통계수학1 과제#3 풀이

*연습문제 1.3 - 문제 6(d) (역함수만 구할 것, $x = f^{-1}(y)$꼴로 역함수를 표시할 것),
문제 11

6(d) $f(x) = y = \sqrt{4 - x^2},\ 0 \le x \le 2$
$x = f^{-1}(y) = \sqrt{4 - y^2},\ 0 \le y \le 2$

11 $f(x) = \frac{1}{8}x - 3$ and $g(x) = x^3 \Rightarrow f^{-1}(x) = 8(x+3)$ and $g^{-1}(x) = \sqrt[3]{x}$

(a) $(f^{-1} \circ g^{-1})(1) = f^{-1}(g^{-1}(1)) = f^{-1}(1) = 32$

(b) $(f^{-1} \circ f^{-1})(6) = f^{-1}(f^{-1}(6)) = f^{-1}(72) = 600$

*연습문제 1.4 - 문제 15 (f의 역함수만 구할 것, $x = f^{-1}(y)$꼴로 역함수를 표시할 것)

답:
$$\begin{aligned}
y &= \ln(x + \sqrt{x^2+1}) \\
e^y &= x + \sqrt{x^2+1} \\
(e^y - x)^2 &= x^2 + 1 \\
e^{2y} - 2xe^y + x^2 &= x^2 + 1 \\
2xe^y &= e^{2y} - 1 \\
x &= \frac{e^{2y}-1}{2e^y} \\
\therefore f^{-1}(y) &= \frac{e^{2y}-1}{2e^y}
\end{aligned}$$

*연습문제 1.6 - 문제 13

답: $\lim_{x\to 0}\frac{\sin x(1-\cos x)}{2x^2} = \lim_{x\to 0}\left[\frac{1}{2} \cdot \frac{\sin x}{x} \cdot \frac{1-\cos x}{x}\right] = \frac{1}{2}(1)(0) = 0$

*연습문제 1.7 - 문제 24 ($x = 0$ 에서 연속이 되기 위한 $f(0)$의 값만 제시할 것)

답:

$$f(x) = \frac{\sqrt{x+c^2} - c}{x},\ c > 0$$

Domain: $x + c^2 \ge 0 \Longrightarrow x \ge -c^2$ and $x \ne 0$, $[-c^2, 0) \cup (0, \infty)$

$$\begin{aligned}
\lim_{x\to 0}\frac{\sqrt{x+c^2}-c}{x} &= \lim_{x\to 0}\frac{\sqrt{x+c^2}-c}{x}\cdot\frac{\sqrt{x+c^2}+c}{\sqrt{x+c^2}+c} \\
&= \lim_{x\to 0}\frac{(x+c^2)-c^2}{x\left[\sqrt{x+c^2}+c\right]} = \lim_{x\to 0}\frac{1}{\sqrt{x+c^2}+c} = \frac{1}{2c}
\end{aligned}$$

Define $f(0) = 1/(2c)$ to make f continuous at $x = 0$.

*연습문제 2.1 - 문제 14(b) (미분가능성만 판단하여 제시), 문제 15

14(b) x=1에서 미분가능

$$f(x) = \begin{cases} (x-1)^3, & x \le 1 \\ (x-1)^2, & x > 1 \end{cases}$$

The derivative from the left is

$$\lim_{x\to1^-} \frac{f(x)-f(1)}{x-1} = \lim_{x\to1^-} \frac{(x-1)^3 - 0}{x-1} = \lim_{x\to1^-} (x-1)^2 = 0.$$

The derivative from the right is

$$\lim_{x\to1^+} \frac{f(x)-f(1)}{x-1} = \lim_{x\to1^+} \frac{(x-1)^2 - 0}{x-1} = \lim_{x\to1^+} (x-1) = 0.$$

These one-sided limits are equal. Therefore, f is differentiable at $x = 1$. $(f'(1) = 0)$

15 x=2에서 미분가능

Note that f is continuous at $x = 2$.

$$f(x) = \begin{cases} x^2+1, & x \le 2 \\ 4x-3, & x > 2 \end{cases}$$

The derivative from the left is

$$\lim_{x\to2^-} \frac{f(x)-f(2)}{x-2} = \lim_{x\to2^-} \frac{(x^2+1) - 5}{x-2} = \lim_{x\to2^-} (x+2) = 4.$$

The derivative from the right is

$$\lim_{x\to2^+} \frac{f(x)-f(2)}{x-2} = \lim_{x\to2^+} \frac{(4x-3) - 5}{x-2} = \lim_{x\to2^+} 4 = 4.$$

The one-sided limits are equal. Therefore, f is differentiable at $x = 2$, $(f'(2) = 4)$.

*연습문제 2.2 - 문제 3(b), 문제 18

3(b) $f(x) = \sqrt{x} - 6\sqrt[3]{x} = x^{1/2} - 6x^{1/3}$

$$f'(x) = \frac{1}{2}x^{-1/2} - 2x^{-2/3} = \frac{1}{2\sqrt{x}} - \frac{2}{x^{2/3}}$$

18 $y = 2x^2 - 3x + 1$

$y = ax^2 + bx + c$

Since the parabola passes through $(0, 1)$ and $(1, 0)$, we have:

$(0, 1)$: $1 = a(0)^2 + b(0) + c \implies c = 1$

$(1, 0)$: $0 = a(1)^2 + b(1) + 1 \implies b = -a - 1$

Thus, $y = ax^2 + (-a-1)x + 1$. From the tangent line $y = x - 1$, we know that the derivative is 1 at the point $(1, 0)$.

$y' = 2ax + (-a-1)$

$1 = 2a(1) + (-a-1)$

$1 = a - 1$

$a = 2$

$b = -a - 1 = -3$ Therefore, $y = 2x^2 - 3x + 1$.

*연습문제 2.3 - 문제 3(c), 문제 9

답:

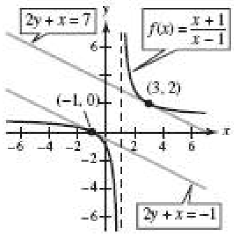
3(c) $f'(x) = \dfrac{2x-5}{2x\sqrt{x}}$

$$f(x) = \frac{2x+5}{\sqrt{x}} = 2x^{1/2} + 5x^{-1/2}$$

$$f'(x) = x^{-1/2} - \frac{5}{2}x^{-3/2} = x^{-3/2}\left[x - \frac{5}{2}\right]$$

$$= \frac{2x-5}{2x\sqrt{x}} = \frac{2x-5}{2x^{3/2}}$$

9 $y=-\frac{1}{2}x-\frac{1}{2}$, $y=-\frac{1}{2}x+\frac{7}{2}$



$$f(x) = \frac{x+1}{x-1}$$

$$f'(x) = \frac{(x-1)-(x+1)}{(x-1)^2} = \frac{-2}{(x-1)^2}$$

$$2y + x = 6 \Rightarrow y = -\frac{1}{2}x + 3; \text{ Slope: } -\frac{1}{2}$$

$$\frac{-2}{(x-1)^2} = -\frac{1}{2}$$

$$(x-1)^2 = 4$$

$$x - 1 = \pm 2$$

$$x = -1, 3; \quad f(-1) = 0, \quad f(3) = 2$$

$$y - 0 = -\frac{1}{2}(x+1) \Rightarrow y = -\frac{1}{2}x - \frac{1}{2}$$

$$y - 2 = -\frac{1}{2}(x-3) \Rightarrow y = -\frac{1}{2}x + \frac{7}{2}$$

*연습문제 2.4 – 문제 14(b)

답: $f''(0)=0$

$$f(x) = \cos(x^2), \quad (0, 1)$$

$$f'(x) = -\sin(x^2)(2x) = -2x\sin(x^2)$$

$$f''(x) = -2x\cos(x^2)(2x) - 2\sin(x^2)$$

$$= -4x^2\cos(x^2) - 2\sin(x^2)$$

$$f''(0) = 0$$

*연습문제 2.5 – 문제 5(b) (주어진 점에서 접선을 구할 것)

답: $y'=0$

$$(x^2+y^2)^2 = 4x^2y$$

$$2(x^2+y^2)(2x+2yy') = 4x^2y' + y(8x)$$

$$4x^3 + 4x^2yy' + 4xy^2 + 4y^3y' = 4x^2y' + 8xy$$

$$4x^2yy' + 4y^3y' - 4x^2y' = 8xy - 4x^3 - 4xy^2$$

$$4y'(x^2y + y^3 - x^2) = 4(2xy - x^3 - xy^2)$$

$$y' = \frac{2xy - x^3 - xy^2}{x^2y + y^3 - x^2}$$

At $(1, 1)$: $y' = 0$

*연습문제 2.6 – 문제 1, 문제 6

답:

1(a) $\frac{1}{5}$

$f(x) = x^3 + 2x - 1,\ f(1) = 2 = a$

$f'(x) = 3x^2 + 2$

$$(f^{-1})'(2) = \frac{1}{f'(f^{-1}(2))} = \frac{1}{f'(1)} = \frac{1}{3(1)^2 + 2} = \frac{1}{5}$$

(b) $\frac{2\sqrt{3}}{3}$

$f(x) = \sin x, f\left(\frac{\pi}{6}\right) = \frac{1}{2} = a$

$f'(x) = \cos x$

$$(f^{-1})'\left(\frac{1}{2}\right) = \frac{1}{f'(f^{-1}(1/2))} = \frac{1}{f'(\pi/6)} = \frac{1}{\cos(\pi/6)} = \frac{1}{\sqrt{3}/2} = \frac{2\sqrt{3}}{3}$$

6 $y = \frac{4\sqrt{3}}{3}x + \frac{\pi}{3} - \frac{2\sqrt{3}}{3}$

$y = 2\arcsin x,\quad \left(\frac{1}{2}, \frac{\pi}{3}\right)$

$$y' = \frac{2}{\sqrt{1 - x^2}}$$

At $\left(\frac{1}{2}, \frac{\pi}{3}\right)$, $y' = \frac{2}{\sqrt{1 - (1/4)}} = \frac{4}{\sqrt{3}}$.

Tangent line: $y - \frac{\pi}{3} = \frac{4}{\sqrt{3}}\left(x - \frac{1}{2}\right)$

$$y = \frac{4}{\sqrt{3}}x + \frac{\pi}{3} - \frac{2}{\sqrt{3}}$$

$$y = \frac{4\sqrt{3}}{3}x + \frac{\pi}{3} - \frac{2\sqrt{3}}{3}$$

*연습문제 3.1 – 문제 4(c)

답: Max (−1,5) Min (0,0)

$y = 3x^{2/3} - 2x, [-1, 1]$

$$y' = 2x^{-1/3} - 2 = \frac{2(1 - \sqrt[3]{x})}{\sqrt[3]{x}}$$

Left endpoint: $(-1, 5)$ Maximum

Critical number: $(0, 0)$ Minimum

Right endpoint: $(1, 1)$