[6장 - 4절]

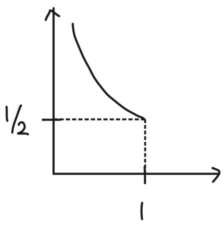
#4

$f(x;\theta) = \theta x^{\theta-1}$, $0<x<1$, $\theta \in \Omega = \{\theta : 0<\theta<\infty\}$

(a)

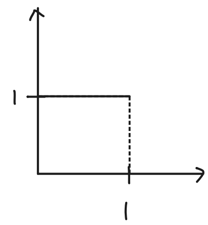
i) $\theta = 1/2$

$f(x;1/2) = 1/2\, x^{-1/2}$, $0<x<1$



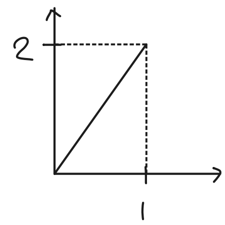
ii) $\theta = 1$

$f(x;1) = x^0 = 1$, $0<x<1$



iii) $\theta = 2$

$f(x;2) = 2x$, $0<x<1$



(b)

$L(\theta) = \theta^n \left(\prod_{i=1}^{n} x_i\right)^{\theta-1}$ $\quad$ $l(\theta) = n\ln\theta + (\theta-1)\sum_{i=1}^{n} x_i$

$l(\theta) = n\ln\theta + (\theta-1)\ln\left(\prod_{i=1}^{n} x_i\right)$

$\frac{d}{d\theta}\ln(\theta) = n\cdot\frac{1}{\theta} + \ln\left(\prod_{i=1}^{n} x_i\right) \overset{\text{PUt}}{=} 0 \rightarrow \theta = \frac{-n}{\ln\left(\prod_{i=1}^{n} x_i\right)}$

$\therefore \hat{\theta}^{MLE} = -\frac{n}{\ln\left(\prod_{i=1}^{n} x_i\right)}$

(c)

$E(X) = \int_0^1 x\cdot\theta x^{\theta-1}\,dx = \int_0^1 \theta x^{\theta}\,dx = \left[\frac{\theta}{\theta+1}x^{\theta+1}\right]_0^1 = \frac{\theta}{\theta+1}$

$\frac{\theta}{\theta+1} = \bar{X}$ $\qquad$ $\hat{\theta}^{MME} = \frac{\bar{X}}{1-\bar{X}}$

i) $\hat{\theta}^{MLE} = -\frac{10}{\ln(0.0256\times\cdots\times 0.0102)} = 0.55$

$\hat{\theta}^{MME} = \frac{0.37401}{1-0.37461} = 0.597$

ii) $\hat{\theta}^{MLE} = -\frac{10}{\ln(0.9960\times\cdots\times 0.8609)} = 2.21$

$\hat{\theta}^{MME} = \frac{0.70592}{1-0.70592} = 2.4$

iii) $\hat{\theta}^{MLE} = -\frac{10}{\ln(0.4698\times\cdots\times 0.2154)} = 0.96$

$\hat{\theta}^{MME} = \frac{0.46368}{1-0.46368} = 0.865$

#5

$f(x;\theta) = \frac{1}{\theta}\cdot e^{(-\frac{x}{\theta})}$, $0<x<\infty$, $0<\theta<\infty$

(a)

$$E(\bar{X}) = E\left(\frac{\sum_{i=1}^{n} X_i}{n}\right)$$

$$= \frac{\sum_{i=1}^{n} \underline{E(X_i)}^{\to \theta}}{n}$$

$$= \theta$$

$\therefore \bar{X}$ is unbiased estimator of θ

(b)

$$Var(X) = E(X^2) - \{E(X)\}^2$$

$$E(X^2) = \int_0^\infty x^2 \frac{1}{\theta} e^{(-\frac{x}{\theta})} dx$$

$$= \left[-x^2 \cdot e^{-\frac{x}{\theta}}\right]_0^\infty + \int_0^\infty e^{-\frac{x}{\theta}} \cdot 2x \, dx = 0 + 2\theta E(X) = 2\theta^2$$

$$Var(X) = 2\theta^2 - \theta^2 = \theta^2$$

$$Var(\bar{X}) = Var\left(\frac{\sum_{i=1}^{n} X_i}{n}\right) = \frac{\sum_{i=1}^{n} Var(X_i)}{n^2}$$

$$= \frac{n\theta^2}{n^2} = \frac{\theta^2}{n}$$

(c)

$$\bar{X} = \frac{3.5+8.1+0.9+4.4+0.5}{5} = 3.48$$

#6

$$E(S^2) = E\left[\frac{1}{n-1}\left(\sum_{i=1}^{n} X_i^2 - n\bar{X}^2\right)\right]$$

$$= \frac{1}{n-1}\left[\sum_{i=1}^{n} E(X_i^2) - nE(\bar{X}^2)\right]$$

$$\Rightarrow E(X^2) = Var(X) + \{E(X)\}^2$$

$$= \sigma^2 + \mu^2$$

$$\Rightarrow E(\bar{X}^2) = Var(\bar{X}) + \{E(\bar{X})\}^2$$

$$= \sigma^2/n + \mu^2$$

$$E(S^2) = \frac{1}{n-1}\left[n(\sigma^2+\mu^2) - \sigma^2 - n\mu^2\right]$$

$$= \frac{1}{n-1}\cdot\sigma^2(n-1) = \sigma^2$$

#10

(a)

$$E(S) = E\left(\sqrt{\frac{\sum_{i=1}^{n}(X_i-\bar{X})^2}{n-1}}\right)$$

$$= \sqrt{\frac{\sigma^2}{n-1}}\; E\left(\sqrt{\frac{\sum_{i=1}^{n}(X_i-\bar{X})^2}{\sigma^2}}\right)$$

$$= \sqrt{\frac{\sigma^2}{n-1}}\; E\left(\sqrt{\underbrace{\frac{(n-1)S^2}{\sigma^2}}_{\sim \chi^2(n-1)}}\right)$$

$$= \sqrt{\frac{\sigma^2}{n-1}} \int_0^\infty \sqrt{x}\cdot \frac{1}{\Gamma\left(\frac{n-1}{2}\right)\cdot 2^{\frac{n-1}{2}}} \cdot x^{\frac{n-1}{2}-1} \cdot e^{-\frac{x}{2}}\,dx$$

$$= \sqrt{\frac{\sigma^2}{n-1}} \int_0^\infty \frac{1}{\Gamma\left(\frac{n-1}{2}\right)\cdot 2^{\frac{n-1}{2}}} \cdot x^{\frac{n}{2}-1} \cdot e^{-\frac{x}{2}}\,dx$$

$$= \sqrt{\frac{\sigma^2}{n-1}} \cdot \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} \int_0^\infty \frac{1}{\Gamma\left(\frac{n}{2}\right)\cdot 2^{\frac{n-1}{2}}} \cdot x^{\frac{n}{2}-1} \cdot e^{-\frac{x}{2}}\,dx$$

$$= \sqrt{\frac{\sigma^2}{n-1}} \cdot \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} \cdot 2^{\frac{1}{2}} \underbrace{\int_0^\infty \frac{1}{\Gamma\left(\frac{n}{2}\right)\cdot 2^{\frac{n}{2}}} \cdot x^{\frac{n}{2}-1} \cdot e^{-\frac{x}{2}}\,dx}_{1}$$

$$= \sqrt{\frac{\sigma^2}{n-1}} \cdot \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} \cdot \sqrt{2}$$

$$= \sigma \frac{\sqrt{2}}{\sqrt{n-1}} \cdot \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} \qquad \left(C = \frac{\sqrt{n-1}}{\sqrt{2}} \cdot \frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} \right)$$

$$E(S) = \frac{\sigma}{C}$$

$$E(CS) = \sigma$$

(b)

i) $n=5$

$$C = \frac{2}{\sqrt{2}} \cdot \frac{\Gamma(2)}{\Gamma\left(\frac{5}{2}\right)} = \frac{2}{\sqrt{2}} \cdot \frac{4}{3\sqrt{\pi}} = 1.064$$

ii) $n=6$

$$C = \frac{\sqrt{5}}{\sqrt{2}} \cdot \frac{\Gamma\left(\frac{5}{2}\right)}{\Gamma(3)} = \sqrt{\frac{5}{2}} \cdot \frac{3\sqrt{\pi}/4}{2} = \frac{3\sqrt{5\pi}}{8\sqrt{2}} = 1.051$$

A1

(a)

$\sigma^2 = \theta$

$$f(x) = \frac{1}{\sqrt{2\pi\theta}} \exp\left(-\frac{(x-\mu_0)^2}{2\theta}\right)$$

$$L(\theta) = \left(\frac{1}{\sqrt{2\pi\theta}}\right)^n \exp\left(-\frac{\sum_{i=1}^{n}(x_i-\mu_0)^2}{2\theta}\right)$$

$$l(\theta) = -\frac{n}{2}\ln(2\pi\theta) - \frac{\sum_{i=1}^{n}(x_i-\mu_0)^2}{2\theta}$$

$$\frac{d}{\partial\theta} l(\theta) = -\frac{n}{2}\cdot\frac{1}{\theta} + \frac{\sum_{i=1}^{n}(x_i-\mu_0)^2}{2\theta^2} \overset{\text{Put}}{=} 0$$

$$-n\theta + \sum_{i=1}^{n}(x_i-\mu_0)^2 = 0$$

$$\theta = \frac{\sum_{i=1}^{n}(x_i-\mu_0)^2}{n}$$

$$\therefore \hat{\sigma}^2 = \frac{\sum_{i=1}^{n}(x_i-\mu_0)^2}{n} \quad (\sigma^2 = \theta)$$

(b)

$$E(\hat{\sigma}^2) = E\left(\frac{1}{n}\sum_{i=1}^{n}(x_i-\mu_0)^2\right)$$

$$= E\left(\frac{\sigma^2}{n}\cdot\frac{\sum_{i=1}^{n}(x_i-\mu_0)^2}{\sigma^2}\right) \qquad \frac{\sum_{i=1}^{n}(x_i-\mu_0)^2}{\sigma^2} \sim \chi^2(n)$$

$$= \frac{\sigma^2}{n}\cdot E\left(\frac{\sum_{i=1}^{n}(x_i-\mu_0)^2}{\sigma^2}\right)$$

$$= \frac{\sigma^2}{n}\cdot n = \sigma^2$$

A2

$$f(x;p) = (1-p)^x p$$

$$L(\theta) = \prod_{i=1}^{n}(1-p)^{x_i}\cdot p$$

$$= (1-p)^{\sum_{i=1}^{n}x_i} p^n$$

$$l(\theta) = \sum_{i=1}^{n}x_i \ln(1-p) + n\ln p$$

$$\frac{d}{d\theta} l(\theta) = \frac{-\sum_{i=1}^{n}x_i}{1-p} + \frac{n}{p} \overset{\text{Put}}{=} 0$$

$$\frac{n}{p} = \frac{\sum_{i=1}^{n}x_i}{1-p}$$

$$p\left(\sum_{i=1}^{n}x_i + n\right) = n$$

$$\hat{p}^{MLE} = \frac{n}{\sum_{i=1}^{n}x_i + n}$$

#A3

1)

$$E(X) = \int_0^{\theta} x \cdot \frac{1}{\theta}\, dx = \frac{1}{\theta} \cdot \frac{1}{2} \left[x^2 \right|_0^{\theta}$$

$$= \frac{1}{2\theta} \cdot \theta^2 = \frac{\theta}{2}$$

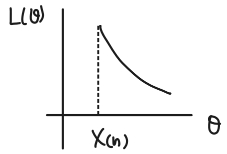
$$\frac{\theta}{2} = \overline{X}$$

$$\tilde{\theta} = 2\overline{X}$$

2)

$$L(\theta) = \left(\frac{1}{\theta}\right)^n I(0 \le X_i \le \theta)$$

$$= \left(\frac{1}{\theta}\right)^n I(0 \le X_{(n)} \le \theta)$$



$\therefore \hat{\theta} = X_{(n)}$

$$E(X) = \frac{\theta - 0}{2}$$

3)

$$E(\tilde{\theta}) = E(2\overline{X}) = 2E(\overline{X}) = 2 \cdot \frac{\theta}{2} = \theta$$

$\therefore$ $\tilde{\theta}$ is an unbiased estimator of θ

4)

i) $\hat{\theta} = X_{(n)}$의 확률밀도함수

$X_{(n)} = Y$

$$F_Y(y) = P(Y \le y) = \{P(X_i \le y)\}^n$$

$$F_Y(y) = \begin{cases} 0 & y \le 0 \\ (y/\theta)^n & 0 < y < \theta \\ 1 & y \ge \theta \end{cases}$$

$$f_Y(y) = \begin{cases} \frac{n}{\theta^n} \cdot y^{n-1} & 0 < y < \theta \\ 0 & o.w \end{cases}$$

ii)

$$E(\hat{\theta}) = E(X_{(n)}) = E(Y)$$

$$= \int_0^{\theta} \frac{n}{\theta^n} \cdot y^n\, dy$$

$$= \frac{n}{\theta^n} \cdot \left[\frac{1}{n+1} y^{n+1} \right|_0^{\theta}$$

$$= \frac{n}{\theta^n} \cdot \frac{1}{n+1} \cdot \theta^{n+1}$$

$$= \frac{n}{n+1} \theta$$

$\therefore$ $\hat{\theta}$ is biased estimator of θ