Week 1-2

19:00 — 21:00

↓

19:15 — 20:45 (90分)

· **Example 1.1.17** $\quad S = \{1, 2, \cdots\} = \mathbb{N}.$

$$A_n = \{1, 3, \cdots, 2n-1\}, \quad B_n = \{n, n+1 \cdots\} = \mathbb{N} \cap \{1, 2, \cdots, n-1\}^c$$

· $A_1 \subseteq A_2 \subseteq \cdots$; increasing set.

$\Rightarrow$ ① $\displaystyle\bigsqcup_{n=1}^{\infty} A_n = \lim_{k\to\infty} \underbrace{\bigsqcup_{n=1}^{k} A_n}_{(A_1 \sqcup A_2 \sqcup \cdots \sqcup A_k) = A_k}$

$\displaystyle = \lim_{k\to\infty} \underbrace{A_k}_{\{1,3,5,\cdots,2k-1\}} = \text{set of odd numbers.}$

② $\displaystyle\bigcap_{n=1}^{\infty} A_n = \lim_{k\to\infty} \underbrace{\bigcap_{n=1}^{k} A_n}_{(A_1 \cap A_2 \cap \cdots \cap A_k) = A_1}$

$\displaystyle = \lim_{k\to\infty} A_1 = A_1 = \{1\}$

· $B_1 \supseteq B_2 \supseteq \cdots$; decreasing set.

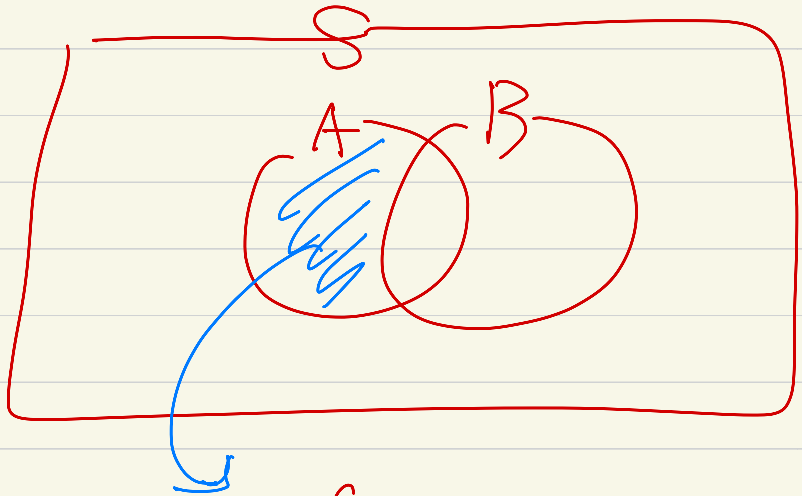
$\Rightarrow$ ③ $\displaystyle\bigsqcup_{n=1}^{\infty} B_n = \lim_{k\to\infty} \underbrace{\bigsqcup_{n=1}^{k} B_n}_{(B_1 \sqcup \cdots \sqcup B_k) = B_1}$

$\displaystyle = \lim_{k\to\infty} B_1 = \{1, 2, \cdots\} = \mathbb{N}.$

④ $\displaystyle\bigcap_{n=1}^{\infty} B_n = \lim_{k\to\infty} \underbrace{\bigcap_{n=1}^{k} B_n}_{(B_1 \cap \cdots \cap B_k) = B_k}$

$\displaystyle = \lim_{k\to\infty} B_k = \lim_{k\to\infty} (\mathbb{N} \cap \{1, 2, \cdots, k-1\}^c)$

$= \mathbb{N} \cap \mathbb{N}^c = \emptyset$



$A \cap B^c$

Example 1.1.8 $S = (0,5)$

$$C_n = \left(1-\frac{1}{n}, 2+\frac{1}{n}\right), \quad D_n = \left(\frac{1}{n}, 3-\frac{1}{n}\right)$$

- $C_1 \supseteq C_2 \supseteq \cdots$: Decreasing.

$\Rightarrow$ ① $\displaystyle\bigcup_{n=1}^{\infty} C_n = \lim_{k\to\infty} \bigcup_{n=1}^{k} C_n = \lim_{k\to\infty} C_1 = (0,3)$

② $\displaystyle\bigcap_{n=1}^{\infty} C_n = \lim_{k\to\infty} \bigcap_{n=1}^{k} C_n = \lim_{k\to\infty} C_k.$

$$= \lim_{k\to\infty}\left(1-\frac{1}{k}, 2+\frac{1}{k}\right) = [1,2].$$

Note : $A \subset B$

$\Leftrightarrow$ 집합 A의 임의의 원소 a는 항상 집합 B에 속한다.

Step1 : $[1,2] \subseteq \displaystyle\bigcap_{n=1}^{\infty} C_n$

Step2 : $[1,2] \supseteq \displaystyle\bigcap_{n=1}^{\infty} C_n.$

Claim : $\bigcap_{n=1}^{\infty} C_n = [1,2]$

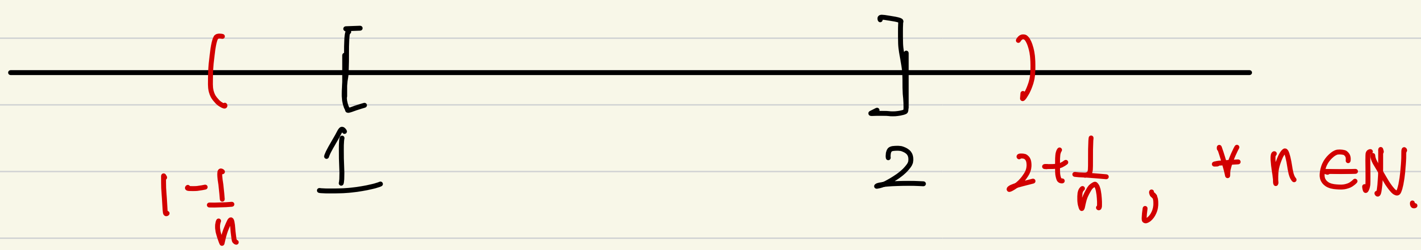
Step 1 : Show $[1,2] \subseteq \bigcap_{n=1}^{\infty} C_n$.

$\Rightarrow$ take any $x \in [1,2]$. then $1 \le x \le 2$.

Since $1-\frac{1}{n} \le 1 \le x \le 2 \le 2+\frac{1}{n}$ for all $n \in \mathbb{N}$

It implies that $x \in C_n$ for all $n \in \mathbb{N}$

$(\therefore)$ $x \in \bigcap_{n=1}^{\infty} C_n$ $\Rightarrow$ $[1,2] \subseteq \bigcap_{n=1}^{\infty} C_n$.



Step 2 Show $\bigcap_{n=1}^{\infty} C_n \subseteq [1,2]$.

$\Rightarrow$ take any $x \in \bigcap_{n=1}^{\infty} C_n$ then $1-\frac{1}{n} < x < 2+\frac{1}{n}$ for all $n \in \mathbb{N}$.

We will show that this $x \in [1,2]$

(B.W.O.C). $x \in \bigcap_{n=1}^{\infty} C_n$ 이면 $x \in [1,2]$

• By way of contrast $\Leftrightarrow$ $x \notin [1,2]$ 이면 $x \notin \bigcap_{n=1}^{\infty} C_n$

Suppose that $x \in [1,2]^c$

then $x < 1$ or $x > 2$.

• If $x < 1$ $\Rightarrow$ $x = 1-\varepsilon$, for some $\varepsilon > 0$.

However for any $\varepsilon > 0$, $\exists N_1 \in \mathbb{N}$ such that

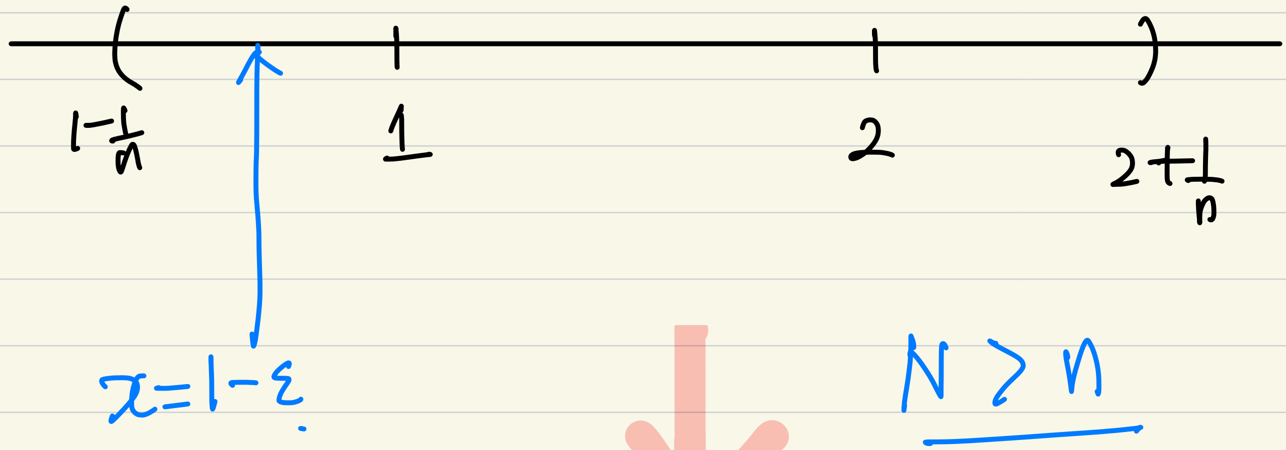
$\varepsilon > \frac{1}{n}$ for all $n \geq N_1$ $\varepsilon = 10^{-5}, 10^{-10} \leftarrow N_1 = 10^5, 10^{10}$

that means $x = 1-\varepsilon < 1-\frac{1}{n}$ for $n \geq N_1$

then $x \notin C_n$ for $n \geq N_1$

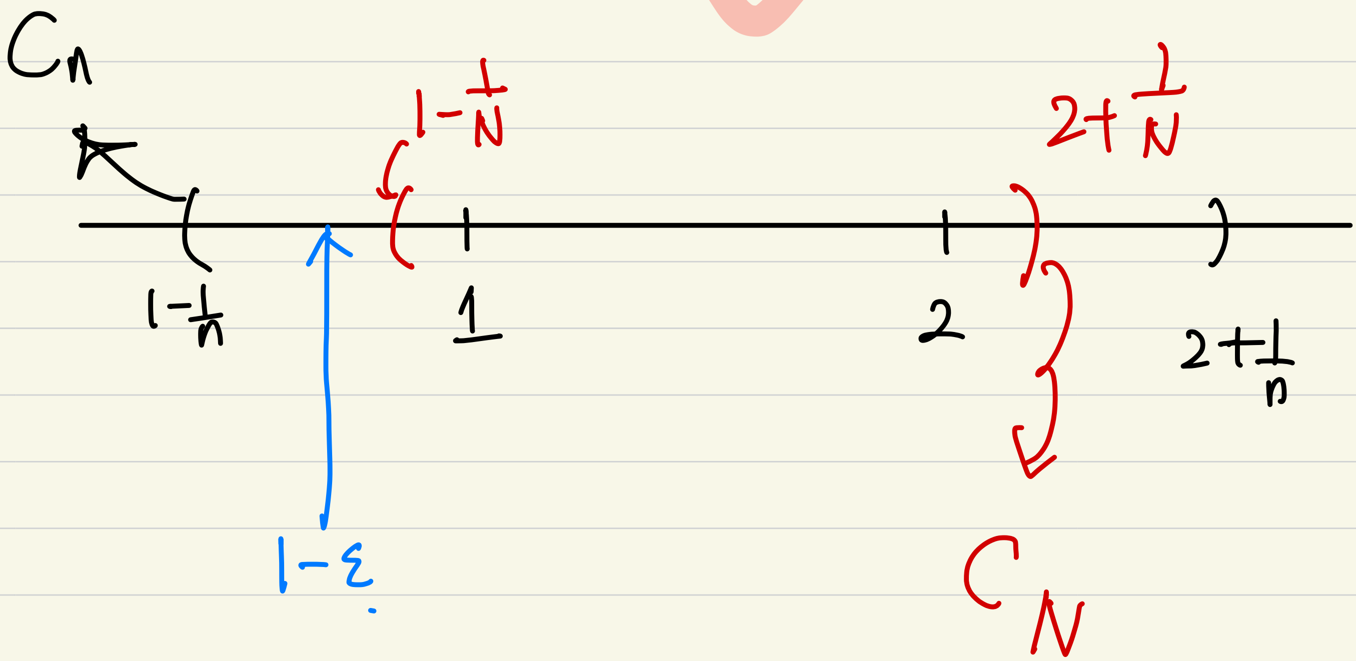
which contradicts to the our starting point

$x \in C_n$ for all $n \in \mathbb{N}$.



$1-\frac{1}{n}$
1
2
$2+\frac{1}{n}$
$x = 1-\varepsilon$

$N > n$



C_n
$1-\frac{1}{N}$
$2+\frac{1}{N}$
$1-\frac{1}{n}$
1
2
$2+\frac{1}{n}$
$1-\varepsilon$
C_N

$x = 2.017 = 2 + \underbrace{0.017}_{\varepsilon}$ $\quad N_2 = \frac{1000}{17} =$

$m \quad \frac{1000}{17} \quad m+1.$

- Similarly If $x > 2$.

  then $\exists\ \varepsilon > 0$ Such that $x = 2 + \varepsilon$.

  However for any $\varepsilon > 0$, $\exists\ N_2 \in \mathbb{N}$ Such that

  $\frac{1}{n} < \varepsilon$ for all $n \geq N_2$.

That implies $2 + \frac{1}{n} < x = 2 + \varepsilon$. for all $n \geq N_2$.

then $x \notin C_n$ for all $n \geq N_2$

which contradicts to the our starting point

$x \in C_n$ for all $n \in \mathbb{N}$.

따라서 $x \in [1,2]$.

By <u>step 1</u> & <u>step 2</u>

$$\bigcap_{n=1}^{\infty} C_n = [1,2] \quad \blacksquare$$

$D_n = \left(\frac{1}{n}, 3-\frac{1}{n}\right)$

$\cdot\ D_1 \subseteq D_2 \subset \cdots$; increasing set

③ $\bigsqcup_{n=1}^{\infty} D_n = \lim_{k\to\infty} \bigsqcup_{n=1}^{k} D_n = \lim_{k\to\infty} D_k = (0,3)$.

$\lim_{k\to\infty}\left(\frac{1}{k}, 3-\frac{1}{k}\right)$

Step 1 : $\bigsqcup_{n=1}^{\infty} D_n \subseteq (0,3)$

$\Rightarrow$ take any $x \in \bigsqcup_{n=1}^{\infty} D_n$. then $\exists\, n \in \mathbb{N}$

such that $x \in D_n = \left(\frac{1}{n}, 3-\frac{1}{n}\right)$

Since $0 < \frac{1}{n} < x < 3-\frac{1}{n} < 3$

It implies that $x \in (0,3)$

Step 2 : $(0,3) \subseteq \bigsqcup_{n=1}^{\infty} D_n$.

$\Rightarrow$ take any $x \in (0,3)$. then $0<x<3$

we will show that $x \in \bigsqcup_{n=1}^{\infty} D_n$.

$\Leftrightarrow \exists\, n \in \mathbb{N}$ such that

$x \in D_n = \left(\frac{1}{n}, 3-\frac{1}{n}\right)$.

· By way of contrast (B.W.O.C)

Suppose that for all $n \in \mathbb{N}$ $x \notin D_n$.

that is $x \leq \frac{1}{n}$ for all $n \in \mathbb{N}$ or

$x \geq 3 - \frac{1}{n}$ for all $n \in \mathbb{N}$.

If $x \leq \frac{1}{n}$ for all $n \in \mathbb{N}$.

If $x > 0$

$x = \varepsilon$, $\varepsilon > 0$

It means that $x \leq 0$.

which contradicts to the our starting point $x \in (0,3)$

If $x \geq 3 - \frac{1}{n}$ for all $n \in \mathbb{N}$

It means that $x \geq 3$.

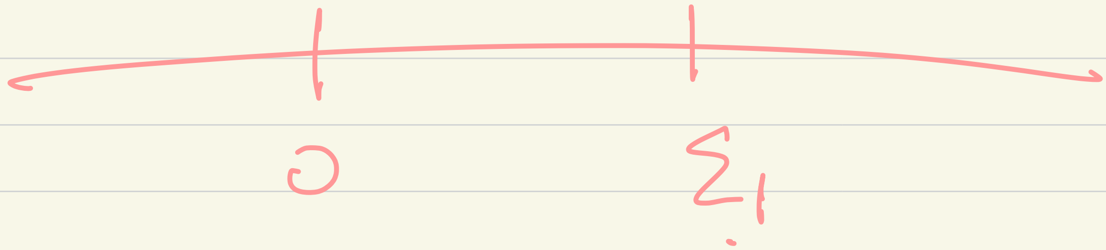
which contradicts to the our starting point $x \in (0,3)$

따라서 $x \in D_n$ for some $n \in \mathbb{N}$.

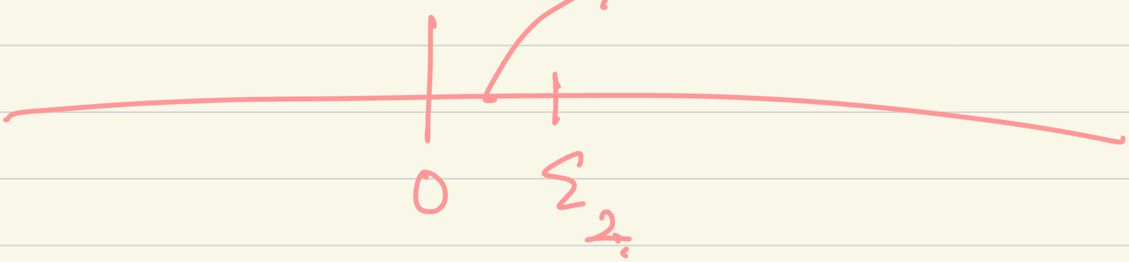
$\Rightarrow x \in \bigcup_{n=1}^{\infty} D_n$

By step 1 & step 2 : $\bigcup_{n=1}^{\infty} D_n = (0,3)$ ☒

$\forall \qquad \forall \ \varepsilon > 0$



$\Rightarrow \exists \ n^* \in \mathbb{N}, \quad \frac{1}{n^*}$



④ $$\bigcap_{n\geq 1}^{\infty} D_n = \lim_{k\to\infty} \bigcap_{n\geq 1}^{k} D_n.$$

$$= \lim_{k\to\infty} D_1 = D_1 = (1,2)$$