*for natural numbers $i, j, k, \ell$ and $\epsilon \in \{0, 1\}$. There are no other differentials.*

Consequently, this spectral sequence is periodic on the element $v_2^3$.

**Remark 4.4.** It would be nice to have an argument for this differential from first principles, but the author is not currently aware of one. He suspects this implies the existence of an interesting coproduct on $H_*\mathrm{tmf}$.

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[1] The class $c_6\alpha_1$ will be dealt with by an Adams differential.

4.2. **algebraic $E_\infty$-term.** We will now describe a few patterns which make up the $E_\infty$-page of the algebraic spectral sequence. We will describe these patterns as certain modules over $\mathrm{Ext}_{A(1)_*}(\mathbb{F}_3)$ along with the monomial of the algebraic spectral sequence which generates it.

(Pattern 1) Since $v_2^3$ is a permanent cycle, we have the free $\mathrm{Ext}_{A(1)_*}(\mathbb{F}_3)$ modules on the powers of $v_2^3$ and the $v_2^3$-multiples of $v_2^2 b_4$, i.e. for all $j \geq 0$,

$$\mathrm{Ext}_{A(1)_*}(\mathbb{F}_3) \cdot \{v_2^{3j}, v_2^{3j-1} b_4\};$$

(Pattern 2) For $j \equiv 0, 1 \mod 3$, we have the patterns

$$\mathrm{Ext}_{A(1)_*}(\mathbb{F}_3)/(\alpha_2) \cdot \{v_2^j b_4\}$$

(Pattern 3) For $j \not\equiv 0 \mod 3$, we have the following patterns

$$\mathrm{Ext}_{A(1)_*}(\mathbb{F}_3)/(\alpha_1, \alpha_2, \beta) \cdot \{v_0 v_2^j\} \oplus \mathrm{Ext}_{A(1)_*}(\mathbb{F}_3)/(\alpha_2, v_0) \cdot \{v_2^j \alpha_2\}.$$

The way we obtained these patterns was by noting that, as a module over $\mathrm{Ext}_{A(1)_*}(\mathbb{F}_3)$, the $E_1$-page of the algebraic spectral sequence is freely generated by the monomials $v_2^j b_4^\epsilon$. In other words, we have an isomorphism of $\mathrm{Ext}_{A(1)_*}(\mathbb{F}_3)$-modules,

$$E_1 \cong \bigoplus_{j \geq 0, \, \varepsilon \in \{0,1\}} \mathrm{Ext}_{A(1)_*}(\mathbb{F}_3) \cdot \{v_2^j b_4^\varepsilon\}.$$

The three patterns arise by partitioning the free modules $\mathrm{Ext}_{A(1)_*}(\mathbb{F}_3) \cdot \{v_2^j b_4^\epsilon\}$ into those which neither receive nor support any differentials (Pattern 1), receive differentials (Pattern 2), or support differentials (Pattern 3).

**Remark 4.5.** In later parts of this paper we will need to refer to these patterns. We will refer to them as *patterns of type $j$ on generator $x$*. So for example, if we look at the pattern on the Adams $E_2$-term generated by the monomial $v_2^4 b_4$, then we will call this a pattern of type 2 on generator $v_2^4 b_4$. For patterns of the third type, we will call these patterns of type 3 on generator $v_2^j$. This may be confusing since $v_2^j$ does not survive the algebraic spectral sequence unless $j$ is a multiple of 3. This terminology stems from the fact that this pattern is the residual piece of a free $\mathrm{Ext}_{A(1)_*}(\mathbb{F}_3)$-module generated by $v_2^j$.

4.3. **algebraic hidden extensions.** As with any spectral sequence, there is the possibility of extension problems. We will show that there is a crucial hidden $v_0$-extension which will play an important role in the next section. Namely, ....

[Margin notes: Label classes | flesh out this section | reference actual differential where this comes up.]
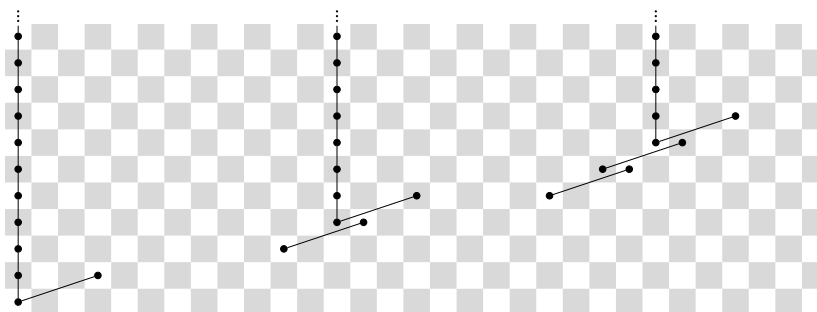
FIGURE 4.2. A depiction of pattern 2
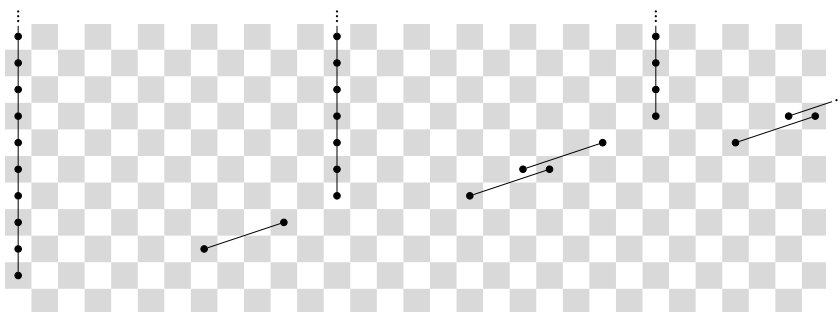


FIGURE 4.3. A depiction of pattern 3

**Proposition 4.6.** *In the algebraic spectral sequence, there is a hidden multiplicative extension*

$$v_0 \cdot (v_2^2 \alpha_2) \doteq v_2 b_4 c_6 \alpha_1,$$

*consequently for every natural number $j$ and $k$, we have the hidden extension*

$$v_0 \cdot v_2^2 c_6^j \beta^k \alpha_2 \doteq v_2 c_6^{j+1} \beta^k b_4 \alpha_2.$$

**Remark 4.7.** One might protest that this is not a hidden extension since $v_2 b_4 c_6 \alpha_1$ is an element in the correct Adams filtration. However, from the perspective of the algebraic spectral sequence, $v_2^2 \alpha_2$ has filtration 0 and $v_2 b_4 c_6 \alpha_2$ has filtration 1. Since $v_0$ has filtration 0, this is in fact a hidden extension.

Before proving this, we will need to show the following.

I also need to show the hidden extension $v_0 \cdot (v_2 \alpha_2) \doteq b_4 c_6 \alpha_1$

**Lemma 4.8.** *In* $\mathrm{Ext}_{A(1)_*}(\mathbb{F}_3)$, *there is the Massey product*

$$c_6\alpha_1 \in \langle v_0, \alpha_2, \alpha_2 \rangle.$$

*and there is zero indeterminacy.*

To prove this, we need to recall May's Convergence theorem. The proof of this fact can be found as Theorem 4.1 of [6], but we will only be interested in the case of a three-fold Massey product. The variant we will use is Theorem 2.2.2 of [4].

**Theorem 4.9** (May's Convergence Theorem). *Let* $\alpha_0, \alpha_1, \alpha_2$ *be elements of* Ext *such that the Massey product* $\langle \alpha_0, \alpha_1, \alpha_2 \rangle$ *is defined. For each* $i$, *let* $a_i$ *be a permanent cycle on the May* $E_r$-*page which detects* $\alpha_i$. *Suppose further that*

  (1) *The Massey product* $\langle a_0, a_1, a_2 \rangle$ *is defined on the* $E_{r+1}$-*page: there are* $a_{01}$ *and* $a_{12}$ *such that* $d_r(a_{01}) = \overline{a_0}a_1$ *and* $d_r(a_{12}) = \overline{a_1}a_2$.
  (2) *If* $(m, s, t)$ *is the tri-degree of either* $a_{01}$ *or* $a_{12}$, *and for for any* $m' \geq m$ *and* $q$ *such that* $m' - q < m - r$, *the differential*

$$d_q : E_q^{m',s,t} \to E_q^{m'-q+1,s+1,t}$$

  *is zero.*

*Then the element* $\overline{a_{01}}a_3 + \overline{a_0}a_{12}$ *is a permanent cycle and detects an element of* $\langle \alpha_0, \alpha_1, \alpha_2 \rangle$.

**Remark 4.10.** The second condition in May's Convergence Theorem is often expressed by saying there are no "crossing differentials."

We will use May's convergence theorem to give a proof for Lemma 4.8. Before doing so, it is helpful to recall the May spectral sequence for $\mathrm{Ext}_{A(1)_*}(\mathbb{F}_3)$. This spectral sequence is obtained by putting a filtration on

$$A(1)_* = P(\zeta_1)/(\zeta_1^3) \otimes E(\overline{\tau}_0, \overline{\tau}_1),$$

is obtained by giving the generators of $A(1)_*$ the following *May filtration*

  • $MF(\overline{\tau}_0) = MF(\zeta_1) = 1$,
  • $MF(\overline{\tau}_1) = 3$.

This produces a filtration on the cobar complex for $A(1)_*$, resulting in the May spectral sequence

$$E_1^{m,s,t} \implies \mathrm{Ext}_{A(1)_*}^{s,t}(\mathbb{F}_3),$$

with the following $E_1$-term,

$$E_1 \cong E(\alpha_1) \otimes P(v_0, v_1, \beta).$$

The coproduct on $A(1)_*$ gives the $d_1$-differential

$$d_1(v_1) = v_0\alpha_1.$$

The rest of the $d_1$-differentials are propagated from this one and the multiplicativity of the May spectral sequence. This makes the class $\alpha_2 := v_1\alpha_1$ a non-zero permanent cycle. One easily shows that

**Lemma 4.11.** *In* $\mathrm{Ext}_{A(1)_*}(\mathbb{F}_3)$, *there is the Massey product*

$$\alpha_2 = \langle v_0, \alpha_1, \alpha_1 \rangle.$$

*Proof of Lemma 4.8.* Since $\alpha_1$ is exterior on the May $E_1$-page, we get the following defining system $d_1(v_1^2) = \overline{v_0}\alpha_2$ and $d_1(0) = \alpha_2^2$ for the Massey product $\langle v_0, \alpha_2, \alpha_2 \rangle$.

Since $\alpha_2^2 = 0$ and $v_0\alpha_2 = 0$ in $\mathrm{Ext}_{A(1)_*}(\mathbb{F}_3)$, the Massey product $\langle v_0, \alpha_2, \alpha_2 \rangle$ is defined in $\mathrm{Ext}_{A(1)_*}(\mathbb{F}_3)$. Furthermore, there is no indeterminacy of this Massey product. So we just need to check the second condition of May's Convergence Theorem, that there are no crossing differentials.. Note that $v_1^2 \in E_1^{6,2,10}$ and that $E_2^{m,2,10} = 0$ for all $m$. So condition (2) is satisfied here. Likewise, note that $(v_1\alpha_1)^2 \in E_1^{8,4,18}$ and that $E_1^{m,3,18} = 0$ for all $m$. Thus there are no nonzero differentials to worry about. So condition (2) is always satisfied here as well. Thus we may apply May's Convergence Theorem to infer that

$$c_6\alpha_1 \in \langle v_0, \alpha_2, \alpha_2 \rangle.$$

It is easy to see that there is no indeterminacy. $\qquad\square$

Keep in mind that May's convergence theorem is actually very general (cf. the discussion preceeding [7, A1.4.10]). It applies to any spectral sequence which arises from a multiplicative filtration on a DGA. In particular, it applies to the Cartan-Eilenberg SS and the algebraic SS we have used. Since the Cartan-Eilenberg SS collapses, and since the algebraic SS only has $d_1$-differentials, the May Convergence Theorem vacuously applies to these spectral sequences. Thus, we derive the following corollary.

**Corollary 4.12.** *In* $\mathrm{Ext}_{A_*}(H_*\mathrm{tmf})$ *there is the Massey*

$$c_6\alpha_1 = \langle v_0, \alpha_2, \alpha_2 \rangle.$$

We will use this corollary to derive the hidden multiplicative extension.

*Proof of Proposition 4.6.* One can check, using the May Convergence Theorem applied to the algebraic spectral sequence, that one has the Massey product

$$v_2^2\alpha_2 \in \langle \alpha_2, \alpha_2, b_4v_2 \rangle,$$

and that this Massey product has no indeterminacy. In order to apply the May Convergence Theorem to this Massey product, we must check that it is defined on $\mathrm{Ext}_{A_*}(\mathrm{tmf})$. Note that $\alpha_2^2 = 0$ in this Ext group, since there

is no nonzero group in the 14 stem. Furthermore, since hidden extensions must always target a class in higher filtration, and since the algebraic spectral sequence only has elements in filtration $0$ and $1$, it follows that there can be no hidden extension for the product of $b_4 v_2$ and $\alpha_2$. Thus the Massey product is defined in Ext(tmf). Using the First Juggling Theorem (cf. [7, A1.4.6]), we have

$$v_0 \cdot (v_2^2 \alpha_2) = v_0 \langle \alpha_2, \alpha_2, b_4 v_2 \rangle \doteq \langle v_0, \alpha_2, \alpha_2 \rangle b_4 v_2 = c_6 \alpha_1 b_4 v_2.$$

$\square$

I didn't check the crossing differentials condition for the massey product.

### 4.4. comparison to the Adams spectral sequence in tmf-modules.

We now make a few remarks comparing the $E_2$-term of the Adams spectral sequence for tmf and the Adams spectral sequence for tmf in tmf-modules as studied by Hill ([3]). The latter is a spectral sequence

$$E_2^{s,t} \operatorname{Ext}_{A_*^{\mathrm{tmf}}}(\mathbb{F}_3) \implies \pi_{t-s} \mathrm{tmf}.$$

where

$$A_*^{\mathrm{tmf}} = \pi_*(H \wedge_{\mathrm{tmf}} H) \cong A(1)_* \otimes E(a_2)$$

where $a_2$ is in degree 9. This class has an interesting coproduct, but this does not concern us here. What is interesting for us, however, is that in order to compute this coproduct, Hill filters $A_*^{\mathrm{tmf}}$, resulting in an algebraic spectral sequence

$$E_1 = \operatorname{Ext}_{E_0 A_*^{\mathrm{tmf}}}(\mathbb{F}_3) \implies \operatorname{Ext}_{A_*^{\mathrm{tmf}}}(\mathbb{F}_3).$$

One easily derives that

$$E_1 = \operatorname{Ext}_{E_0 A_*^{\mathrm{tmf}}}(\mathbb{F}_3) \cong \operatorname{Ext}_{A(1)_*}(\mathbb{F}_3) \otimes P(\widetilde{c_4})$$

where $\widetilde{c_4}$ is the class represented in the cobar complex of $E_0 A^t m f_*$ by $[a_2]$. In particular, $\widetilde{c_4} \in E_1^{1,9}$. It turns out that this $E_1$-page is isomorphic to the $E_1$-term of our algebraic spectral sequence, but with various elements in ours in the "wrong" filtration. We provide a short dictionary relating various names in our spectral sequence to Hill's algebraic spectral sequence.

| alg'c SS | Hill's alg'c SS |
|:--------:|:---------------:|
| $b_4$ | $\widetilde{c_4}$ |
| $v_2$ | $\widetilde{c_4}^2$ |
| $b_4 v_2$ | $\widetilde{c_4}^3$ |
| $\vdots$ | $\vdots$ |

In particular, the element that Hill calls $\widetilde{c_4}^{2\ell + \varepsilon}$ corresponds to the element we call $b_4^\varepsilon v_2^\ell$. Moreover, Hill is able to derive a differential $d_1(\widetilde{c_4}) = \alpha_2$. Algebraic manipulation then yields the following differentials $d_2(\alpha_2 \widetilde{c_4}^2) = v_1^3 \beta$ and $d_2(v_0 \widetilde{c_4}^2) = v_1^3 \alpha_1$. These all correspond to various differentials

we encounter in this paper as well, but interestingly, not all of them are algebraic differentials. On the one hand, the differential $d_1(\widetilde{c}_4^2) = \widetilde{c}_4 \alpha_2$ corresponds to our algebraic differential $d_1(v_2) = b_4 \alpha_2$. But the differential $d_1(\widetilde{c}_4) = \alpha_2$ corresponds to an *Adams $d_2$-differential* $d_2(b_4) = \alpha_2$.

In particular, half of Hill's algebraic differentials are seen in our algebraic spectral sequence, but the other half arise as Adams differentials. It is this discrepancy that makes the tmf-relative Adams spectral sequence more computable as opposed to the absolute Adams spectral sequence.

## 5. ADAMS DIFFERENTIALS

In this section, we will determine the differentials in the Adams spectral sequence for tmf. Since tmf is a commutative ring spectrum, the Adams spectral sequence is multiplicative. Begin by noting that there are several classes which are permanent cycles for degree reasons.

**Lemma 5.1.** *The classes $v_0, \alpha_1, \alpha_2, \beta,$ and $c_6$ are all permanent cycles for the Adams spectral sequence. Consequently, the differentials in the Adams spectral sequence are linear over $\mathrm{Ext}_{A(1)_*}(\mathbb{F}_3)$.*

This observation is very useful for our calculation for the following reason. In the last section, we have expressed the Adams $E_2$-term as a direct sum of certain patterns which were modules over $\mathrm{Ext}_{A(1)_*}(\mathbb{F}_3)$. This observation implies that the only nonzero differentials in the Adams spectral sequence will originate on the monomials which generate these patterns. We will also make frequent use of the following facts about $\pi_*\mathrm{tmf}$.

**Theorem 5.2** (cf. [2]). *The homotopy groups of tmf are 72 periodic. Furthermore, the torsion in $\pi_*\mathrm{tmf}$ is concentrated in stems 3, 10, 13, 20, 27, 30, 37, and 40 modulo 72.*

We will begin by determining all of the length 2 differentials first.

5.1. **Adams $d_2$-differentials.** As was mentioned previously, the Adams differentials are all linear over $\mathrm{Ext}_{A(1)_*}(\mathbb{F}_3)$, which means we only have to figure which of the following families of monomials support Adams $d_2$-differentials: For any natural number $j$

(1) $v_2^{3j}$,
(2) $v_2^j b_4$,
(3) $v_0 v_2^j$, and
(4) $v_2^j \alpha_2$.

From our charts, one sees that $v_2^3$ can support a length 3 differential at minimum. Thus, $v_2^{3j}$ is a $d_2$-cycle for all $j$. Moreover, there are several

multiplicative relations on the Adams $E_2$-term which we get from the previous section. For example, we have $v_2^3 \cdot (v_2 b_4) = v_2^4 b_4$. Consequently, we have

**Proposition 5.3.** *The Adams $d_2$-differentials for* tmf *are linear over* $\mathrm{Ext}_{A(1)_*}(\mathbb{F}_3) \otimes P(v_2^3)$. *Thus, we only need to determine which of the monomials $b_4$, $v_0 v_2$, $v_2 \alpha_2$, $b_4 v_2$, $v_0 v_2^2$, $v_2^2 \alpha_2$, and $b_4 v_2^2$ support $d_2$-differentials.*

**Corollary 5.4.** *The Adams $E_3$-term is periodic on $v_2^3$.*

**Proposition 5.5.** *There is an Adams $d_2$-differential*

(5.6)                               $$d_2(b_4) \doteq \alpha_2.$$

*Proof 1.* From the known computation of $\pi_* \mathrm{tmf}$, it is seen that $\pi_7 \mathrm{tmf} = 0$. Thus, the class $\alpha_2$ must die. The only possibility is the claimed differential. ☐

*Proof 2.* Recall that in the Adams $E_2$-term for tmf, we have the Massey product

$$\beta = \langle \alpha_1, \alpha_1, \alpha_1 \rangle.$$

In the Adams spectral sequence for the sphere, there is the same Toda bracket. This is because, $\alpha_1$ and $\beta$ are the Hurewicz images of the classes of the same name in the homotopy groups of $S^0$. This can be seen by considering the induced map on $E_2$-terms

$$\mathrm{Ext}(S^0) \to \mathrm{Ext}(\mathrm{tmf})$$

and using cobar representatives. It is known from the Adams spectral sequence for the sphere that there is a $d_2$-differential whose target is $v_0 \beta$ (cf. [7]). In particular, this forces $v_0 \beta = 0$ in $\pi_* \mathrm{tmf}$. This forces the Adams differential

$$d_2(b_4 \alpha_1) \doteq v_0 \beta = \alpha_1 \alpha_2.$$

However, as $\alpha_1$ is a permanent cycle for the ASS for tmf, we must have that

$$d_2(b_4) \doteq \alpha_2$$

as stated. ☐

**Remark 5.7.** This is one of the Adams differentials which occurs as an algebraic differential in [3].

We can draw an interesting consequence from the second argument provided above (we learned this from Mike Hill and Mark Behrens).

**Corollary 5.8.** *The element $b_4 \alpha_1$ is the image of $h_1 \in \text{Ext}(S^0)$ under the map*

$$\text{Ext}(S^0) \to \text{Ext}(\text{tmf}),$$

*and consequently we have the hidden comodule extension in $H_* \text{tmf}$,*

$$\alpha(\zeta_1^3) = \zeta_1^3 \otimes 1 + \zeta_1 \otimes b_4 + 1 \otimes \zeta_1^3$$

*Proof.* In the Adams spectral sequence for the sphere, it is the class $h_1$ which supports a $d_2$-differential killing $v_0 \beta$. Naturality of the Adams spectral sequence implies that $h_1$ maps to $b_4 \alpha_1$.

In the cobar complex for $S^0$, the element $h_1$ is represented by $\zeta_1^3$. On the other hand, we have represented $b_4 \alpha_1$ in the cobar complex by $[\zeta_1] b_4$. Thus there must be an element of $H_* \text{tmf}$ which bounds the difference between $[\zeta_1^3]$ and $[\zeta_1] b_4$. The only possibility is

$$d([]\zeta_1^3) = [\zeta_1^3] + [\zeta_1] b_4.$$

This implies the claimed coaction.  $\square$

Check signs

**Proposition 5.9.** *There is an Adams $d_2$-differential*

(5.10) $$d_2(v_0 v_2) \doteq c_6 \alpha_1.$$

*Proof 1.* It is know that $\pi_{15}(\text{tmf}) = 0$. The only nonzero class in this stem on the Adams $E_2$-term for tmf is $c_6 \alpha_1$. Thus, this class must die. The only possibility is the claimed differential.  $\square$

*Proof 2.* We provide a second proof which doesn't rely on a priori knowledge of $\pi_* \text{tmf}$. Recall the Massey product for $c_6 \alpha_1$ we found in Corollary 4.12. (this is the part I need to check more carefully I think I have a rigorous argument, now. What follows is that argument) Since $\alpha_2$ projects to $0$ on the $E_3$-page, we have that the Massey product projects to $0$ at $E_3$. One also checks that the indeterminacy for this Massey product on $E_3$ is $0$. It is also the case that there is no room for crossing differentials in this range. Thus Moss' Convergence Theorem implies that $c_6 \alpha_1$ must project to $0$ in $E_\infty$. This implies that $c_6 \alpha_1$ must be killed by a $d_2$-differential. The only possibility is the claimed differential.  $\square$

One would like to conclude from this that there is a length 2 differential from $b_4 v_2$ to $v_2 \alpha_2$. However, one must be cautious. Even though $b_4 v_2$ was a product in the $E_1$-term of the algebraic spectral sequence of the last section, it is no longer decomposable (as $v_2$ supported an algebraic differential). In fact, this differential does not occur. For example, if we rename $b_4$ by $\widetilde{c}_4$ and $v_2$ by $\widetilde{c}_4^2$, then $b_4 v_2$ corresponds to $\Delta$, the modular discriminant, as in [3]. In any of the computations for $\pi_* \text{tmf}$, there is a differential

$d_2(\Delta) = \alpha_1\beta^2$. This suggests that $b_4v_2$ should support a length 3 differential. On the other hand, $\pi_{23}\mathrm{tmf} = 0$, and on $E_2(\mathrm{tmf})$, there are the nonzero classes $v_2\alpha_2$, $\alpha_1\beta^2$, and $b_4c_6\alpha_1$. The previous proposition already takes into account $b_4c_6\alpha_1$ (Actually, no it doesn't, at least not a priori. This is because $v_0 \cdot v_2\alpha_2 = 0$ on the $E_\infty$-page of the algebraic SS. However, with the hidden extension noted above, this is fixed.). This suggests that $v_2\alpha_2$ will support a differential.

**Proposition 5.11.** *In $\pi_*\mathrm{tmf}$, one has that $c_6\beta = 0$. Consequently, there is an Adams $d_2$-differential*

$$(5.12) \qquad\qquad d_2(v_2\alpha_2) \doteq c_6\beta.$$

*Proof.* By the previous proposition, we can form the Massey product $\langle c_6, \alpha_1, \alpha_1 \rangle_{E_3}$ on the $E_3$-page. By the juggling lemma, [7, Appendix 1], we have the

$$c_6\beta = c_6\langle \alpha_1, \alpha_1, \alpha_1 \rangle = \langle c_6, \alpha_1, \alpha_1 \rangle\alpha_1.$$

From the previous proposition, we infer that the Massey product $\langle c_6, \alpha_1, \alpha_1 \rangle$ contains $0$. It is also easy to see that this Massey product has zero indeterminacy. Thus $c_6\beta = 0$ in $E_3(\mathrm{tmf})$. Thus $c_6\beta$ must be the target of a $d_2$-differential. The only possible source is $v_2\alpha_2$. Do we need to use Moss' convergence theorem here??? I think this argument is fine.                               $\square$

> double check this argument is completely Kosher.

The next monomials we need to consider are $b_4v_2$, $v_0v_2^2$, and $v_2^2\alpha_2$, in that order. By inspection of the chart, each of these classes have only one possible target on the $E_2$-page. However, one finds from the previous propositions that each of these potential targets actually supports a differential. Thus $b_4v_2$, $v_0v_2^2$, and $v_2^2\alpha_2$ are $d_2$-cycles. Thus we move on to the monomial $b_4v_2^2$.

**Proposition 5.13.** *There is a $d_2$-differential*

$$(5.14) \qquad\qquad d_2(b_4v_2^2) \doteq v_2^2\alpha_2.$$

*Consequently, we also derive the the $d_2$-differential*

$$(5.15) \qquad\qquad d_2(v_0b_4v_2^2) \doteq v_2c_6b_4\alpha_1$$

*Proof.* It is known that $\pi_{39}\mathrm{tmf}$ is zero, but on the $E_2$-term, there are the nonzero classes $v_2^2\alpha_2$ and $v_2c_6b_4\alpha_1$ which are not killed by previously established $d_2$-differentials. The only way for $v_2^2\alpha_2$ to be killed is by a $d_2$-differential given by the claimed differential. The hidden $v_0$-extension established in Proposition ?? gives us the second differential.                               $\square$

> Is there a way to establish this differential using Massey products / Today brackets???

We can draw from this differential another $d_2$-differential.

**Corollary 5.16.** *There is the following $d_2$-differential*

(5.17)
$$d_2(v_2^2 b_4 \alpha_2) \doteq v_2 c_6 b_4 \beta.$$

*Proof.* From the previous proposition and Proposition 4.6, we deduce the differential
$$d_2(v_2^2 b_4 \beta) \doteq v_2^2 \alpha_2 \beta.$$
However, we also conclude from Proposition 4.6 that
$$v_0(v_2^2 \alpha_2 \beta) \doteq c_6 b_4 v_2 \beta \alpha_1.$$
This implies the differential
$$d_2(v_2^2 v_0 b_4 \beta) \doteq c_6 b_4 v_2 \beta \alpha_1.$$
However, we also have
$$v_2^2 v_0 b_4 \beta = (v_2^2 b_4 \alpha_2) \cdot \alpha_1.$$
Since $\alpha_1$ is a permanent cycle, we deduce the claimed differential.    □

<span style="color:red">I just realized something... the differential should also be extractable from the differential $d_2(b_4 v_2^2) = v_2^2 \alpha_2$ and then multiplying by $\alpha_2$. But the problem is that it looks like that $(v_2^2 \alpha_2)\alpha_2 = 0$. I guess there has to be a hidden $\alpha_2$-extension.</span>

<span style="background-color:orange">figure out this problem...</span>

This completes the determination of the Adams $d_2$-differential. Before proceeding onto computing the $d_3$-differentials, we will give a description of the Adams $E_3$-term based on the differentials we just found.

## 5.2. Determining the Adams $E_3$-term.

We now set out to determine the patterns that make up the Adams $E_3$-term for tmf. To get things going, first note that the pattern of type 2 on generator $b_4$ and the pattern of type 3 on generator $v_2$ support differentials into the pattern $\mathrm{Ext}_{A(1)_*}(\mathbb{F}_3) \cdot \{1\}$ (see Remark 4.5 for an explanation of this terminology). More specifically, the differential (5.6) propagates to give the following $d_2$-differentials for $k, j, \ell \in \mathbb{N}$ and $\varepsilon_1 \in \{0, 1\}$;

$$d_2(v_0^\ell c_6^j \alpha_1^{\varepsilon_1} \beta^k b_4) \doteq \begin{cases} c_6^j \alpha_1^{\varepsilon_1} \alpha_2 \beta^k & \ell = 0 \\ 0 & \ell \neq 0 \end{cases}.$$

Similarly, the differentials (5.10) and (5.12) respectively propagate to give the differentials

$$d_2(c_6^j v_0^\ell \cdot (v_0 v_2)) \doteq \begin{cases} c_6^{j+1} \alpha_1 & \ell = 0 \\ 0 & \ell \neq 0 \end{cases}$$

and

$$d_2(c_6^j \beta^k \alpha_1^{\varepsilon_1} \cdot (v_2 \alpha_2)) \doteq c_6^{j+1} \beta^{k+1} \alpha_1^{\varepsilon_1}.$$

Observe that any monomial in $\mathrm{Ext}_{A(1)_*}(\mathbb{F}_3)$ involving an $\alpha_2$ or a $c_6\alpha_1$ is hit by a differential. So from $\mathrm{Ext}_{A(1)_*}(\mathbb{F}_3)$ we obtain the module

$$\mathrm{Ext}_{A(1)_*}(\mathbb{F}_3)/(\alpha_2, c_6\beta, c_6\alpha_1) \cdot \{1\}.$$

The patterns supported by $b_4$ and $v_2$ do not receive any Adams $d_2$-differentials, so all we must do is determine what remains of these patterns after applying the Adams $d_2$-differentials. It follows from these differentials that what remains of the pattern on $b_4$ is the submodule

$$\mathrm{Ext}_{A(1)_*}(\mathbb{F}_3)/(\alpha_1, \alpha_2, \beta) \cdot \{v_0 b_4\}$$

and what remains of the pattern on $v_2$ is

$$\mathrm{Ext}_{A(1)_*}(\mathbb{F}_3)/(\alpha_1, \alpha_2, \beta) \cdot \{v_0^2 v_2\}.$$

The next pattern we need to consider is the pattern of type 2 on $b_4 v_2$. Since $b_4 v_2$ is a $d_2$-cycle, this entire pattern consists of $d_2$-cycles. Because of the hidden $v_0$-extension in Prop 4.6, we will consider this pattern in tandem with the half of the pattern of type 3 on $v_2^2$ generated by $v_2^2 \alpha_2$. This half also consists only of $d_2$-cycles. Thus, the combined pattern only receives differentials. It receives its differentials from the free pattern $\mathrm{Ext}_{A(1)_*}(\mathbb{F}_3) \cdot \{v_2^2 b_4\}$. The differentials (5.14), (5.15), and (5.17) propagate to give the following differentials:

$$d_2(v_0^j c_6^k \beta^\ell \alpha_1^{\varepsilon_1} \cdot (v_2^2 b_4)) \doteq \begin{cases} c_6^k \beta^\ell \alpha_1^{\varepsilon_1} \cdot (v_2^2 \alpha_2) & j = 0 \\ c_6^{k+1} \beta^\ell \alpha_1 \cdot (v_2 b_4) & j = 1, \varepsilon_1 = 0 \\ 0 & else \end{cases}$$

$$d_2(c_6^k \beta^\ell (b_4 v_2^2) \cdot \alpha_2) \doteq c_6^{k+1} \beta^{\ell+1}(v_2 b_4).$$

Note that any monomial in the pattern $\mathrm{Ext}_{A(1)_*}(\mathbb{F}_3)/(\alpha_2) \cdot \{b_4 v_2\}$ which contains a $c_6\alpha_1$ or $c_6\beta$ is hit by a differential. Hence, this pattern yields the following module

$$\mathrm{Ext}_{A(1)_*}(\mathbb{F}_3)/(\alpha_2, c_6\beta, c_6\alpha_1) \cdot \{b_4 v_2\}.$$

What remains of the patterns on $b_4 v_2^2$ is the submodules

$$\mathrm{Ext}_{A(1)_*}/(\alpha_1, \alpha_2, \beta) \cdot \{v_0^2 v_2^2 b_4\}.$$

The pattern $\mathrm{Ext}_{A(1)_*}(\mathbb{F}_3)/(\alpha_1, \alpha_2, \beta) \cdot \{v_0 v_2^2\}$ consists entirely of $d_2$-cycles and receives no differentials.

As we have already mentioned, the Adams $E_3$-term for tmf is periodic on $v_2^3$. Combining all of these observations proves the following identification of the Adams $E_3$-term.

**Proposition 5.18.** *The Adams $E_3$-term is given as a module over $\mathrm{Ext}_{A(1)_*}(\mathbb{F}_3)$*
*as the infinite direct sum of the following types of modules,*

**Pattern 1'** *For all natural numbers $j$, we have the modules*

$$\mathrm{Ext}_{A(1)_*}(\mathbb{F}_3)/(\alpha_2, c_6\beta, c_6\alpha_1) \cdot \{v_2^{3j}, v_2^{3j+1}b_4\}$$

**Pattern 2'** *For all natural numbers $j$, we have the modules*

$$\mathrm{Ext}_{A(1)_*}(\mathbb{F}_3)/(\alpha_1, \alpha_2, \beta) \cdot \{v_0 v_2^{3j} b_4, v_0^2 v_2^{3j+1}, v_0 v_2^{3j+2}, v_0^2 v_2^{3j+2} b_4\},$$

*and the ring structure is inherited from the Adams $E_2$-term.*

Here is a picture of the Adams $E_3$-term. I think I am on board with this now

We will now make some suggestively rename some of classes. Firstly, we will now use $c_4$ to denote $v_0 b_4$, as this is the class that $v_0 b_4$ detects. Similarly, we rewrite $v_0^2 v_2$ as $c_4^2$, again because that is what this class detects. We will also rewrite $v_2 b_4$ for $\Delta$ and we rewrite $v_0 v_2^2$ as $c_4\Delta$. Finally, we will write $v_2^3$ as $\Delta^2$. The reader should note that, at the $E_3$-page, we *do not* have that $(b_4 v_2^2)^2 = v_2^9$. In fact, $b_4 v_2^2$ is not in the correct filtration for this to happen. However, since $\pi_* \mathrm{tmf}$ is periodic on $\Delta^3$, this does suggest re-expressing the $E_3$-term in the following way. Let $M$ denote the $\mathrm{Ext}_{A(1)_*}(\mathbb{F}_3)$-module

(5.19)
$$M := \mathrm{Ext}_{A(1)_*}(\mathbb{F}_3)/(\alpha_2, c_6\beta, c_6\alpha_1) \cdot \{1, v_2 b_4, v_2^3\}$$
$$\oplus \mathrm{Ext}_{A(1)_*}(\mathbb{F}_3)/(\alpha_1, \alpha_2, \beta) \cdot \{v_0 b_4, v_0^2 v_2, v_0 v_2^2, v_0^2 v_2^2 b_4, v_0^2 v_2^4\}.$$

The following now follows from the previous proposition.

**Corollary 5.20.** *The $E_3$-term for the Adams spectral sequence for* tmf *is isomorphic as an $\mathrm{Ext}_{A(1)_*}(\mathbb{F}_3)$-module as*

(5.21)
$$E_3(\mathrm{tmf}) \cong \bigoplus_{k \geq 0} M \cdot \{v_2^{9k}\} \oplus \bigoplus_{j \geq 0} M \cdot \{b_4 v_2^{9j+4}\}$$

**Remark 5.22.** The copies of $M$ generated by $v_2^{9j}$ correspond to the powers of $\Delta$ whose powers are congruent to 0 modulo 6, while the copies of $M$ generated by $b_4 v_2^{9j+4}$ correspond to the powers of $\Delta$ congruent to 3 modulo 6.

At this point the Adams $E_3$-term isomorphic to "isomorphic" to the Adams-Novikov $E_2$-term but with elements in different filtrations. In particular, $\Delta$ and $\Delta^2$ are in weird filtrations....

I think at this point the Adams $E_3$-term is isomorphic to the Adams-Novikov $E_2$-term for tmf. What is weird though is that $\Delta^2$ will support a shorter differential then $\Delta$...

[Margin notes:]

double check this is completely correct

make adams chart for $e_3$-page.

rephrase some of this paragraph

figure out a better way to express this.

**Remark 5.23.**

> write down comparison with Hill's spectral sequence $E_3$-page here...

### 5.3. Higher Adams differentials.

The Adams $E_3$-term for tmf is much sparser than the $E_2$-term. This greatly reduces the possiblity of higher Adams differentials. We will now determine the $d_3$ and $d_4$ differentials in the Adams spectral sequence for tmf. Recall that the unit map

$$S^0 \to \mathrm{tmf}$$

for tmf induces a map in Ext taking the class $\beta$ to $\beta$ and $\alpha_1$ to $\alpha_1$. This is used in the calculation [1] to derive higher Adams-Novikov differentials. We will also use it to derive higher Adams differentials.

**Proposition 5.24.** *There is an Adams $d_3$-differential*

$$d_3(v_2^3) \doteq b_4 v_2^2 \beta^2 \alpha_1.$$

*Proof.* It is know that $\pi_{47}(\mathrm{tmf}) = 0$. The only non-zero class in that stem on the $E_3$-term is $b_4 v_2 \beta^2 \alpha_1$. This forces the stated differential. $\square$

**Remark 5.25.** The author has made attempts to give an argument for this differential from first principles. But as of the writing of this short article, he has been unable to find one.

**Proposition 5.26.** *There is an Adams $d_4$-differential*

$$d_4(b_4 v_2) \doteq \beta^2 \alpha_1.$$

*Proof.* Since the classes $\alpha_1$ and $\beta$ are both in the Hurewicz image of tmf, so are all the monomials $\beta^n \alpha_1^{\varepsilon_1}$. In the stable homotopy of $S^0$, the class $\beta^3 \alpha_1$ is zero. Since the corresponding class in $E_3(\mathrm{tmf})$ is not zero, it must be hit by a differential. The only possible class which could support a differential to $\beta^3 \alpha_1$ is $b_4 v_2 \beta$. Thus we have the $d_4$-differential

$$d_4(b_4 v_2 \beta) \doteq \beta^3 \alpha_1.$$

As the Adams differentials are linear over $\beta$, we infer

$$d_4(b_4 v_2) \doteq \beta^2 \alpha_1.$$

$\square$

> Resolve whether or not this can be given as a corollary?

**Corollary 5.27.** *There is an Adams $d_3$-differential*

$$d_3(v_2^3) \doteq b_4 v_2 \beta^2 \alpha_1.$$

*Proof.* On the $E_3$-term of the Adams spectral sequence for tmf, the class $\beta^2 \alpha_1$ is nonzero, and we have the multiplicative relation

$$(\beta^2 \alpha_1) \cdot (b_4 v_2) \doteq b_4 v_2 \beta^2 \alpha_1.$$

As we have seen, in $\pi_* \mathrm{tmf}$, we must have $\beta^2 \alpha_1 = 0$, and hence we must have $b_4 v_2 \beta^2 \alpha_1 = 0$ in $\pi_* \mathrm{tmf}$. Since the latter class is nonzero on the

$E_3$-term, it must be hit by a class. The only possibility is the stated differential. I am not totally sure this argument is valid. I am skeptical because even though the multiplicative relation I mention on $E_3$ is true, $b_4 v_2$ supports a differential, whereas $b_4 v_2 \beta^2 \alpha_1$ does not. Thus, on $E_4$, ths product $b_4 v_2 \beta^2 \alpha_1$ is not decomposable. Actually, this is valid. That is because the previous proposition establishes an *length 4* differential. actually no that doesn't resolve it. This is because $b_4 v_2$ is not a permanent cycleI think it might be useful if I instead use the product

$$b_4 v_2 \alpha_1 \cdot \beta^2 = b_4 v_2 \beta^2 \alpha_1.$$

. I am going to put this aside for awhile. Another idea is to use the Toda bracket for $b = b_4 v_2 \alpha_1$ below to make an argument. □

**Remark 5.28.** One might like to think that there is the $d_4$-differential

$$d_4(b_4 v_2^4) \doteq v_2^3 \beta^2 \alpha_1,$$

because one can multiply the differential on $b_4 v_2$ to get this differential. But this is not the case since $v_2^3$ supports a shorter differential. This is an important occurrence in this spectral sequence because $b_4 v_2^4$ is detecting the class $\Delta^3$ and the homotopy groups of tmf are famously periodic on $\Delta^3$.

For degree reasons, these are the only possible $d_3$ and $d_4$ differentials. We will now produce the last differential in the Adams spectral sequence. In order to do that, we will need the following observation.

**Lemma 5.29.** *The class $b := b_4 v_2 \alpha_1$ is given on the $E_5$-page as the following Massey product*

$$b = \langle \beta^2, \alpha_1, \alpha_1 \rangle.$$

*Thus, by Moss' Convergence Theorem, the class $b$ in $\pi_* \text{tmf}$ is given by the corresponding Toda bracket.*

*Proof.* The differential $d_4(\Delta) = \beta^2 \alpha_1$ gives a defining system for the Massey product on the $E_5$-page, and there is zero indeterminacy. Furthermore, the Toda bracket $\langle \beta^2, \alpha_1, \alpha_1 \rangle$ is defined. Since there are no differentials up to the 30 stem after the $E_4$-page, there are no crossing differentials to worry about. So by Moss' convergence theorem, $b$ is given by the associated Toda bracket. □

**Proposition 5.30** (compare [1]). *There is the following hidden multiplicative extension in $\pi_* \text{tmf}$,*

$$b \cdot \alpha_1 \doteq \beta^3.$$

*Proof.* Recall that $\beta$ is given by the Toda bracket $\langle \alpha_1, \alpha_1, \alpha_1 \rangle$. So, by the first juggling lemma (cf. [7, Appendix 1]), it follows that

$$b \cdot \alpha_1 = \langle \beta^2, \alpha_1, \alpha_1 \rangle \alpha_1 = \beta^2 \langle \alpha_1, \alpha_1, \alpha_1 \rangle = \beta^3.$$

□

**Corollary 5.31.** *The class $\beta^5$ is 0 in the homotopy groups of* tmf. *Thus, there is a $d_6$-differential*

$$d_6(v_2^3 \alpha_1) \doteq \beta^5.$$

*Proof.* Using the multiplicative extension of the previous proposition, we have

$$\beta^5 = \beta^2 \beta^3 = \beta^2 b \alpha_1.$$

Since $\beta^2 \alpha_1 = 0$, we have that $\beta^5 = 0$ in $\pi_* \text{tmf}$. This forces the claimed differential. □

We can now make the following observation.

**Lemma 5.32.** *All differentials of length at least 3 established thus far respect the decomposition* (5.21). *That is, any differentials which are propagated from any of the previous differentials have as its source and target elements within the same copy of $M$. Furthermore, it follows for degree reasons that the pattern $M$ can support no other differentials.*

> while I am certain that this lemma is true, I'm not sure if its a trivial consequence of everything done thus far. It does follow easily when one just looks at the damn charts.

The second part of this lemma asserts that any differentials of length at least 7 must be between *distinct* copies of $M$. Our next goal is to show that this cannot happen. Towards this end we will show first that $b_4 v_2^2$ and $v_2^9$ are permanent cycles.

**Lemma 5.33.** *The class $b_4 v_2^4$ is a permanent cycle.*

*Proof.* The only possible differential that $b_4 v_2^4$ could have supported was a $d_4$-differential to $v_2^3 \beta \alpha_1$. But we have already shown that this class supports a $d_6$-differential. □

We will now argue that there are no more differentials in this spectral sequence. Towards this end, we will show that $v_2^9$ is a permanent cycle of this spectral.

**Proposition 5.34.** *The class $v_2^9$ is a permanent cycle.*

*Proof.* Begin by noticing that $v_2^9$ is a $d_3$-cycle. So the smallest length differential it could support is one of length at least four. Observe that all of the $v_0$-torsion free classes are in even degrees. As $v_2^9$ is in an even degree, it follows that $v_2^9$ cannot support a differential into a $v_0$-torsion free class. Thus, if $v_2^9$ is to support a differential, the target must be $v_0$-torsion. We

see in the $E_3$-page that all of the $v_0$-torsion is simple. Thus any differential supported by $v_2^9$ would have to have as its target a class of the form

$$b_4^\varepsilon v_2^j \beta^k \alpha_1^{\varepsilon'}$$

where $j$ is congruent to $0$ or $1$ modulo $3$. This just follows from our computation of the Adams $E_3$-term for tmf. More specifically, all the $v_0$-torsion is concentrated in Pattern 1'. The $(t-s,s)$-bidegree of such elements is

$$(12j + 10n + 3\varepsilon' + 8\varepsilon, j + 2n + \varepsilon').$$

An elementary number theory argument on the bidegrees shows that the only possible targets of a differential supported by $v_2^9$ is $\beta^{14}\alpha_1$ or $v_2^{10}\beta^2\alpha_1$. Either way, $v_2^9$ would have to support a differential of length at least 6 to hit either of these classes. However, from previous propositions, both of these classes either support or are hit by a differential of length at most 6. Thus $v_2^9$ cannot hit either of these classes, and hence must be a permanent cycle. $\qquad\square$

We can now derive that the Adams spectral sequence for tmf collapses at $E_7$. Towards this end, let $\overline{M}$ denote subquotient obtained from $M$ by incorporating the $d_3$ to $d_6$-differentials. Then we have the decomposition

$$(5.35) \qquad E_7(\mathrm{tmf}) \cong \bigoplus_{k\geq 0} \overline{M} \cdot \{v_2^{9k}\} \oplus \bigoplus_{j\geq 0} \overline{M} \cdot \{b_4 v_2^{9j+4}\}.$$

**Proposition 5.36.** *The Adams spectral sequence for* tmf *collapses at $E_7$.*

*Proof.* The direct sum decomposition (5.35) arises from the patterns $\overline{M} \cdot \{1\}$ and $\overline{M} \cdot \{b_4 v_2^4\}$ and iteratively multiplying these two patterns by $v_2^9$. Since $b_4 v_2^4$ and $v_2^9$ are permanent cycles, and since differentials of length at least 7 must be between distinct copies of $\overline{M}$, it follows that there are no further differentials. Consequently, the Adams spectral sequence collapses at $E_7$. $\qquad\square$

put chart here.

### 5.4. Hidden extensions.
In the previous subsection we showed that the Adams spectral sequence for tmf collapses at $E_7$ and we completely computed this page via (5.35). We've already established one hidden extension in Proposition 5.30, which corresponds to the single hidden extension occurring in the Adams-Novikov spectral sequence for tmf.

However, there are several relations in $\pi_*\mathrm{tmf}$ which are apparent on the Adams-Novikov $E_\infty$-page appearing in the $0$-line, but which are hidden from the perspective of the Adams spectral sequence. We make several observations.

**Lemma 5.37.** *The class $c_4$ in $E_\infty$ detects the class of the corresponding name in $\pi_8$tmf (hence justifying the notation). Similarly, $c_6$ detects the class of the corresponding name in $\pi_{12}$tmf*

Looking at our chart for $E_\infty$, we find that there is a single $v_0$-tower in the 16-stem which is generated by $v_0^2 v_2$. This implies the following,

**Lemma 5.38.** *The class $v_0^2 v_2$ detects the class $c_4^2$ in $\pi_{16}$tmf and we have the hidden extension $c_4 \cdot c_4 \doteq v_0^2 v_2$.*

We can also say which classes are detecting the various classes involving $\Delta$ in $\pi_*$tmf. We will make rename some classes in order to give more stream lined expressions. We will rename $b_4 v_2$ by $v_2^{3/2}$. Thus, for example, the class $v_2^{9/2}$ refers to $b_4 v_2^4$. We will refer to classes $(v_2^{3/2})^k$, when $k = 2\ell$, we interpret this to mean $v_2^{3\ell}$, while when $k = 2\ell + 1$ we interpret this as $b_4 v_2 \cdot v_2^{3\ell} = b_4 v_2^{3\ell+1}$. At the moment, we have introduced this notation more for convenience, it is *not* reflective of a multiplicative structure on any page of this spectral sequence. Indeed, on $E_3$, the square of $b_4 v_2$ is 0. However, this notation is motivated by a certain hidden extension which will appear shortly.

Note that from the results of the previous section, we have

**Lemma 5.39.** *When $j \equiv 1, 2 \mod 3$, the classes $(v_2^{3/2})^j$ support a differential, and in this case the classes $v_0(v_2^{3/2})^j$ is a permanent cycle.*

We can determine what these classes detect in $\pi_*$(tmf).

**Corollary 5.40.** *For $j \equiv 1, 2 \mod 3$, the classes $v_0(v_2^{3/2})^j$ detect $3\Delta^j$. For $j \equiv 0 \mod 3$, the class $(v_2^{3/2})^j$ detects $\Delta^j$.*

Because these correspond to multiples of powers of $\Delta$, this implies a family of hidden extensions.

**Corollary 5.41.** *In $E_\infty$, we have the following hidden extensions for every $\ell \geq 0$,*
$$(v_0 v_2^{3/2}) \cdot v_0(v_2^{3/2})^{2\ell+1} = v_0^2 v_2^{3\ell+3}$$
*We have the hidden extensions for odd $j$*
$$(v_2^{3/2})^3 \cdot (v_2^{3/2})^{3j} = (v_2^{3/2})^{3(j+1)}.$$

This corollary justifies our choice of notation.

The theory of modular forms tells us that there is the famous relation
$$c_4^3 - c_6^2 = 1728\Delta = 2^3 3^3 \Delta.$$
This implies a hidden extension in the $E_\infty$-term.

**Lemma 5.42.** *There is a hidden extension in $E_\infty(\mathrm{tmf})$ given by*

$$c_4 \cdot (v_0^2 v_2) \doteq v_0^3 b_4 v_2 + c_6^2$$

These hidden extensions of course propogate themselves throughout the $E_\infty$-term. There are no hidden extensions beyond the ones mentioned above.

**Remark 5.43.** It is rather unsatisfying that the way these hidden extensions were determined was by using the known multiplicative structure in $\pi_*\mathrm{tmf}$. It would be nice to have arguments from first principles. It would seem that this would require knowing Massey product descriptions of various classes, such as $c_4, v_2^{3/2}$, and so on. But the author was unable to find such descriptions.

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