THE ADAMS SPECTRAL SEQUENCE FOR 3-LOCAL tmf

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ABSTRACT. The purpose of this short article is to record the computation for 3-local tmf via the Adams spectral sequence.

CONTENTS

1. INTRODUCTION

In this paper, we will carry out a computation of the homotopy groups of tmf. The homotopy groups of tmf have been known for quite awhile now. For example, the computation of $\pi_*\mathrm{tmf}$ was explicitly written up in [1], though it was known even earlier to Hopkins and his collaborators (cf. [6]). The usual approach to calculating $\pi_*\mathrm{tmf}$ is via the Adams-Novikov spectral sequence (also referred to as the descent spectral sequence in this context). One advantage of this approach is that the Adams-Novikov E_2-term can be computed using the theory of elliptic curves.

However, there are occasions where one wants to know the Adams spectral sequence for computing the homotopy groups of a spectrum. That is, one may want to know the Adams E_2-term, all differentials, and all hidden extensions. The purpose of this paper is to record the Adams spectral sequence for 3-local topological modular forms.

The reader may wonder why one would be interested in knowing the Adams spectral sequence for tmf, especially since the Adams-Novikov spectral sequence is far more efficient for this purpose. We should mention that the analogous calculation at the prime 2 is being carried out by Rognes and Bruner ([2]). It is the author's understanding that their interest in that spectral sequence stemmed from their work on the topological Hochschild homology of tmf. We speculate knowing the Adams spectral sequence at the prime 3 might be useful for similar reasons.

1.1. **Outline of the paper.** Recall that the Adams spectral sequence is a convergent spectral sequence of the form

$$\mathrm{Ext}_{A_*}(\mathbb{F}_3, H_*\mathrm{tmf}) \Longrightarrow \pi_*\mathrm{tmf}_3^\wedge.$$

Thus, a necessary input is $H_*\mathrm{tmf}$. This was determined, for example, in [9], where Rezk shows there is a short exact sequence of comodules

$$0 \to \Sigma^8\mathscr{B} \to H_*\mathrm{tmf} \to \mathscr{B} \to 0$$

where $\mathscr{B}$ is a certain subalgebra of the dual Steenrod algebra A_*. This is the starting point of our calculation. We view this short exact sequence as giving a multiplicative filtration of $H_*\mathrm{tmf}$ by comodules, yielding an algebraic spectral sequence

$$E_1^{*,*,*} = \mathrm{Ext}_{A_*}(E_0H_*\mathrm{tmf}) \cong \mathrm{Ext}_{A_*}(\mathscr{B}) \otimes E(b_4) \Longrightarrow \mathrm{Ext}_{A_*}(H_*\mathrm{tmf}).$$

In §2 we recall these details and establish a change-of-rings formula for $\mathrm{Ext}_{A_*}(\mathscr{B})$. In §3 we use a Cartan-Eilenberg spectral sequence to compute $\mathrm{Ext}_{A_*}(\mathscr{B})$. An expert in these affairs can safely ignore this section; we include it for those who are trying to learn more computational methods in homotopy theory. In §4, we determine the Adams E_2-term. Finally, in §5, we establish the Adams differentials and derive $\pi_*\mathrm{tmf}$.

Conventions. In this article, we will implicitly assume that all spectra are 3-complete. Thus tmf refers, from here on out, to the 3-completion of the spectrum of topological modular forms. We will always denote mod 3 Eilenberg-MacLane spectrum by H. Given a Hopf algebra Γ and a comodule C over Γ, we will abbreviate $\mathrm{Ext}_\Gamma(\mathbb{F}_3, C)$ by $\mathrm{Ext}_\Gamma(C)$. In the case when $\Gamma = A_*$, we will write $\mathrm{Ext}(C)$. We will always employ Adams indexing unless specifically stated otherwise. We let ζ_n denote $\chi\xi_n$ and $\overline{\tau}_n$ denote $\chi\tau_n$ in the dual Steenrod algebra.

Acknowledgements. The author would like to thank Mark Behrens for encouraging him to write up this computation. He would also like to thank Hood Chatham for creating such a wonderful LaTeXpackage for drawing spectral sequences as well as for assistance in drawing some of the charts in this paper.

2. THE MOD 3 HOMOLOGY OF tmf

In this section we recall necessary facts about the mod 3 homology of tmf. In [9], it is shown that, as an algebra, the homology of tmf is given by

$$H_*\mathrm{tmf} \cong E(b_4) \otimes \mathscr{B}$$

where $|b_4| = 8$ and

$$\mathscr{B} := \mathbb{F}_3[\zeta_1^3, \zeta_n \mid n \geq 2] \otimes E(\overline{\tau}_n \mid n \geq 3).$$

One can easily check that $\mathscr{B}$ is a comodule algebra over A_*. Furthermore, Rezk shows that there is nontrivial extension of comodules

$$0 \longrightarrow \Sigma^8\mathscr{B} \longrightarrow H_*\mathrm{tmf} \longrightarrow \mathscr{B} \to 0. \tag{2.1}$$

Applying $\mathrm{Ext}_{A_*}(-)$ to this short exact sequence of comodules yields a long exact sequence in Ext. We regard this as a convergent spectral sequence

$$\mathrm{Ext}(\Sigma^8\mathscr{B}) \oplus \mathrm{Ext}(\mathscr{B}) \implies \mathrm{Ext}(H_*\mathrm{tmf}).$$

The fact that 2.1 is a nontrivial extension implies that this spectral sequence does not immediately collapse. Determining the differentials in this spectral sequence is the subject of section 4. Thus, it is apparent that we need to compute the Ext groups of $\mathscr{B}$. We will simplify this by establishing a change-of-rings formula.

Definition 2.2. Let Γ be the Hopf algebra

$$\Gamma := \mathbb{F}_3[\zeta_1]/(\zeta_1^3) \otimes E(\overline{\tau}_0, \overline{\tau}_1, \overline{\tau}_2)$$

with the induced coproduct from the dual Steenrod algebra.

Example 2.3. In the dual Steenrod algebra, the coproduct on $\overline{\tau}_2$ is given by

$$\psi(\overline{\tau}_2) = \overline{\tau}_2 \otimes 1 + \overline{\tau}_0 \otimes \zeta_2 + \overline{\tau}_1 \otimes \zeta_1^3 + 1 \otimes \overline{\tau}_2.$$

Thus, in Γ, $\overline{\tau}_2$ is a Hopf algebra primitive. On the other hand,

$$\psi(\overline{\tau}_1) = \overline{\tau}_1 \otimes 1 + \overline{\tau}_0 \otimes \zeta_1 + 1 \otimes \overline{\tau}_1.$$

Thus this Hopf algebra is not primitively generated..

The proof of the following proposition is standard.

Proposition 2.4. *There is an isomorphism*

$$\mathscr{B} \cong A_* \,\square_\Gamma\, \mathbb{F}_3.$$

Corollary 2.5. *There is a change-of-rings isomorphism*

$$\mathrm{Ext}(\mathscr{B}) \cong \mathrm{Ext}_\Gamma(\mathbb{F}_3).$$

Thus we must compute the cohomology of the Hopf algebra Γ. This is done in the next section.

3. Computing the cohomology of Γ

In the last section we showed that the Ext groups of $\mathscr{B}$ are $\mathrm{Ext}_\Gamma(\mathbb{F}_3)$. Since Γ is a finite Hopf algebra, there is hope of computing its cohomology. Recall that $A(1)$ is the subalgebra of the Steenrod algebra generated by the Bockstein β and $\mathscr{P}^1$. Its dual is

$$A(1)_* \cong \mathbb{F}_3[\zeta_1]/(\zeta_1^3) \otimes E(\overline{\tau}_0, \overline{\tau}_1).$$

In paticular, $A(1)_*$ is a sub-Hopf algebra of Γ. The following proposition relies on the material in the first appendix of [8]. We recommend the reader look at Definition A1.1.15. The following lemma is easily checked.

Lemma 3.1. *The following*

$$A(1)_* \to \Gamma \to E(\overline{\tau}_2)$$

is a cocentral extension of Hopf algebras over $\mathbb{F}_3$.

When one has an extension of Hopf algebras, one can consider the Cartan-Eilenberg spectral sequence. In general, if

$$(D, \Phi) \to (A, \Gamma) \to (A, \Sigma)$$

is an extension of Hopf algebroids, N is a left comodule over Γ, then there is a natural convergent spectral sequence of the form

$$E_2^{f,s,t} = \mathrm{Ext}_\Phi^{f,t}(D, \mathrm{Ext}_\Sigma^s(A,N)) \implies \mathrm{Ext}_\Gamma^{f+s,t}(A,N).$$

Here, f denotes the filtration degree and the differentials are of the form

$$d_r : E_r^{f,s,t} \to E_r^{f+r,s-r+1,t}.$$

See A1.3.14 and A1.3.15 of [8] for details on this spectral sequence. Applied to our extension of Hopf algebras, this spectral sequence takes on the form

$$E_2^{f,s,t} = \mathrm{Ext}_{A(1)_*}^{f,t}(\mathbb{F}_3, \mathrm{Ext}_{E(\overline{\tau}_2)}^s(\mathbb{F}_3, \mathbb{F}_3)) \implies \mathrm{Ext}_\Gamma^{f+s,t}(\mathbb{F}_3). \tag{3.2}$$

Since $E(\overline{\tau}_2)$ is a primitively generated exterior Hopf algebra, we have that

$$\mathrm{Ext}_{E(\overline{\tau}_2)}(\mathbb{F}_3) \cong \mathbb{F}_3[v_2]$$

where the (s,t)-bidegree of v_2 is $(1,17)$. Note that since $\mathbb{F}_3$ is a comodule algebra, the Cartan-Eilenberg spectral sequence is multiplicative.

In order to determine the E_2-page of this spectral sequence, we need to understand the coaction of $A(1)_*$ on $\mathbb{F}_3[v_2]$. As $\mathbb{F}_3[v_2]$ is a comodule algebra over $A(1)_*$, it is enough to determine the coaction on v_2.

Lemma 3.3. *Under the canonical $A(1)_*$-coaction on $\mathrm{Ext}_{E(\overline{\tau}_2)}(\mathbb{F}_3)$, the element v_2 is a comodule primitive.*

Proof. Observe that the largest degree element of $A(1)_*$ is $\zeta_1^2\overline{\tau}_0\overline{\tau}_1$, which has degree 13. Since

$$\mathrm{Ext}_{E(\overline{\tau}_2)}(\mathbb{F}_3) \cong \mathbb{F}_3[v_2], \quad |v_2| = (1,17)$$

the coaction on v_2 must be $1 \otimes v_2$ for degree reasons. □

Corollary 3.4. *The E_2-term of the Cartan-Eilenberg SS is given by*

$$E_2 \cong \mathrm{Ext}_{A(1)_*}(\mathbb{F}_3) \otimes \mathbb{F}_3[v_2].$$

and the filtration degree of v_2 is 0.

Thus we must determine the cohomology of $A(1)_*$. An elementary exercise in using the May spectral sequence (cf. [8]) will yield

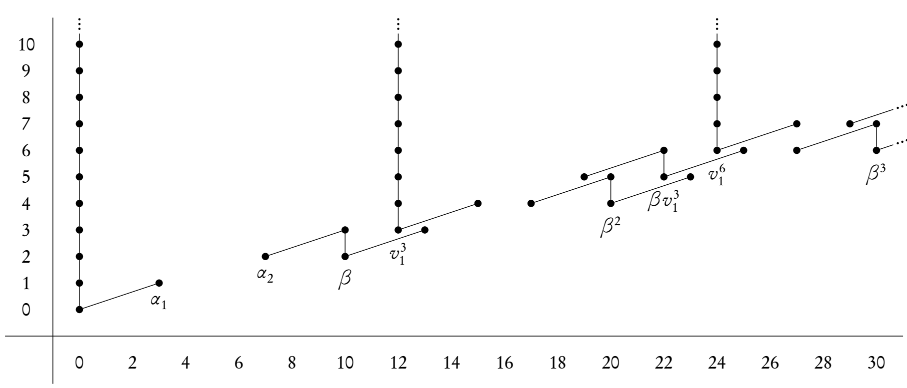
Proposition 3.5. *The algebra $\mathrm{Ext}_{A(1)_*}(\mathbb{F}_3)$ is given by*

$$\mathbb{F}_3[v_0, v_1^3, \beta] \otimes E(\alpha_1, \alpha_2)/(v_0\alpha_1, v_0\alpha_2, \alpha_1\alpha_2 - v_0\beta)$$

where the (s,t)-bidegrees of the generators are given by

- $|\alpha_1| = (1,4)$
- $|\beta| = (2,12)$,
- $|\alpha_2| = (2,9)$

A chart for this Ext group is given below.



From now on, we will write c_6 for v_1^3. For degree reasons, this spectral sequence collapses. Thus we can infer the following.

Corollary 3.6. *The cohomology of the Hopf algebra Γ is given by*

$$\mathbb{F}_3[v_0, c_6, v_2, \beta] \otimes E(\alpha_1, \alpha_2)/(v_0\alpha_1, v_0\alpha_2, h_0\alpha_2 - v_0\beta).$$

4. Determining the Adams E_2-term

In this section we will determine the E_2-term of the Adams spectral sequence converging to $\pi_*\mathrm{tmf}$. The way this will be achieved is by applying the functor Ext(—) to the short exact sequence (2.1) to obtain a long exact sequence. Regarding this as a spectral sequence provides us with

$$E_1 = \mathrm{Ext}(\Sigma^8\mathscr{B}) \oplus \mathrm{Ext}(\mathscr{B}) \implies \mathrm{Ext}(\mathrm{tmf}).$$

For the purposes of this paper, we will refer to this spectral sequence as the *algebraic spectral sequence*.

4.1. **algebraic differentials.** The results of the previous section show that, as an algebra, the E_1-term of the algebraic spectral sequence is given by

$$E_1 = \mathrm{Ext}_\Gamma(\mathbb{F}_3) \otimes E(b_4).$$

Since this is a spectral sequence derived from a long exact sequence, there can only be d_1-differentials. Note that in Adams indexing, the d_1-differential looks like an Adams d_1-differential. Results of the previous sections imply the following.

Lemma 4.1. *The short exact sequence* (2.1) *gives a multiplicative filtration of $H_*\mathrm{tmf}$ by A_*-comodules. Thus the algebraic spectral sequence is a multiplicative spectral sequence. Moreover, there is an isomorphism of A_*-comodule algebras*

$$E_0H_*\mathrm{tmf} \cong \mathscr{B} \otimes E(b_4)$$

with b_4 a comodule primitive and has filtration degree 1. Thus

$$E_1 \cong \mathrm{Ext}_\Gamma(\mathbb{F}_3) \otimes E(b_4)$$

as a graded ring.

Below is a chart for the E_1-page of the algebraic spectral sequence. The classes in blue are those in the coset for b_4 in the E_1-page. In other words, they have filtration 1 with respect to the filtration on $H_*\mathrm{tmf}$. Moreover, because the filtration on $H_*\mathrm{tmf}$ only changes when going from the 0th filtration to the first filtration, this spectral sequence only has a d^1-differential. This differential is just the connecting homomorphism arising from the long exact sequence. In particular, this means that all differentials originate from a black class and target a blue class.

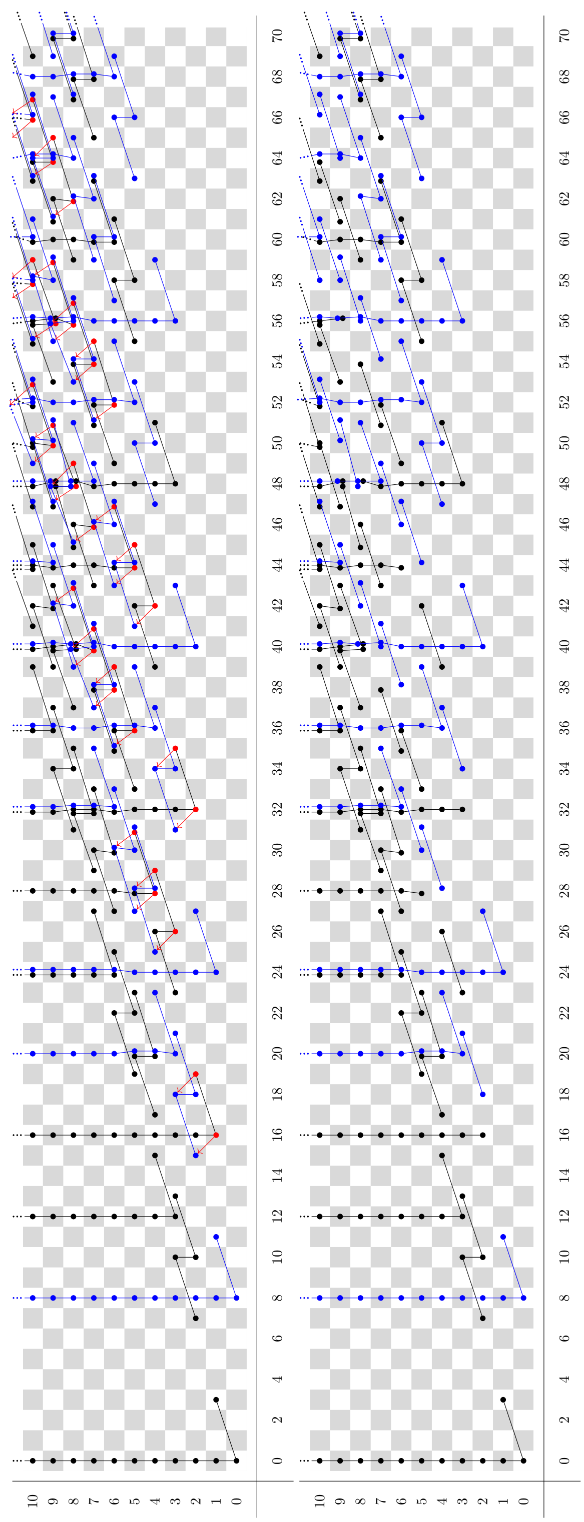


FIGURE 4.1. The E_1 and E_2-page of the algebraic spectral sequence.

We will now determine the differentials in the algebraic spectral sequence. First, we make the following simple observation.

Lemma 4.2. *For degree reasons, the classes $\alpha_1, \alpha_2, b_4, \beta$, and c_6 are permanent cycles of the algebraic spectral sequence.*

This observation and the multiplicativity of the spectral sequence eliminate many possible differentials.

From the known computation of $\pi_*\mathrm{tmf}$ (cf. [1]), we see that $\pi_{15}\mathrm{tmf} = 0$. In the E_1-term of the algebraic spectral sequence, there are two classes in stem 15; the class $b_4\alpha_2$ and the class $c_6\alpha_1$. Both of these classes must die, but the multiplicativity of the spectral sequence implies that a differential must originate on v_2. The structure of the spectral sequence thus forces the following differential[1]

$$d_1(v_2) \doteq b_4\alpha_2.$$

Multiplicativity of the spectral sequence and the previous lemma yields the following result.

Proposition 4.3. *The algebraic spectral sequence has the following d_1-differentials*

$$d_1(v_2^i v_0^j c_6^k \beta^\ell \alpha_1^\epsilon) \doteq v_2^{i-1} v_0^j c_6^k \beta^\ell \alpha_1^\epsilon b_4 \alpha_2 \qquad i \not\equiv 0 \mod 3$$

for natural numbers i, j, k, ℓ and $\epsilon \in \{0,1\}$. There are no other differentials.

Consequently, this spectral sequence is periodic on the element v_2^3.

Remark 4.4. It would be nice to have an argument for this differential from first principles, but the author is not currently aware of one. He suspects this implies the existence of an interesting coproduct on $H_*\mathrm{tmf}$.

4.2. **algebraic E_∞-term.** We will now describe a few patterns which make up the E_∞-page of the algebraic spectral sequence. We will describe these patterns as certain modules over $\mathrm{Ext}_{A(1)_*}(\mathbb{F}_3)$ along with the monomial of the algebraic spectral sequence which generates it.

(Pattern 1) Since v_2^3 is a permanent cycle, we have the free $\mathrm{Ext}_{A(1)_*}(\mathbb{F}_3)$ modules on the powers of v_2^3 and the v_2^3-multiples of $v_2^2 b_4$, i.e. for all $j \geq 0$,

$$\mathrm{Ext}_{A(1)_*}(\mathbb{F}_3) \cdot \{v_2^{3j}, v_2^{3j-1} b_4\};$$

(Pattern 2) For $j \equiv 0, 1 \mod 3$, we have the patterns

$$\mathrm{Ext}_{A(1)_*}(\mathbb{F}_3)/(\alpha_2) \cdot \{v_2^j b_4\}$$

(Pattern 3) For $j \not\equiv 0 \mod 3$, we have the following patterns

$$\mathrm{Ext}_{A(1)_*}(\mathbb{F}_3)/(\alpha_1, \alpha_2, \beta) \cdot \{v_0 v_2^j\} \oplus \mathrm{Ext}_{A(1)_*}(\mathbb{F}_3)/(\alpha_2, v_0) \cdot \{v_2^j \alpha_2\}.$$

[1]The class $c_6\alpha_1$ will be dealt with by an Adams differential.

The way we obtained these patterns was by noting that, as a module over $\mathrm{Ext}_{A(1)_*}(\mathbb{F}_3)$, the E_1-page of the algebraic spectral sequence is freely generated by the monomials $v_2^j b_4^\epsilon$. In other words, we have an isomorphism of $\mathrm{Ext}_{A(1)_*}(\mathbb{F}_3)$-modules,

$$E_1 \cong \bigoplus_{j\geq 0,\, \varepsilon\in\{0,1\}} \mathrm{Ext}_{A(1)_*}(\mathbb{F}_3)\cdot\{v_2^j b_4^\varepsilon\}.$$

The three patterns arise by partitioning the free modules $\mathrm{Ext}_{A(1)_*}(\mathbb{F}_3)\cdot\{v_2^j b_4^\epsilon\}$ into those which neither receive nor support any differentials (Pattern 1), receive differentials (Pattern 2), or support differentials (Pattern 3).

Remark 4.5. In later parts of this paper we will need to refer to these patterns. We will refer to them as *patterns of type j on generator x*. So for example, if we look at the pattern on the Adams E_2-term generated by the monomial $v_2^4 b_4$, then we will call this a pattern of type 2 on generator $v_2^4 b_4$. For patterns of the third type, we will call these patterns of type 3 on generator v_2^j. This is potentially confusing since v_2^j does not survive the algebraic spectral sequence unless j is a multiple of 3. This terminology stems from the fact that this pattern is the residual piece of a free $\mathrm{Ext}_{A(1)_*}(\mathbb{F}_3)$-module generated by v_2^j.

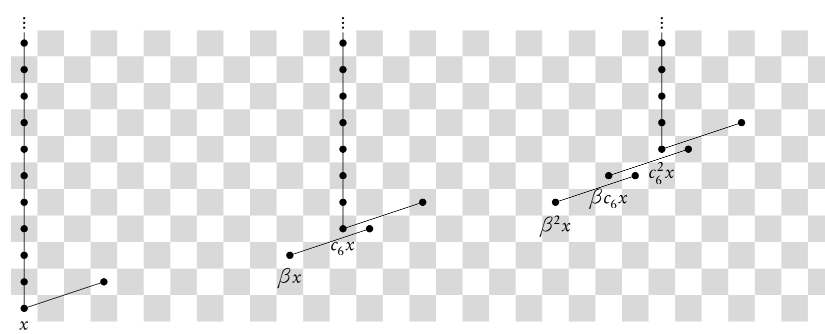


FIGURE 4.2. A depiction of pattern 2 on a generator x

4.3. **algebraic hidden extensions.** As with any spectral sequence, there is the possibility of extension problems. We will show that there is a crucial hidden v_0-extension which will play an important role in the next section. Namely,

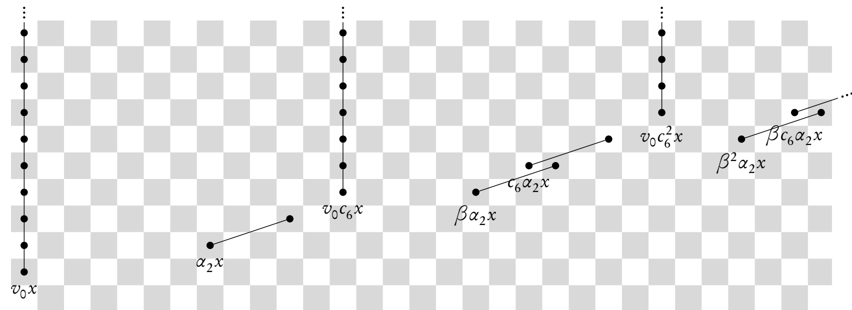


FIGURE 4.3. A depiction of pattern 3 on a generator x

Proposition 4.6. *In the algebraic spectral sequence, there is a hidden multiplicative extension*

$$v_0\cdot(v_2^2\alpha_2)\doteq v_2b_4c_6\alpha_1,$$

consequently for every natural number j and k, we have the hidden extension

$$v_0\cdot v_2^2c_6^j\beta^k\alpha_2\doteq v_2c_6^{j+1}\beta^k b_4\alpha_2.$$

Remark 4.7. One might protest that this is not a hidden extension since $v_2b_4c_6\alpha_1$ is an element in the correct Adams filtration. However, from the perspective of the algebraic spectral sequence, $v_2^2\alpha_2$ has filtration 0 and $v_2b_4c_6\alpha_2$ has filtration 1. Since v_0 has filtration 0, this is in fact a hidden extension.

Remark 4.8. The reason why we need to prove this hidden extension is because one can see, from the known computation of $\pi_*\mathrm{tmf}$, that there are differentials

$$d_2(b_4v_2^2)\doteq v_2^2\alpha_2$$

and

$$d_2(v_0b_4v_2^2)\doteq b_4c_6\alpha_1.$$

Since the Adams differentials are linear with respect to v_0, in order for for this to be consistent, we must have the claimed hidden extension.

Before proving this, we will need to show the following.

Lemma 4.9. *In* $\mathrm{Ext}_{A(1)_*}(\mathbb{F}_3)$, *there is the Massey product*

$$c_6\alpha_1\in\langle v_0,\alpha_2,\alpha_2\rangle.$$

and there is zero indeterminacy.

To prove this, we need to recall May's Convergence theorem. The proof of this fact can be found as Theorem 4.1 of [7], but we will only be interested in the case of a three-fold Massey product. The variant we will use is Theorem 2.2.2 of [5].

Theorem 4.10 (May's Convergence Theorem)**.** *Let $\alpha_0, \alpha_1, \alpha_2$ be elements of* Ext *such that the Massey product $\langle \alpha_0, \alpha_1, \alpha_2 \rangle$ is defined. For each i, let a_i be a permanent cycle on the May E_r-page which detects α_i. Suppose further that*

(1) *The Massey product $\langle a_0, a_1, a_2 \rangle$ is defined on the E_{r+1}-page: there are a_{01} and a_{12} such that $d_r(a_{01}) = \overline{a_0} a_1$ and $d_r(a_{12}) = \overline{a_1} a_2$.*
(2) *If (m, s, t) is the tri-degree of either a_{01} or a_{12}, and for for any $m' \geq m$ and q such that $m' - q < m - r$, the differential*

$$d_q : E_q^{m', s, t} \to E_q^{m'-q+1, s+1, t}$$

is zero.

Then the element $\overline{a_{01}} a_3 + \overline{a_0} a_{12}$ is a permanent cycle and detects an element of $\langle \alpha_0, \alpha_1, \alpha_2 \rangle$.

Remark 4.11. The second condition in May's Convergence Theorem is often expressed by saying there are no "crossing differentials."

We will use May's convergence theorem to give a proof for Lemma 4.9. Before doing so, it is helpful to recall the May spectral sequence for $\mathrm{Ext}_{A(1)_*}(\mathbb{F}_3)$. This spectral sequence is obtained by putting a filtration on

$$A(1)_* = P(\zeta_1)/(\zeta_1^3) \otimes E(\overline{\tau}_0, \overline{\tau}_1),$$

is obtained by giving the generators of $A(1)_*$ the following *May filtration*

- $MF(\overline{\tau}_0) = MF(\zeta_1) = 1$,
- $MF(\overline{\tau}_1) = 3$.

This produces a filtration on the cobar complex for $A(1)_*$, resulting in the May spectral sequence

$$E_1^{m,s,t} \implies \mathrm{Ext}^{s,t}_{A(1)_*}(\mathbb{F}_3),$$

with the following E_1-term,

$$E_1 \cong E(\alpha_1) \otimes P(v_0, v_1, \beta).$$

The coproduct on $A(1)_*$ gives the d_1-differential

$$d_1(v_1) = v_0 \alpha_1.$$

The rest of the d_1-differentials are propagated from this one and the multiplicativity of the May spectral sequence. This makes the class $\alpha_2 := v_1 \alpha_1$ a non-zero permanent cycle. One easily shows that

Lemma 4.12. *In* $\mathrm{Ext}_{A(1)_*}(\mathbb{F}_3)$*, there is the Massey product*

$$\alpha_2 = \langle v_0, \alpha_1, \alpha_1 \rangle.$$

Proof of Lemma 4.9. Since α_1 is exterior on the May E_1-page, we get the following defining system $d_1(v_1^2) = \overline{v_0}\alpha_2$ and $d_1(0) = \alpha_2^2$ for the Massey product $\langle v_0, \alpha_2, \alpha_2 \rangle$.

Since $\alpha_2^2 = 0$ and $v_0\alpha_2 = 0$ in $\mathrm{Ext}_{A(1)_*}(\mathbb{F}_3)$, the Massey product $\langle v_0, \alpha_2, \alpha_2 \rangle$ is defined in $\mathrm{Ext}_{A(1)_*}(\mathbb{F}_3)$. Furthermore, there is no indeterminacy of this Massey product. So we just need to check the second condition of May's Convergence Theorem, that there are no crossing differentials. Note that $v_1^2 \in E_1^{6,2,10}$ and that $E_2^{m,2,10} = 0$ for all m. So condition (2) is satisfied here. Likewise, note that $(v_1\alpha_1)^2 \in E_1^{8,4,18}$ and that $E_1^{m,3,18} = 0$ for all m. Thus there are no nonzero differentials to worry about. So condition (2) is always satisfied here as well. Thus we may apply May's Convergence Theorem to infer that

$$c_6\alpha_1 \in \langle v_0, \alpha_2, \alpha_2 \rangle.$$

It is easy to see that there is no indeterminacy. □

Remark 4.13. Keep in mind that May's convergence theorem is actually very general (cf. the discussion preceeding [8, A1.4.10]). It applies to any spectral sequence which arises from a multiplicative filtration on a DGA. In particular, it applies to the Cartan-Eilenberg SS and the algebraic SS we have used. Since the Cartan-Eilenberg SS collapses, and since the algebraic SS only has d_1-differentials, the May Convergence Theorem vacuously applies to these spectral sequences.

Thus, we derive the following corollary.

Corollary 4.14. *In* $\mathrm{Ext}_{A_*}(H_*\mathrm{tmf})$ *there is the Massey*

$$c_6\alpha_1 = \langle v_0, \alpha_2, \alpha_2 \rangle.$$

We will use this corollary to derive the hidden multiplicative extension.

Proof of Proposition 4.6. One can check, using the May Convergence Theorem applied to the algebraic spectral sequence, that one has the Massey product

$$v_2^2\alpha_2 \in \langle \alpha_2, \alpha_2, b_4v_2 \rangle,$$

and that this Massey product has no indeterminacy. In order to apply the May Convergence Theorem to this Massey product, we must check that it is defined on $\mathrm{Ext}_{A_*}(\mathrm{tmf})$. Note that $\alpha_2^2 = 0$ in this Ext group, since there is no nonzero group in the 14 stem. Furthermore, since hidden extensions must always target a class in higher filtration, and since the algebraic spectral sequence only has elements in filtration 0 and 1, it follows that there can be no hidden extension for the product of $b_4 v_2$ and α_2. Thus the Massey product is defined in Ext(tmf) (cf. Remark 4.13). Using the First Juggling Theorem (cf. [8, A1.4.6]), we have

$$v_0 \cdot (v_2^2 \alpha_2) = v_0 \langle \alpha_2, \alpha_2, b_4 v_2 \rangle \doteq \langle v_0, \alpha_2, \alpha_2 \rangle b_4 v_2 = c_6 \alpha_1 b_4 v_2$$

yielding the desired extension. □

We will also have occasion to use the following hidden extension.

Corollary 4.15. *In the algebraic SS, there is the Massey product*

$$v_2 \alpha_2 = \langle \alpha_2, \alpha_2, b_4 \rangle$$

and consequently the hidden extension

$$v_0 \cdot (v_2 \alpha_2) \doteq b_4 \cdot (c_6 \alpha_1).$$

Proof. A defining system for the Massey product $\langle \alpha_2, \alpha_2, b_4 \rangle$ on the E_2 page is given by $d_1(0) = \alpha_2^2$ and $d_1(v_2) = \alpha_2 b_4$. Observe that there is no indeterminacy. So by May's convergence theorem we have the Massey product

$$v_2 \alpha_2 \doteq \langle \alpha_2, \alpha_2, b_4 \rangle.$$

Since this Massey product and $\langle v_0, \alpha_2, \alpha_2 \rangle$ are both strictly defined, we get from the First Juggling Theorem [8, A1.4.6(c)] the following equalities

$$b_4 c_6 \alpha_1 = \langle v_0, \alpha_2, \alpha_2 \rangle b_4 = v_0 \langle \alpha_2, \alpha_2, b_4 \rangle = v_0 (v_2 \alpha_2).$$

□

4.4. **comparison to the Adams spectral sequence in** tmf**-modules.** We now make a few remarks comparing the E_2-term of the Adams spectral sequence for tmf and the Adams spectral sequence for tmf in tmf-modules as studied by Hill ([4]). The latter is a spectral sequence

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{A_*^{\mathrm{tmf}}}(\mathbb{F}_3) \implies \pi_{t-s}\mathrm{tmf}.$$

where

$$A_*^{\mathrm{tmf}} = \pi_*(H \wedge_{\mathrm{tmf}} H) \cong A(1)_* \otimes E(a_2)$$

where a_2 is in degeree 9. This class has an interesting coproduct, but this does not concern us here. What is interesting for us, however, is that in order to compute this coproduct, Hill filters A_*^{tmf}, resulting in an algebraic spectral sequence

$$E_1 = \mathrm{Ext}_{E_0 A_*^{\mathrm{tmf}}}(\mathbb{F}_3) \implies \mathrm{Ext}_{A_*^{\mathrm{tmf}}}(\mathbb{F}_3).$$

One easily derives that

$$E_1 = \mathrm{Ext}_{E_0 A_*^{\mathrm{tmf}}}(\mathbb{F}_3) \cong \mathrm{Ext}_{A(1)_*}(\mathbb{F}_3) \otimes P(\widetilde{c}_4)$$

where $\widetilde{c}_4$ is the class represented in the cobar complex of $E_0A^t\,mf_*$ by $[a_2]$. In particular, $\widetilde{c}_4 \in E_1^{1,9}$. It turns out that this E_1-page is isomorphic to the E_1-term of our algebraic spectral sequence, but with various elements in ours in the "wrong" filtration. We provide a short dictionary relating various names in our spectral sequence to Hill's algebraic spectral sequence.

alg'c SS	Hill's alg'c SS
b_4	$\widetilde{c}_4$
v_2	$\widetilde{c}_4{}^2$
b_4v_2	$\widetilde{c}_4{}^3$
$\vdots$	$\vdots$

In particular, the element that Hill calls $\widetilde{c}_4{}^{2\ell+\varepsilon}$ corresponds to the element we call $b_4^\varepsilon v_2^\ell$. Moreover, Hill is able to derive a differential $d_1(\widetilde{c}_4) = \alpha_2$. Algebraic manipulation then yields the following differentials $d_2(\alpha_2\widetilde{c}_4{}^2) = v_1^3\beta$ and $d_2(v_0\widetilde{c}_4{}^2) = v_1^3\alpha_1$. These all correspond to various differentials we encounter in this paper as well, but interestingly, not all of them are algebraic differentials. On the one hand, the differential $d_1(\widetilde{c}_4{}^2) = \widetilde{c}_4\alpha_2$ corresponds to our algebraic differential $d_1(v_2) = b_4\alpha_2$. But the differential $d_1(\widetilde{c}_4) = \alpha_2$ corresponds to an *Adams* d_2-differential $d_2(b_4) = \alpha_2$.

In particular, half of Hill's algebraic differentials are seen in our algebraic spectral sequence, but the other half arise as Adams differentials. It is this discrepancy that makes the tmf-relative Adams spectral sequence more computable as opposed to the absolute Adams spectral sequence.

5. Adams differentials

In this section, we will determine the differentials in the Adams spectral sequence for tmf. Since tmf is a commutative ring spectrum, the Adams spectral sequence is multiplicative. Begin by noting that there are several classes which are permanent cycles for degree reasons.

Lemma 5.1. *The classes $v_0, \alpha_1, \alpha_2, \beta$, and c_6 are all permanent cycles for the Adams spectral sequence. Consequently, the differentials in the Adams spectral sequence are linear over* $\mathrm{Ext}_{A(1)_*}(\mathbb{F}_3)$.

This observation is very useful for our calculation for the following reason. In the last section, we have expressed the Adams E_2-term as a direct sum of certain patterns which were modules over $\mathrm{Ext}_{A(1)_*}(\mathbb{F}_3)$. This observation implies that the only nonzero differentials in the Adams spectral sequence will originate on the monomials which generate these patterns. We will also make frequent use of the following facts about $\pi_*\mathrm{tmf}$.

Theorem 5.2 (cf. [3]). *The homotopy groups of* tmf *are 72 periodic. Furthermore, the torsion in* π_*tmf *is concentrated in stems 3, 10, 13, 20, 27, 30, 37, and 40 modulo 72.*

We will begin by determining all of the length 2 differentials first.

5.1. **Adams d_2-differentials.** As was mentioned previously, the Adams differentials are all linear over $\mathrm{Ext}_{A(1)_*}(\mathbb{F}_3)$, which means we only have to figure which of the following families of monomials support Adams d_2-differentials: For any natural number j

(1) v_2^{3j},
(2) $v_2^j b_4$,
(3) $v_0 v_2^j$ for $j \equiv 1, 2 \mod 3$, and
(4) $v_2^j \alpha_2$ for $j \equiv 1, 2 \mod 3$.

From our charts, one sees that v_2^3 can support a length 3 differential at minimum. Thus, v_2^{3j} is a d_2-cycle for all j. Moreover, there are several multiplicative relations on the Adams E_2-term which we get from the previous section. For example, we have $v_2^3 \cdot (v_2 b_4) = v_2^4 b_4$. Consequently, we have

Proposition 5.3. *The Adams d_2-differentials for* tmf *are linear over* $\mathrm{Ext}_{A(1)_*}(\mathbb{F}_3) \otimes P(v_2^3)$. *Thus, we only need to determine which of the monomials b_4, $v_0 v_2$, $v_2 \alpha_2$, $b_4 v_2$, $v_0 v_2^2$, $v_2^2 \alpha_2$, and $b_4 v_2^2$ support d_2-differentials.*

Corollary 5.4. *The Adams E_3-term is periodic on v_2^3.*

Proposition 5.5. *There is an Adams d_2-differential*

$$d_2(b_4) \doteq \alpha_2. \tag{5.6}$$

Proof 1. From the known computation of π_*tmf, it is seen that $\pi_7\mathrm{tmf} = 0$. Thus, the class α_2 must die. The only possibility is the claimed differential. □

Proof 2. Recall that in the Adams E_2-term for tmf, we have the Massey product

$$\beta = \langle \alpha_1, \alpha_1, \alpha_1 \rangle.$$

In the Adams spectral sequence for the sphere, there is the same Toda bracket. This is because, α_1 and β are the Hurewicz images of the classes of the same name in the homotopy groups of S^0. This can be seen by considering the induced map on E_2-terms

$$\mathrm{Ext}(S^0) \to \mathrm{Ext}(\mathrm{tmf})$$

and using cobar representatives. It is known from the Adams spectral sequence for the sphere that there is a d_2-differential whose target is $v_0\beta$ (cf. [8]). In particular, this forces $v_0\beta = 0$ in $\pi_*\mathrm{tmf}$. This forces the Adams differential

$$d_2(b_4\alpha_1) \doteq v_0\beta = \alpha_1\alpha_2.$$

However, as α_1 is a permanent cycle for the ASS for tmf, we must have that

$$d_2(b_4) \doteq \alpha_2$$

as stated. □

Remark 5.7. This is one of the Adams differentials which occurs as an algebraic differential in [4].

We can draw an interesting consequence from the second argument provided above (we learned this from Mike Hill and Mark Behrens).

Corollary 5.8. *The element $b_4\alpha_1$ is the image of $h_1 \in \mathrm{Ext}(S^0)$ under the map*

$$\mathrm{Ext}(S^0) \to \mathrm{Ext}(\mathrm{tmf}),$$

and consequently we have the hidden comodule extension in $H_*\mathrm{tmf}$,

$$\alpha(\zeta_1^3) = \zeta_1^3 \otimes 1 - \zeta_1 \otimes b_4 + 1 \otimes \zeta_1^3$$

Proof. In the Adams spectral sequence for the sphere, it is the class h_1 which supports a d_2-differential killing $v_0\beta$. Naturality of the Adams spectral sequence implies that h_1 maps to $b_4\alpha_1$.

In the cobar complex for S^0, the element h_1 is represented by ζ_1^3. On the other hand, we have represented $b_4\alpha_1$ in the cobar complex by $[\zeta_1]b_4$. Thus there must be an element of $H_*\mathrm{tmf}$ which bounds the difference between $[\zeta_1^3]$ and $[\zeta_1]b_4$. The only possibility is

$$d([\,]\zeta_1^3) = [\zeta_1^3] - [\zeta_1]b_4.$$

This implies the claimed coaction. □

Proposition 5.9. *There is an Adams d_2-differential*

$$d_2(v_0v_2) \doteq c_6\alpha_1. \tag{5.10}$$

Proof 1. It is known that $\pi_{15}(\mathrm{tmf}) = 0$. The only nonzero class in this stem on the Adams E_2-term for tmf is $c_6\alpha_1$. Thus, this class must die. The only possibility is the claimed differential. □

Proof 2. We provide a second proof which does not rely on a priori knowledge of $\pi_*\mathrm{tmf}$. Recall the Massey product for $c_6\alpha_1$ we found in Corollary 4.14. Since α_2 projects to 0 on the E_3-page, we have that the Massey product projects to 0 at E_3. One also checks that the indeterminacy for this Massey product on E_3 is 0. It is also the case that there is no room for crossing differentials in this range. Thus Moss' Convergence Theorem implies that $c_6\alpha_1$ must project to 0 in E_∞. This implies that $c_6\alpha_1$ must be killed by a d_2-differential. The only possibility is the claimed differential. □

One would like to conclude from this that there is a length 2 differential from b_4v_2 to $v_2\alpha_2$. However, one must be cautious. Even though b_4v_2 was a product in the E_1-term of the algebraic spectral sequence of the last section, it is no longer decomposable (as v_2 supported an algebraic differential). In fact, this differential does not occur. As explained in subsection 4.4, the classes b_4 and v_2 correspond to Hill's classes $\widetilde{c}_4$ and $\widetilde{c}_4^{\,2}$ respectively. Also, b_4v_2 corresponds to Δ, the modular discriminant. In any of the computations for $\pi_*\mathrm{tmf}$, there is a differential $d_?(\Delta)=\alpha_1\beta^2$. This suggests that b_4v_2 ought to support a length 3 differential to $\alpha_1\beta^2$. On the other hand, $\pi_{23}\mathrm{tmf}=0$, and on $E_2(\mathrm{tmf})$, there are the nonzero classes $v_2\alpha_2, \alpha_1\beta^2$, and $b_4c_6\alpha_1$. Also, Proposition 5.5 implies that $d_2(c_6b_4\alpha_1)=c_6\alpha_1\alpha_2$, taking care of the class $c_6b_4\alpha_1$. This suggests that $v_2\alpha_2$ will support a differential.

Proposition 5.11. *In $\pi_*\mathrm{tmf}$, one has that $c_6\beta=0$. Consequently, there is an Adams d_2-differential*

$$d_2(v_2\alpha_2)\doteq c_6\beta. \tag{5.12}$$

We give two proofs.

Proof 1. By the previous proposition, we can form the Massey product $\langle c_6,\alpha_1,\alpha_1\rangle_{E_3}$ on the E_3-page. By the juggling lemma, [8, Appendix 1], we have that

$$c_6\beta=c_6\langle\alpha_1,\alpha_1,\alpha_1\rangle=\langle c_6,\alpha_1,\alpha_1\rangle\alpha_1.$$

From the previous proposition, we infer that the Massey product $\langle c_6,\alpha_1,\alpha_1\rangle$ contains 0. It is also easy to see that this Massey product has zero indeterminacy. Thus $c_6\beta=0$ in $E_3(\mathrm{tmf})$. Thus $c_6\beta$ must be the target of a d_2-differential. The only possible source is $v_2\alpha_2$. □

Proof 2. From Proposition 5.5, we deduce that

$$d_2(b_4c_6\alpha_1)=c_6\alpha_1\alpha_2=v_0c_6\beta.$$

The hidden extension 4.15 then implies the stated differential. □

The next monomials we need to consider are $b_4v_2, v_0v_2^2$, and $v_2^2\alpha_2$, in that order. By inspection of the chart, each of these classes have only one possible target on the E_2-page. However, one finds from the previous propositions that each of these potential targets actually supports a differential. Thus $b_4v_2, v_0v_2^2$, and $v_2^2\alpha_2$ are d_2-cycles. Thus we move on to the monomial $b_4v_2^2$.

Proposition 5.13. *There is a d_2-differential*

$$d_2(b_4v_2^2) \doteq v_2^2\alpha_2. \tag{5.14}$$

as well as the d_2-differential

$$d_2(v_0b_4v_2^2) \doteq v_2c_6b_4\alpha_1 \tag{5.15}$$

Proof 1. It is known that $\pi_{39}\mathrm{tmf}$ is zero (cf. [1, 9]), but on the E_2-term, there are the nonzero classes $v_2^2\alpha_2$ and $v_2c_6b_4\alpha_1$ which are not killed by previously established d_2-differentials. The only way for $v_2^2\alpha_2$ to be killed is by a d_2-differential given by the claimed differential. The hidden v_0-extension established in Proposition 4.6 gives us the second differential. □

Proof 2. We have already established the differential $d_2(v_0v_2) = c_6\alpha_1$. Since v_2b_4 is a d_2-cycle, we have that

$$d_2((v_0v_2)v_2b_4) = v_2b_4c_6\alpha_1.$$

However, in the algebraic spectral sequence, we had the relation

$$(v_0v_2)v_2b_4 = v_0(b_4v_2^2).$$

The multiplicativity of the spectral sequence and the hidden extension Proposition 4.6 implies the differential $d_2(b_4v_2^2) = v_2^2\alpha_2$. □

We can draw from this differential another d_2-differential.

Corollary 5.16. *There is the following d_2-differential*

$$d_2(v_2^2b_4\alpha_2) \doteq v_2c_6b_4\beta. \tag{5.17}$$

Proof. From the previous proposition we deduce the differential

$$d_2(v_2^2b_4\beta) \doteq v_2^2\alpha_2\beta.$$

However, we have from Proposition 4.6 that

$$v_0(v_2^2\alpha_2\beta) \doteq c_6b_4v_2\beta\alpha_1.$$

This implies the differential

$$d_2(v_2^2v_0b_4\beta) \doteq c_6b_4v_2\beta\alpha_1.$$

However, we also have

$$v_2^2v_0b_4\beta = (v_2^2b_4\alpha_2)\cdot\alpha_1.$$

Since α_1 is a permanent cycle, multiplicativity of the spectral sequence implies the claimed differential. □

This completes the determination of the Adams d_2-differential. Below, in Figure 5.1, we depict that Adams E_2-term along with the d_2-differentials. The reader will notice that we have use several different colors in the chart. Here is a key to the use of these colors.

Color	Pattern	generator
black	pattern 1	1
blue	pattern 2	$b4$
lime green	pattern 3	v_2
dark magenta	pattern 2	$v_2 b_4$
lavender rose	pattern 3	v_2^2
teal	pattern 1	$v_2^2 b_4$
sea green	pattern 1	v_2^3
maroon	pattern 2	$v_2^3 b_4$
dark violet	pattern 3	v_2^4
blue gray	pattern 2	$v_2^4 b_4$

.

Before proceeding onto computing the d_3-differentials, we will give a description of the Adams E_3-term based on the differentials we just found.

5.2. **Determining the Adams E_3-term.** We now set out to determine the patterns that make up the Adams E_3-term for tmf. To get things going, first note that the pattern of type 2 on generator b_4 and the pattern of type 3 on generator v_2 support differentials into the pattern $\mathrm{Ext}_{A(1)_*}(\mathbb{F}_3)\cdot\{1\}$ (see Remark 4.5 for an explanation of this terminology). More specifically, the differential (5.6) propagates to give the following d_2-differentials for $k, j, \ell \in \mathbb{N}$ and $\varepsilon_1 \in \{0,1\}$;

$$d_2(v_0^\ell c_6^j \alpha_1^{\varepsilon_1} \beta^k b_4) \doteq \begin{cases} c_6^j \alpha_1^{\varepsilon_1} \alpha_2 \beta^k & \ell = 0 \\ 0 & \ell \neq 0 \end{cases}.$$

Similarly, the differentials (5.10) and (5.12) respectively propagate to give the differentials

$$d_2(c_6^j v_0^\ell \cdot (v_0 v_2)) \doteq \begin{cases} c_6^{j+1} \alpha_1 & \ell = 0 \\ 0 & \ell \neq 0 \end{cases}$$

and

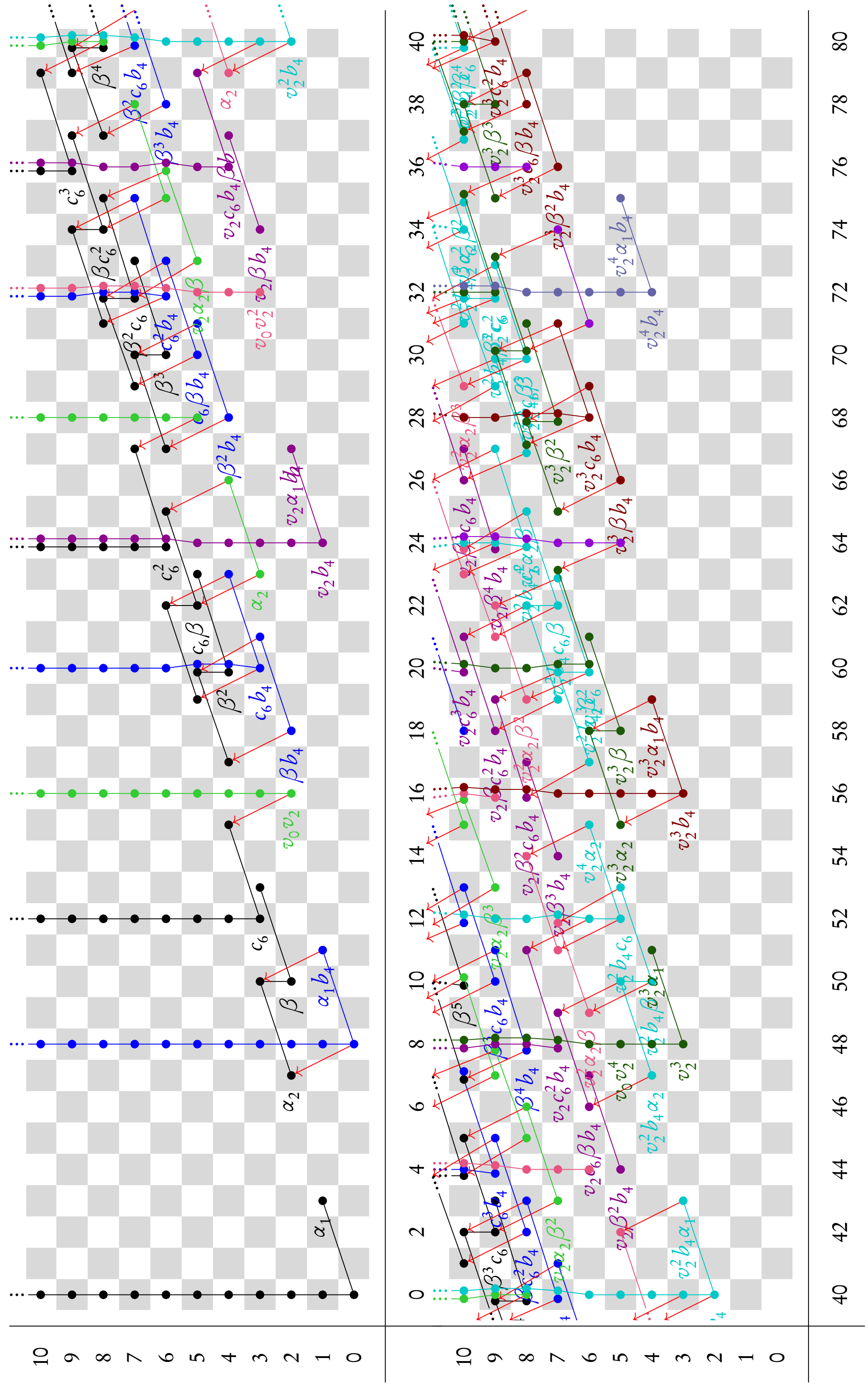
$$d_2(c_6^j \beta^k \alpha_1^{\varepsilon_1} \cdot (v_2 \alpha_2)) \doteq c_6^{j+1} \beta^{k+1} \alpha_1^{\varepsilon_1}.$$

Observe that any monomial in $\mathrm{Ext}_{A(1)_*}(\mathbb{F}_3)$ involving an α_2 or a $c_6\alpha_1$ is hit by a differential. So from $\mathrm{Ext}_{A(1)_*}(\mathbb{F}_3)$ we obtain the module

$$\mathrm{Ext}_{A(1)_*}(\mathbb{F}_3)/(\alpha_2, c_6\beta, c_6\alpha_1)\cdot\{1\}.$$

The patterns supported by b_4 and v_2 do not receive any Adams d_2-differentials, so all we must do is determine what remains of these patterns after applying the Adams d_2-differentials. It follows from these differentials that what remains of the pattern on b_4 is the submodule

$$\mathrm{Ext}_{A(1)_*}(\mathbb{F}_3)/(\alpha_1, \alpha_2, \beta)\cdot\{v_0 b_4\}$$



(A) (B)

FIGURE 5.1. Adams E_2-page in stems 0-80 with d_2-differentials

and what remains of the pattern on v_2 is

$$\mathrm{Ext}_{A(1)_*}(\mathbb{F}_3)/(\alpha_1, \alpha_2, \beta) \cdot \{v_0^2 v_2\}.$$

The next pattern we need to consider is the pattern of type 2 on $b_4 v_2$. Since $b_4 v_2$ is a d_2-cycle, this entire pattern consists of d_2-cycles. Because of the hidden v_0-extension in Prop 4.6, we will consider this pattern in tandem with the half of the pattern of type 3 on v_2^2 generated by $v_2^2\alpha_2$. This half also consists only of d_2-cycles. Thus, the combined pattern only receives differentials. It receives its differentials from the free pattern $\mathrm{Ext}_{A(1)_*}(\mathbb{F}_3)\cdot \{v_2^2 b_4\}$. The differentials (5.14), (5.15), and (5.17) propagate to give the following differentials:

$$d_2(v_0^j c_6^k \beta^\ell \alpha_1^{\epsilon_1} \cdot (v_2^2 b_4)) \doteq \begin{cases} c_6^k \beta^\ell \alpha_1^{\varepsilon_1} \cdot (v_2^2\alpha_2) & j=0 \\ c_6^{k+1}\beta^\ell \alpha_1 \cdot (v_2 b_4) & j=1, \varepsilon_1 = 0 \\ 0 & else \end{cases}$$

$$d_2(c_6^k \beta^\ell (b_4 v_2^2)\cdot \alpha_2) \doteq c_6^{k+1}\beta^{\ell+1}(v_2 b_4).$$

Note that any monomial in the pattern $\mathrm{Ext}_{A(1)_*}(\mathbb{F}_3)/(\alpha_2)\cdot\{b_4 v_2\}$ which contains a $c_6\alpha_1$ or $c_6\beta$ is hit by a differential. Hence, this pattern yields the following module

$$\mathrm{Ext}_{A(1)_*}(\mathbb{F}_3)/(\alpha_2, c_6\beta, c_6\alpha_1)\cdot\{b_4 v_2\}.$$

It also follows that what remains of the pattern on $b_4 v_2^2$ is the submodule

$$\mathrm{Ext}_{A(1)_*}/(\alpha_1,\alpha_2,\beta)\cdot\{v_0^2 v_2^2 b_4\}.$$

The other half of pattern 3 on v_2, i.e. $\mathrm{Ext}_{A(1)_*}(\mathbb{F}_3)/(\alpha_1,\alpha_2,\beta)\cdot\{v_0 v_2^2\}$, consists entirely of d_2-cycles and receives no differentials. Thus this survives in full to the E_3-page.

As we have already mentioned, the Adams E_3-term for tmf is periodic on v_2^3. Combining all of these observations proves the following identification of the Adams E_3-term.

Proposition 5.18. *The Adams E_3-term is given as a module over* $\mathrm{Ext}_{A(1)_*}(\mathbb{F}_3)$ *as the infinite direct sum of the following types of modules,*

Pattern 1' *For all natural numbers j, we have the modules*

$$\mathrm{Ext}_{A(1)_*}(\mathbb{F}_3)/(\alpha_2, c_6\beta, c_6\alpha_1)\cdot\{v_2^{3j}, v_2^{3j+1}b_4\}$$

Pattern 2' *For all natural numbers j, we have the modules*

$$\mathrm{Ext}_{A(1)_*}(\mathbb{F}_3)/(\alpha_1,\alpha_2,\beta)\cdot\{v_0 v_2^{3j} b_4, v_0^2 v_2^{3j+1}, v_0 v_2^{3j+2}, v_0^2 v_2^{3j+2} b_4\},$$

and the ring structure is inherited from the Adams E_2-term.

We give the Adams chart for the E_3-term below in Figure 5.2.

As we will see later, $v_0 b_4$ will detect the class c_4. Similarly, the class $v_0^2 v_2$ will detect c_4^2. On the other hand, there are certain important classes in the Adams-Novikov spectral sequence which support differentials. Namely, the class Δ. In the ASS, the class $v_2 b_4$ corresponds to Δ while the class v_2^3 corresponds to Δ^2. The reader should note that, at the E_3-page, we *do not* have that $(b_4 v_2^2)^2 = v_2^9$. In fact, $b_4 v_2^2$ is not in the correct filtration for this to happen. This, in fact, is what makes the Adams spectral sequence more difficult than the analogous calculation in [4]. However, since $\pi_*\mathrm{tmf}$ is periodic on Δ^3, this does suggest re-expressing the E_3-term in the following way. Let M denote the $\mathrm{Ext}_{A(1)_*}(\mathbb{F}_3)$-module

$$\begin{aligned} M := {} & \mathrm{Ext}_{A(1)_*}(\mathbb{F}_3)/(\alpha_2, c_6\beta, c_6\alpha_1)\cdot\{1, v_2 b_4, v_2^3\} \\ & \oplus \mathrm{Ext}_{A(1)_*}(\mathbb{F}_3)/(\alpha_1,\alpha_2,\beta)\cdot\{v_0 b_4, v_0^2 v_2, v_0 v_2^2, v_0^2 v_2^2 b_4, v_0^2 v_2^4\}. \end{aligned} \tag{5.19}$$

The following now follows from the previous proposition.

Corollary 5.20. *There is an isomorphism of* $\mathrm{Ext}_{A(1)_*}(\mathbb{F}_3)$*-modules*

$$E_3(\mathrm{tmf}) \cong \bigoplus_{k\geq 0} M\cdot\{v_2^{9k}\} \oplus \bigoplus_{j\geq 0} M\cdot\{b_4 v_2^{9j+4}\}. \tag{5.21}$$

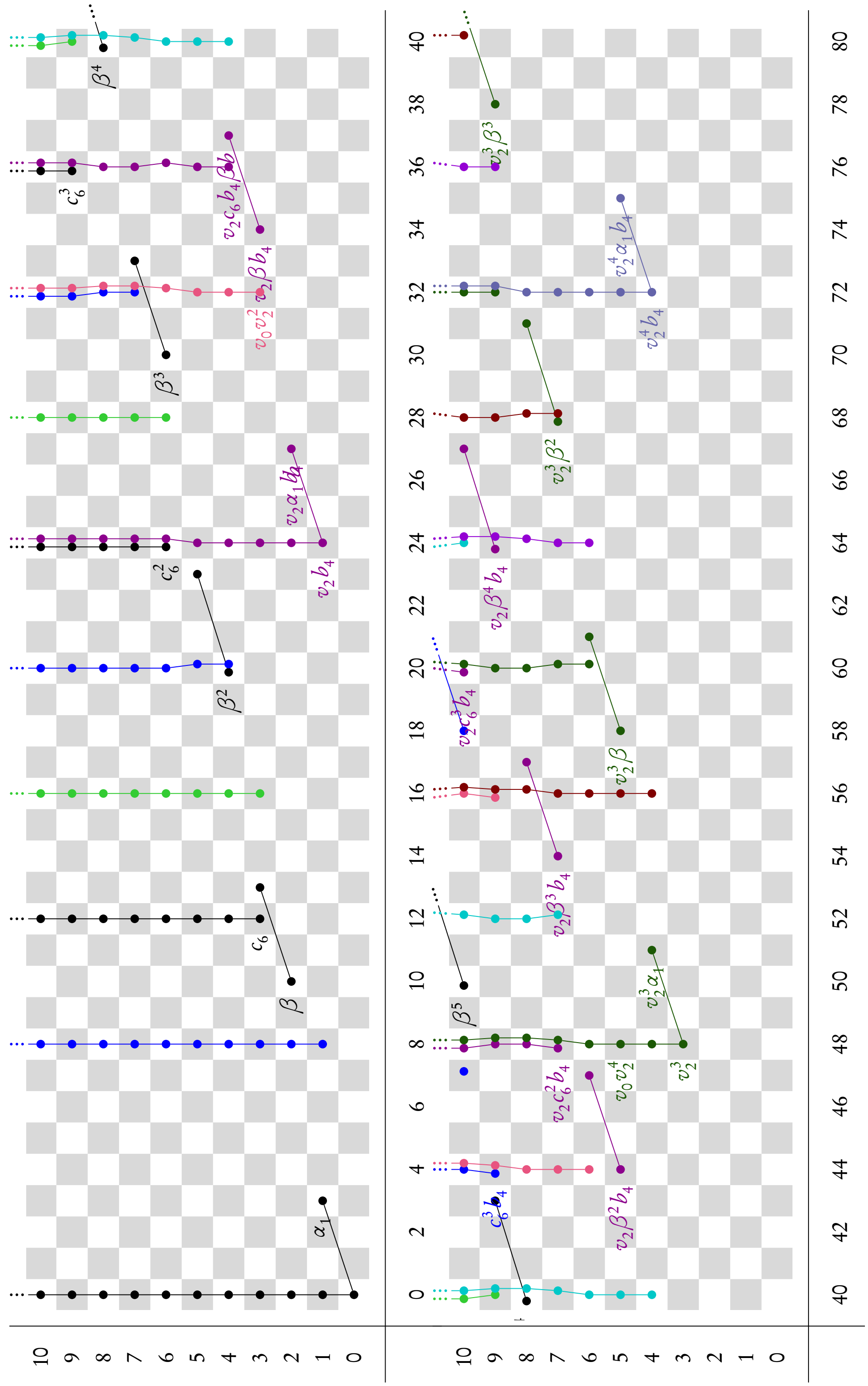
Remark 5.22. We will see that the classes v_2^{9j} detect the powers of Δ whose powers are congruent to 0 modulo 6, while the classes $b_4 v_2^{9j+4}$ detect the powers of Δ whose powers are congruent to 3 modulo 6.

At this point the Adams E_3-term isomorphic to "isomorphic" to the Adams-Novikov E_2-term but with te elements in the "wrong" filtrations. All of the later differentials correspond to the usual differentials in the Adams-Novikov spectral sequence, and in fact we could deduce them from that spectral sequence. However, we will try and provide several arguments from first principles below.

5.3. **Higher Adams differentials.** The Adams E_3-term for tmf is much sparser than the E_2-term. This greatly reduces the possiblity of higher Adams differentials. We will now determine the d_3 and d_4 differentials in the Adams spectral sequence for tmf. Recall that the unit map

$$S^0 \to \mathrm{tmf}$$

for tmf induces a map in Ext taking the class β to β and α_1 to α_1. This is used in the calculation [1] to derive higher Adams-Novikov differentials. We will also use it to derive higher Adams differentials.



(A) (B)

FIGURE 5.2. Adams E_3-page in stems 0-80

Proposition 5.23. *There is an Adams d_3-differential*

$$d_3(v_2^3) \doteq b_4 v_2 \beta^2 \alpha_1.$$

Proof. It is know that $\pi_{47}(\mathrm{tmf}) = 0$. The only non-zero class in that stem on the E_3-term is $b_4 v_2 \beta^2 \alpha_1$. This forces the stated differential. □

Remark 5.24. The author has made attempts to give an argument for this differential from first principles. But as of the writing of this short article, he has been unable to find one.

Proposition 5.25. *There is an Adams d_4-differential*

$$d_4(b_4 v_2) \doteq \beta^2 \alpha_1.$$

Proof. Since the classes α_1 and β are both in the Hurewicz image of tmf, so are all the monomials $\beta^n \alpha_1^{\varepsilon_1}$. In the stable homotopy of S^0, the class $\beta^3 \alpha_1$ is zero. Since the corresponding class in $E_3(\mathrm{tmf})$ is not zero, it must be hit by a differential. The only possible class which could support a differential to $\beta^3 \alpha_1$ is $b_4 v_2 \beta$. Thus we have the d_4-differential

$$d_4(b_4 v_2 \beta) \doteq \beta^3 \alpha_1.$$

As the Adams differentials are linear over β, we infer

$$d_4(b_4 v_2) \doteq \beta^2 \alpha_1.$$

□

Remark 5.26. Recall that $b_4 v_2$ corresponds to the class Δ in the relative Adams spectral sequence (or in the Adams-Novikov spectral sequence). In particular, this differential corresponds to the d_2-differential $d_2(\Delta) = \alpha \beta^2$ in [4]. In that spectral sequence, this implies the differential $d_2(\Delta^2) \doteq \Delta \alpha \beta^2$. However, in the ASS, the class $b_4 v_2$ squares to 0, so we cannot establish such a d_2-differential. Rather it corresponds to the d_3-differential we established in Proposition 5.23. It is interesting to note these d_2-differentials in the relative ASS get decoupled in the ASS.

Remark 5.27. One might like to think that there is the d_4-differential

$$d_4(b_4 v_2^4) \doteq v_2^3 \beta^2 \alpha_1,$$

because one can multiply the differential on $b_4 v_2$ to get this differential. But this is not the case since v_2^3 supports a shorter differential. This is an important occurrence in this spectral sequence because $b_4 v_2^4$ is detecting the class Δ^3 and the homotopy groups of tmf are famously periodic on Δ^3.

For degree reasons, these are the only possible d_3 and d_4 differentials. We will now produce the last differential in the Adams spectral sequence. In order to do that, we will need the following observation.

Lemma 5.28. *The class $b := b_4 v_2 \alpha_1$ is given on the E_5-page as the following Massey product*
$$b = \langle \beta^2, \alpha_1, \alpha_1 \rangle.$$
Thus, by Moss' Convergence Theorem, the class b in $\pi_*\mathrm{tmf}$ is given by the corresponding Toda bracket.

Proof. The differential $d_4(\Delta) = \beta^2\alpha_1$ gives a defining system for the Massey product on the E_5-page, and there is zero indeterminacy. Furthermore, the Toda bracket $\langle \beta^2, \alpha_1, \alpha_1 \rangle$ is defined. Since there are no differentials up to the 30 stem after the E_4-page, there are no crossing differentials to worry about. So by Moss' convergence theorem, b is given by the associated Toda bracket. □

Proposition 5.29 (compare with [1]). *There is the following hidden multiplicative extension in $\pi_*\mathrm{tmf}$,*
$$b \cdot \alpha_1 \doteq \beta^3.$$

Proof. Recall that β is given by the Toda bracket $\langle \alpha_1, \alpha_1, \alpha_1 \rangle$. So, by the first juggling lemma (cf. [8, Appendix 1]), it follows that
$$b \cdot \alpha_1 = \langle \beta^2, \alpha_1, \alpha_1 \rangle \alpha_1 = \beta^2 \langle \alpha_1, \alpha_1, \alpha_1 \rangle = \beta^3.$$
□

Corollary 5.30. *The class β^5 is 0 in the homotopy groups of* tmf*. Thus, there is a d_6-differential*
$$d_6(v_2^3\alpha_1) \doteq \beta^5.$$

Proof. Using the multiplicative extension of the previous proposition, we have
$$\beta^5 = \beta^2\beta^3 = \beta^2 b \alpha_1.$$
Since $\beta^2\alpha_1 = 0$, we have that $\beta^5 = 0$ in $\pi_*\mathrm{tmf}$. This forces the claimed differential. □

We can now make the following observation.

Lemma 5.31. *All differentials of length at least 3 established thus far respect the decomposition* (5.21)*. That is, any differentials which are propagated from any of the previous differentials have as its source and target elements within the same copy of M. Furthermore, it follows for degree reasons that the pattern M can support no other internal differentials.*

Proof. As we have mentioned earlier, the Adams differentials are linear over $\mathrm{Ext}_{A(1)_*}(\mathbb{F}_3)$. So this follows because the differentials on the basis elements respect the decomposition. □

The second part of this lemma asserts that any differentials of length at least 7 must be between *distinct* copies of M. Our next goal is to show that this cannot happen. Towards this end we will show that $b_4 v_2^2$ and v_2^9 are permanent cycles.

Lemma 5.32. *The class $b_4 v_2^4$ is a permanent cycle.*

Proof. The only possible differential that $b_4 v_2^4$ could have supported was a d_4-differential to $v_2^3 \beta \alpha_1$. But we have already shown that this class supports a d_6-differential. □

Now we move on to showing v_2^9 is a permanent cycle.

Proposition 5.33. *The class v_2^9 is a permanent cycle.*

Proof. Begin by noticing that v_2^9 is a d_3-cycle. So the smallest length differential it could support is one of length at least four. Observe that all of the v_0-torsion free classes are in even degrees. As v_2^9 is in an even degree, it follows that v_2^9 cannot support a differential into a v_0-torsion free class. Thus, if v_2^9 is to support a differential, the target must be v_0-torsion. We see in the E_3-page that all of the v_0-torsion is simple. Thus any differential supported by v_2^9 would have to have as its target a class of the form

$$b_4^{\varepsilon} v_2^j \beta^k \alpha_1^{\varepsilon'}$$

where j is congruent to 0 or 1 modulo 3. This just follows from our computation of the Adams E_3-term for tmf. More specifically, all the v_0-torsion is concentrated in Pattern 1'. The $(t-s,s)$-bidegree of such elements is

$$(12j + 10n + 3\varepsilon' + 8\varepsilon, j + 2n + \varepsilon').$$

An elementary number theory argument on the bidegrees shows that the only possible targets of a differential supported by v_2^9 is $\beta^{14}\alpha_1$ or $v_2^{10}\beta^2\alpha_1$. Either way, v_2^9 would have to support a differential of length at least 6 to hit either of these classes. However, from previous propositions, both of these classes either support or are hit by a differential of length at most 6. Thus v_2^9 cannot hit either of these classes, and hence must be a permanent cycle. □

We can now derive that the Adams spectral sequence for tmf collapses at E_7. Towards this end, let $\overline{M}$ denote subquotient obtained from M by incorporating the d_3 to d_6-differentials. Then we have the decomposition

$$E_7(\mathrm{tmf}) \cong \bigoplus_{k\geq 0} \overline{M} \cdot \{v_2^{9k}\} \oplus \bigoplus_{j \geq 0} \overline{M} \cdot \{b_4 v_2^{9j+4}\}. \tag{5.34}$$

Proposition 5.35. *The Adams spectral sequence for* tmf *collapses at E_7.*

Proof. The direct sum decomposition (5.34) arises from the patterns $\overline{M} \cdot \{1\}$ and $\overline{M} \cdot \{b_4 v_2^4\}$ and iteratively multiplying these two patterns by v_2^9. Since $b_4 v_2^4$ and v_2^9 are permanent cycles, and since differentials of length at least 7 must be between distinct copies of $\overline{M}$, it follows that there are no further differentials. Consequently, the Adams spectral sequence collapses at E_7. □

We provide the Adams E_3-term along with all higher differentials as well as a chart for the $E_7 = E_\infty$-term below in Figures 5.3 and 5.4

5.4. **Hidden extensions.** In the previous subsection we showed that the Adams spectral sequence for tmf collapses at E_7 and we completely computed this page via (5.34). We've already established one hidden extension in Proposition 5.29, which corresponds to the single hidden extension occurring in the Adams-Novikov spectral sequence for tmf.

However, there are several relations in $\pi_*\mathrm{tmf}$ which are apparent on the Adams-Novikov E_∞-page appearing in the 0-line, but which are hidden from the perspective of the Adams spectral sequence. We make several observations.

Lemma 5.36. *The class $v_0 b_4$ in E_∞ detects the class c_4 in $\pi_8\mathrm{tmf}$. Similarly, c_6 detects the class of the corresponding name in $\pi_{12}\mathrm{tmf}$*

Proof. These are the only classes in those degrees in the homotopy of tmf and in the E_∞-page of the ASS. □

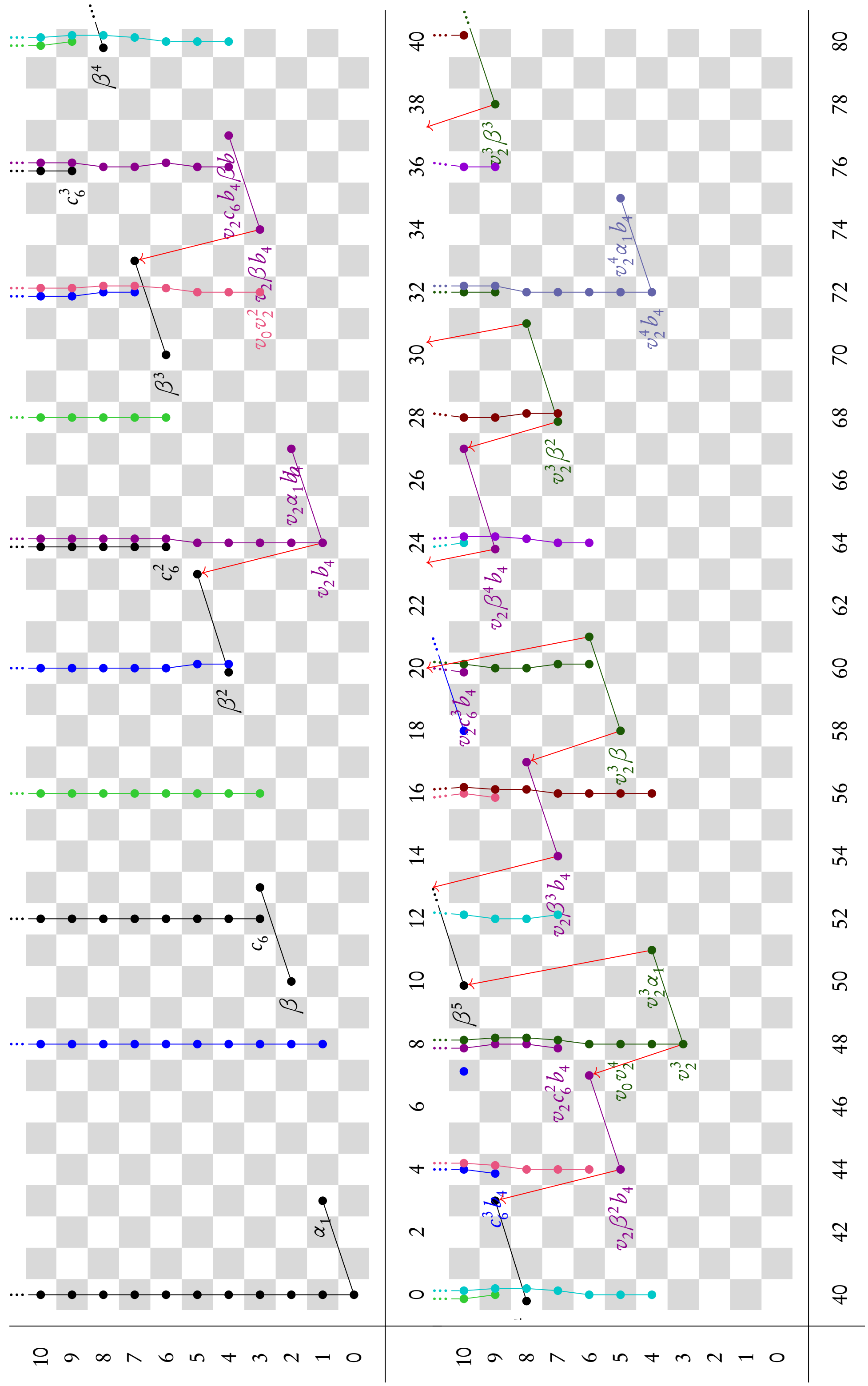
Looking at our chart for E_∞, we find that there is a single v_0-tower in the 16-stem which is generated by $v_0^2 v_2$. This implies the following,

Lemma 5.37. *The class $v_0^2 v_2$ detects the class c_4^2 in $\pi_{16}\mathrm{tmf}$ and we have the hidden extension $c_4 \cdot c_4 \doteq v_0^2 v_2$.*

We can also say which classes are detecting the various classes involving Δ in $\pi_*\mathrm{tmf}$. We will rename some classes in order to give more streamlined expressions. We will rename $b_4 v_2$ by $v_2^{3/2}$. Thus, for example, the class $v_2^{9/2}$ refers to $b_4 v_2^4$. We will use the expression $(v_2^{3/2})^k$, when $k = 2\ell$, to mean $v_2^{3\ell}$, while when $k = 2\ell + 1$ this expression stands for $b_4 v_2 \cdot v_2^{3\ell} = b_4 v_2^{3\ell+1}$. At the moment, we have introduced this notation more for convenience, it is *not* reflective of a multiplicative structure on any page of this spectral sequence. Indeed, on E_∞, the square of $b_4 v_2$ is 0. However, this notation is motivated by a certain hidden extension which will appear shortly.

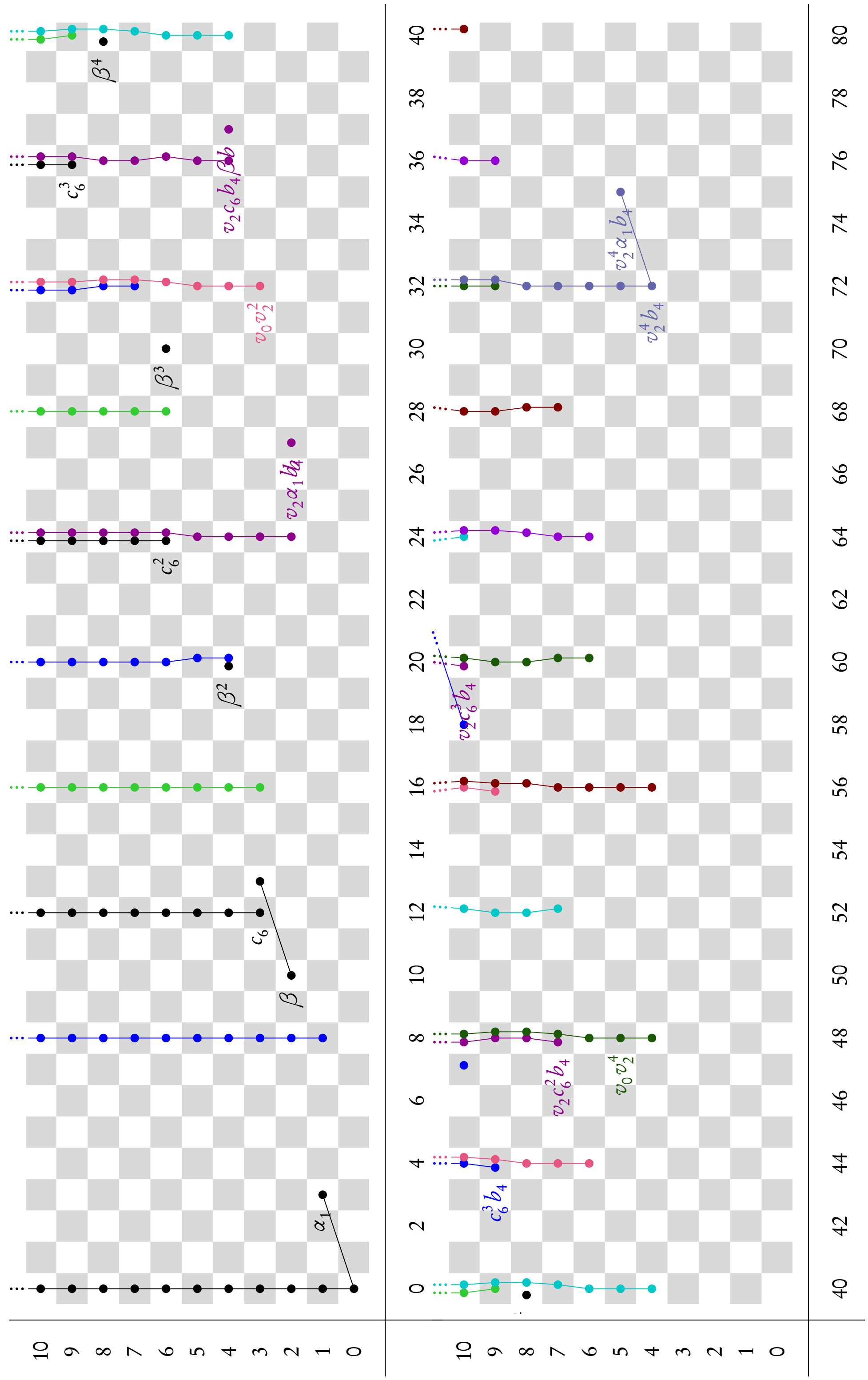
Note that from the results of the previous section, we have

Lemma 5.38. *When $j \equiv 1, 2 \mod 3$, the classes $(v_2^{3/2})^j$ support a differential, and in this case the classes $v_0(v_2^{3/2})^j$ are permanent cycles.*



(A) (B)

FIGURE 5.3. Adams E_3-page in stems 0-80 with d_3 to d_6-differentials



(A) (B)

FIGURE 5.4. Adams E_∞-page in stems 0-80

We can determine what these classes detect in $\pi_*(\mathrm{tmf})$.

Corollary 5.39. *For $j \equiv 1,2 \mod 3$, the classes $v_0(v_2^{3/2})^j$ detect $3\Delta^j$. For $j \equiv 0 \mod 3$, the class $(v_2^{3/2})^j$ detects Δ^j.*

Because these correspond to multiples of powers of Δ, this implies a family of hidden extensions.

Corollary 5.40. *In E_∞, we have the following hidden extensions for every $\ell \geq 0$,*

$$(v_0 v_2^{3/2}) \cdot v_0 (v_2^{3/2})^{2\ell+1} = v_0^2 v_2^{3\ell+3}$$

We have the hidden extensions for odd j

$$(v_2^{3/2})^3 \cdot (v_2^{3/2})^{3j} = (v_2^{3/2})^{3(j+1)}.$$

This corollary justifies our choice of notation. Finally, the theory of modular forms tells us that there is the famous relation

$$c_4^3 - c_6^2 = 1728\Delta = 2^3 3^3 \Delta.$$

This implies a hidden extension in the E_∞-term.

Lemma 5.41. *There is a hidden extension in $E_\infty(\mathrm{tmf})$ given by*

$$c_4 \cdot (v_0^2 v_2) \doteq v_0^3 b_4 v_2 + c_6^2$$

These hidden extensions, of course, propagate themselves throughout the E_∞-term. There are no hidden extensions beyond the ones mentioned above.

Remark 5.42. It is rather unsatisfying that the way these hidden extensions were determined by using the known multiplicative structure in $\pi_*\mathrm{tmf}$. It would be nice to have arguments from first principles. It would seem that this would require knowing Massey product descriptions of various classes, such as $c_4, v_2^{3/2}$, and so on. But the author was unable to find such descriptions.

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