<수리통계학Ⅱ> 6장/5장[점근분포] 과제

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▶ 교재 연습문제 : #6.6-2, #5.8-1, #5.8-3, #5.9-2 &

[A1] $X_1, X_2, \cdots, X_n$ 이 확률밀도함수(pdf)가 $f(x;\theta) = \theta x^{\theta-1}, 0 < x < 1$ 인 모집단에서 추출된 확률표본일 때,

1) $\hat{\theta} = -n/\sum_{i=1}^{n} \ln X_i$ 이 MLE(최대가능도추정량)임을 보여라

2) MLE $\hat{\theta} = -n/\sum_{i=1}^{n} \ln X_i$ 이 비편향추정량임을 보여라.

3) $n \to \infty$ 일 때, MLE $\hat{\theta} = -n/\sum_{i=1}^{n} \ln X_i$ 의 점근분포(asypmtotic dist.)를 구하라.

$$\rightarrow \ln f(\underline{x};\theta) = -n \ln\sqrt{2\pi\sigma^2} - \Sigma \frac{(x_i-\mu)^2}{2\sigma^2}$$

$$= -\frac{n}{2}\ln 2\pi\theta - \Sigma\frac{(x_i-\mu)^2}{2\theta}$$

$$\rightarrow \frac{\partial}{\partial\theta}\ln f(\underline{x};\theta) = -\frac{n}{2\theta} - \Sigma\frac{(x_i-\mu)^2}{2}\cdot\frac{1}{\theta^2} = -\frac{n}{2\theta} - \frac{n}{2\theta} = -\frac{n}{\theta}$$

$$\rightarrow \frac{\partial^2}{\partial\theta^2}\ln f(\underline{x};\theta) = \frac{n}{\theta^2}$$

수리통계학II(002) 과제6 (6장/5장 점근분포) 소프트웨어학부 컴퓨터과학전공 2016133 이유진

#6.6-2

$X_1 \cdots X_n$ r.s from $N(\mu, \sigma^2=\theta) \rightarrow \theta=\sigma^2$의 최우추정량 $\hat{\theta} = \sum \frac{(X_i-\mu)^2}{n}$

(a) $X_1 \cdots X_n$의 fisher information을 구한다.

정규분포 $f(x;\theta) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right] \rightarrow \ln f(x;\theta) = -\frac{1}{2}\ln 2\pi\sigma^2 - \frac{(x-\mu)^2}{2\sigma^2} = -\frac{1}{2}\ln 2\pi\theta - \frac{(x-\mu)^2}{2\theta}$

$\rightarrow \frac{\partial}{\partial\theta}\ln f(x;\theta) = \frac{(x-\mu)^2-\theta}{2\theta^2} \rightarrow \frac{\partial^2}{\partial\theta^2}\ln f(x;\theta) = \frac{\theta-2(x-\mu)^2}{2\theta^3}$

$\sum \frac{(X_i-\mu)^2}{n} = \theta$인 것을 상기하면 $E\left[\frac{\theta-2(X-\mu)^2}{2\theta^3}\right] = -\frac{1}{2\theta^2} \rightarrow I_n(\theta) = -nE\left[\frac{\partial^2}{\partial\theta^2}\ln f(x;\theta)\right] = \frac{n}{2\theta^2}$

$\therefore$ 라오-크래머 하한값은 $\frac{1}{I_n(\theta)} = \frac{2\theta^2}{n}$

(b) $\hat{\theta} \approx N(\theta, \frac{2\theta^2}{n})$

(c) $\theta=\sigma^2$인 경우 식을 정리하면 $\frac{n\hat{\theta}}{\theta} = \frac{n \cdot \Sigma(X_i-\mu)^2}{n \cdot \sigma^2} = \frac{\Sigma(X_i-\mu)^2}{\sigma^2}$

$\rightarrow X \sim N(\mu, \sigma^2)$ 일때 $\frac{(X-\mu)^2}{\sigma^2} \sim \chi^2(1)$이므로 $\frac{n\hat{\theta}}{\theta} \sim \chi^2(n)$ 이다 ($\because$ χ^2분포의 가법성)

#5.8-1

(a) $P(23<X<43) = P(|X-33|<10) = P(|X-33|<2.5\sigma) \geq 1-\frac{1}{2.5^2} = 0.84$

(b) $P(|X-33| \geq 14) = P(|X-\mu| \geq 3.5\sigma) \leq \frac{1}{3.5^2} = 0.082$

#5.8-3

(a) $Y \sim b(100, 0.25)$ 일때 $P\left(\left|\frac{Y}{100}-0.25\right|<0.05\right) \geq 1-\frac{0.25\times0.75}{100\times0.05^2} = 0.25$

(b) $Y \sim b(500, 0.25)$ 일때 $P\left(\left|\frac{Y}{500}-0.25\right|<0.05\right) \geq 1-\frac{0.25\times0.75}{500\times0.05^2} = 0.85$

(c) $Y \sim b(1000, 0.25)$ 일때 $P\left(\left|\frac{Y}{1000}-0.25\right|<0.05\right) \geq 1-\frac{0.25\times0.75}{1000\times0.05^2} = 0.925$

#5.9-2

표본분산의 공식은 $S^2 = \frac{1}{n-1}\sum_{1}^{n}(X_i-\bar{X})^2 \rightarrow \frac{n-1}{\sigma^2}S^2 \sim \chi^2(n-1)$ 임을 알고있다

$\chi^2(n)$의 mgf는 $M(t) = (1-2t)^{-\frac{n}{2}}$

$\chi^2(n-1)$의 mgf는 $M(t) = (1-2t)^{-\frac{n-1}{2}}$

$\rightarrow S^2$의 mgf는 $M_{S^2}(t) = \left(1-2t\cdot\frac{\sigma^2}{n-1}\right)^{-\frac{n-1}{2}}$

$n\rightarrow\infty$이면 $M_{S^2}(t) \rightarrow e^{\sigma^2 t}, -\infty<t<\infty$ 이다.

#A1

$X_1 \cdots X_n$ r.s from $f(x;\theta) = \theta x^{\theta-1}$, $0<x<1$

(1) $L(x;\theta) = \prod_{1}^{n} \theta x_i^{\theta-1} \rightarrow \ln L(x;\theta) = n\ln\theta + \sum_{1}^{n} \ln(x_i^{\theta-1}) = n\ln\theta + (\theta-1)\sum \ln x_i$

$\frac{d}{d\theta} \ln L(x;\theta) = \frac{n}{\theta} + \sum \ln x_i = 0$, 정리하면 $\theta = \frac{-n}{\sum \ln x_i} \rightarrow$ MLE $\hat{\theta} = \frac{-n}{\sum \ln X_i}$

(2) $Y = -\ln X$라 하면 $X = e^{-Y}$이다 $\rightarrow f_Y(y) = f_X(e^{-y})\,|-e^{-y}| = \theta(e^{-y})^{\theta-1}e^{-y} = \theta e^{-\theta y} \Rightarrow Y \sim Exp(\theta)$

$U = \sum_{1}^{n} Y_i \sim \Gamma(n,\theta)$이므로

$E(\hat{\theta}) = E(\frac{n}{U}) = nE(\frac{1}{U}) = n\int_0^\infty \frac{\theta^n}{\Gamma(n)u} u^{n-1}e^{-\theta u}\,du$

$= n\cdot\frac{\theta}{n}\cdot \underbrace{\int_0^\infty \frac{\theta^{n-1}}{\Gamma(n-1)u} u^{n-1}e^{-\theta u}\,du}_{\Gamma(n-1,\theta)} = n\cdot\frac{\theta}{n} = \theta \Rightarrow E(\hat{\theta}) = \theta$ 이므로 비편향추정량

(3) $n\to\infty$일때 $\hat{\theta} = \frac{-n}{\sum \ln X_i}$의 점근분포는 $\hat{\theta}$가 MLE이므로 $\hat{\theta} \approx N(\theta, \frac{1}{I_n(\theta)})$이다.

$f(x;\theta) = \theta x^{\theta-1}$

$\ln f(x;\theta) = \ln\theta + (\theta-1)\ln x$

$\frac{\partial}{\partial\theta}\ln f(x;\theta) = \frac{1}{\theta} + \ln x$

$\frac{\partial^2}{\partial\theta^2}\ln f(x;\theta) = -\frac{1}{\theta^2}$

$\rightarrow I_n(\theta) = -nE[\frac{\partial^2}{\partial\theta^2}\ln f(x;\theta)] = -nE(-\frac{1}{\theta^2}) = \frac{n}{\theta^2}$

$\therefore \hat{\theta} \approx N(\theta, \frac{\theta^2}{n})$ 이다.