Theorem 1.1.4

For A, $\quad P(A) \leq 1$

Proof 1.1.4: Theorem

Since $A \subset \mathcal{S}$, using the result of the Theorem 1.1.3,

$$P(A) \leq P(\mathcal{S}) = 1$$

□

Theorem 1.1.5

For any event $A\&B$,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Proof 1.1.5: Theorem

Note that, $A \cup B = A \cup (A^c \cap B)$ with $A \cap (A^c \cap B) = \phi$ implies

$$P(A \cup B) = P(A) + P(A^c \cap B) \tag{1.1}$$

$B = (A \cap B) \cup (A^c \cap B)$ with $(A \cap B) \cap (A^c \cap B) = \phi$ implies

$$P(B) = P(A \cap B) + P(A^c \cap B) \tag{1.2}$$

by combining (1.1) and (1.2) we get the result. □

Theorem 1.1.6

Inclusion-Exclusion (포함-배제 공식) :

$$P(C_1 \cup C_2 \cup C_3 \cup \cdots \cup C_k) = p_1 - p_2 + p_3 - \cdots + (-1)^{k+1} p_k \tag{1.3}$$

where,

$$\begin{aligned} p_1 &= P(C_1) + P(C_2) + \cdots + P(C_k) \\ p_2 &= P(C_1 \cap C_2) + P(C_1 \cap C_3) + \cdots P(C_{k-1} \cap C_k) \\ &\vdots \\ p_k &= P(C_1 \cap C_2 \cap \cdots \cap C_k) \end{aligned}$$

Proof 1.1.6: Theorem

Note that (1.3) holds when $k = 1, 2$

Let's assume that (1.3) holds when $k = n - 1$ using mathematical induction. That is,

$$P\left(\bigcup_{i=1}^{n-1} C_i\right) = \sum_{i=1}^{n-1}(-1)^{i+1}a_i \tag{1.4}$$

where,

$$\begin{aligned}
a_1 &= P(C_1) + P(C_2) + \cdots + P(C_{n-1}) \\
a_2 &= P(C_1 \cap C_2) + P(C_1 \cap C_3) + \cdots P(C_{n-2} \cap C_{n-1}) \\
&\vdots \\
a_{n-1} &= P(C_1 \cap C_2 \cap \cdots \cap C_{n-1})
\end{aligned}$$

Define $D_i = C_i \cap C_n$, $i = 1, \cdots, n-1$ then

$$P(D_i) = P(C_i \cap C_n)$$

,

$$P(D_i \cap D_j) = P(C_i \cap C_j \cap C_n)$$
$$P(D_i \cap D_j \cap D_k) = P(C_i \cap C_j \cap C_k \cap C_n)$$

and so on.

Since we assume that (1.3) holds when $k = n - 1$,

$$P\left(\bigcup_{i=1}^{n-1}\left(C_i \bigcap C_n\right)\right) = P\left(\bigcup_{i=1}^{n-1} D_i\right) = \sum_{i=1}^{n-1}(-1)^{i+1}b_i \tag{1.5}$$

where,

$$\begin{aligned}
b_1 &= P(D_1) + P(D_2) + \cdots + P(D_{n-1}) \\
b_2 &= P(D_1 \cap D_2) + P(D_1 \cap D_3) + \cdots P(D_{n-2} \cap D_{n-1}) \\
&\vdots \\
b_{n-1} &= P(D_1 \cap D_2 \cap \cdots \cap D_{n-1})
\end{aligned}$$

Then, for $k = n$ in (1.3)

$$\begin{aligned}
P\left(\bigcup_{i=1}^{n} C_i\right) &= P\left(\bigcup_{i=1}^{n-1} C_i \bigcup C_n\right) \\
&= P\left(\bigcup_{i=1}^{n-1} C_i\right) + P(C_n) - P\left(\left(\bigcup_{i=1}^{n-1} C_i\right) \bigcap C_n\right) \\
&= P\left(\bigcup_{i=1}^{n-1} C_i\right) + P(C_n) - P\left(\bigcup_{i=1}^{n-1}\left(C_i \bigcap C_n\right)\right) \\
&= \sum_{i=1}^{n}(-1)^{i+1}p_i
\end{aligned} \tag{1.6}$$

where,

$$p_1 = P(C_1) + P(C_2) + \cdots + P(C_n)$$
$$p_2 = P(C_1 \cap C_2) + P(C_1 \cap C_3) + \cdots P(C_{n-1} \cap C_n)$$
$$\vdots$$
$$p_n = P(C_1 \cap C_2 \cap \cdots \cap C_n)$$

□

Definition 1.1.14 (단조 집합열 (Monotone sequence of events)). *A*

- *비감소 (nondecreasing) 집합열: $n = 1, 2, \cdots$ 에 대하여 $A_n \subset A_{n+1}$ 을 만족하는 집합열.*
- *감소 (decreasing) 집합열: $n = 1, 2, \cdots$ 에 대하여 $A_{n+1} \subset A_n$ 을 만족하는 집합열.*

Theorem 1.1.7

•$\{C_n\}$ 이 비감소 (nondecreasing) 집합열인 경우

$$\lim_{n\to\infty} P(C_n) = P(\lim_{n\to\infty} C_n) = P(\cup_{n=1}^{\infty} C_n) \tag{1.7}$$

•$\{D_n\}$ 이 감소 (decreasing) 집합열인 경우

$$\lim_{n\to\infty} P(D_n) = P(\lim_{n\to\infty} D_n) = P(\cap_{n=1}^{\infty} D_n) \tag{1.8}$$

Proof 1.1.7: Theorem

• Proof for the (1.7):

Let $R_1 = C_1$ and $R_n = C_n \cap C_{n-1}^c$. 따라서 $\cup_{n=1}^{\infty} C_n = \cup_{n=1}^{\infty} R_n$ 이며 $R_n \cap R_m = \phi$ for $n \neq m$. 따라서

$$\begin{aligned} P(\lim_{n\to\infty} C_n) = P\left(\bigcup_{n=1}^{\infty} C_n\right) &= P\left(\bigcup_{n=1}^{\infty} R_n\right) = \sum_{n=1}^{\infty} P(R_n) = \lim_{n\to\infty} \sum_{i=1}^{n} P(R_i) \\ &= \lim_{n\to\infty} \left\{P(C_1) + \sum_{j=2}^{n} [P(C_j) - P(C_{j-1})]\right\} = \lim_{n\to\infty} P(C_n) \end{aligned}$$

• Proof for the (1.8):(By your self)

□

- 두 가지 중요한 확률 부등식 (**Boole's inequality, Bonferroni's inequality**)

Theorem 1.1.8: Boole's inequality (확률의 합과 합집합 확률의 관계)

Let $\{A_n\}$ be an arbitrary sequence of events, then

$$\mathbb{P}\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} P\left(A_n\right) \tag{1.9}$$

Proof 1.1.8: Theorem

Pf): Let $B_n = \bigcup_{i=1}^{n} A_i$. Then $\{B_n\}$ is an nondecreasing sequence of events that go up to $\bigcup_{n=1}^{\infty} B_n$. Also, for all $j, B_j = B_{j-1} \cup A_j$. Hence,

$$P\left(B_j\right) \leq P\left(B_{j-1}\right) + P\left(A_j\right)$$

Hence, using the above inequality and the fact that $P\left(B_1\right) = P\left(A_1\right)$, we have

Th1.1.7 $\longrightarrow \lim_{n\to\infty} P(B_n)$

$$P\left(\bigcup_{n=1}^{\infty} A_n\right) = P\left(\bigcup_{n=1}^{\infty} B_n\right) = \lim_{n\to\infty}\left\{P\left(B_1\right) + \sum_{j=2}^{n}\left[P\left(B_j\right) - P\left(B_{j-1}\right)\right]\right\}$$

$$\leq \lim_{n\to\infty} \sum_{j=1}^{n} P\left(A_j\right) = \sum_{n=1}^{\infty} P\left(A_n\right)$$

$P(B_1)$
$+ P(B_2) - P(B_1)$
$+ P(B_3) - P(B_2)$
$\vdots$
$P(B_n) - P(B_{n-1})$

$= P(B_n)$

$$P\left(\bigcup_{i=1}^{\infty} B_n\right) = \lim_{n\to\infty} P(B_n)$$

$$= \lim_{n\to\infty}\Big(\underset{\substack{\parallel \\ P(A_1)}}{P(B_1)} + \sum_{j=2}^{n} \underbrace{P(B_j) - P(B_{j-1})}_{\leq P(A_j)}\Big)$$

$$= \lim_{n\to\infty} \sum_{j=1}^{n} P(A_j)$$

□

Theorem 1.1.9: Bonferroni's inequality (확률의 합과 교집합의 관계)

Let $\{A_n\}$ be an arbitrary sequence of events. Then for any $k \in \{1, 2, \cdots\} = \mathbb{N}$,

$$\mathbb{P}\left(\bigcap_{i=1}^{k} A_i\right) \geq 1 - k + \sum_{i=1}^{k} \mathbb{P}(A_i) \tag{1.10}$$

Proof 1.1.9: Theorem

Pf): The inequality can be proven by induction as follows For $k = 1$:

$$\mathbb{P}(A_1) \geq 1 - 1 + \mathbb{P}(A_1) = \mathbb{P}(A_1)$$

which is always true. Assume the inequality is true for some k, i.e.

$$\mathbb{P}\left(\bigcap_{i=1}^{k} A_i\right) \geq 1 - k + \sum_{i=1}^{k} \mathbb{P}(A_i)$$

Consider $\mathbb{P}\left(\bigcap_{i=1}^{k+1} A_i\right)$

$$\begin{aligned}
\mathbb{P}\left(\bigcap_{i=1}^{k+1} A_i\right) &= \mathbb{P}\left[\left(\bigcap_{i=1}^{k} A_i\right) \cap A_{k+1}\right] \\
&= \mathbb{P}\left(\bigcap_{i=1}^{k} A_i\right) + \mathbb{P}(A_{k+1}) - \mathbb{P}\left[\left(\bigcap_{i=1}^{k} A_i\right) \cup A_{k+1}\right]
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathbb{P}\left(\bigcap_{i=1}^{k+1} A_i\right) &\geq 1 - k + \sum_{i=1}^{k} \mathbb{P}(A_i) + \mathbb{P}(A_{k+1}) - \mathbb{P}\left[\left(\bigcap_{i=1}^{k} A_i\right) \cup A_{k+1}\right] \\
&\geq 1 - k + \sum_{i=1}^{k+1} \mathbb{P}(A_i) - \mathbb{P}\left[\left(\bigcap_{i=1}^{k} A_i\right) \cup A_{k+1}\right]
\end{aligned}$$

From the definition, the last term cannot exceed 1 so

$$\begin{aligned}
\mathbb{P}\left(\bigcap_{i=1}^{k+1} A_i\right) &\geq 1 - k + \sum_{i=1}^{k+1} \mathbb{P}(A_i) - 1 \\
&\geq 1 - (k+1) + \sum_{i=1}^{k+1} \mathbb{P}(A_i)
\end{aligned}$$

This means that if the statement is true for k, it is also true for $k+1$. By induction, we have proven the Bonferroni's inequality. □