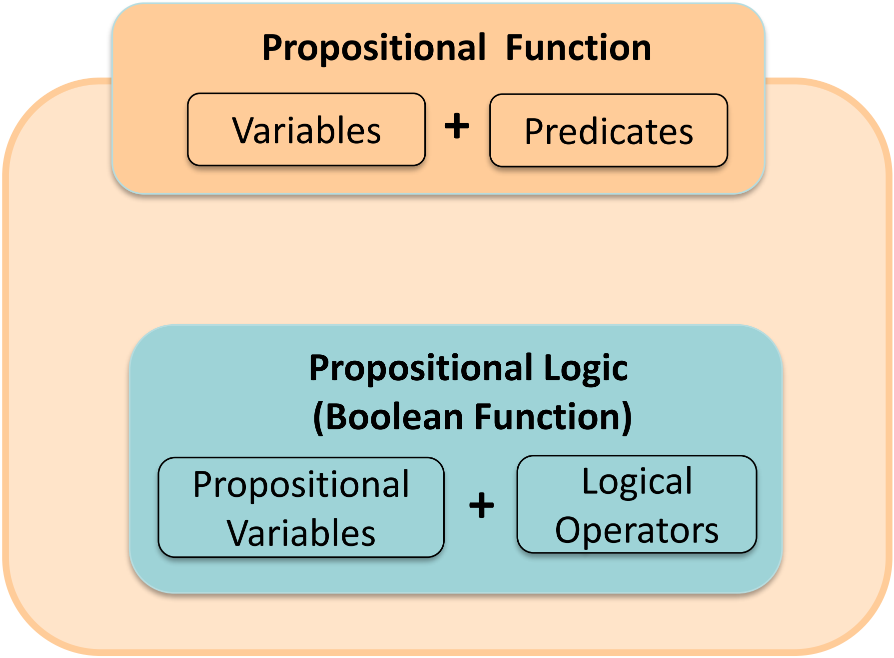
Propositional Functions

Relationship between Propositional Logic and Propositional Function



Propositional Functions

Let's consider an English sentence as below.

"x is greater than 3"

- x: variable
- "is greater than 3" : predicate
- $P(x) = x > 3$
- P(x): propositional function P at x

Propositional Functions

Consider $P(x) = x < 5$

- $P(x)$ has no truth values (x is not given a value).
- $P(1)$ is true: The proposition $1<5$ is true.
- $P(10)$ is false: The proposition $10<5$ is false.

$P(x)$ will create a proposition when given a value.

Let $P(x)$ = "x is a multiple of 5"

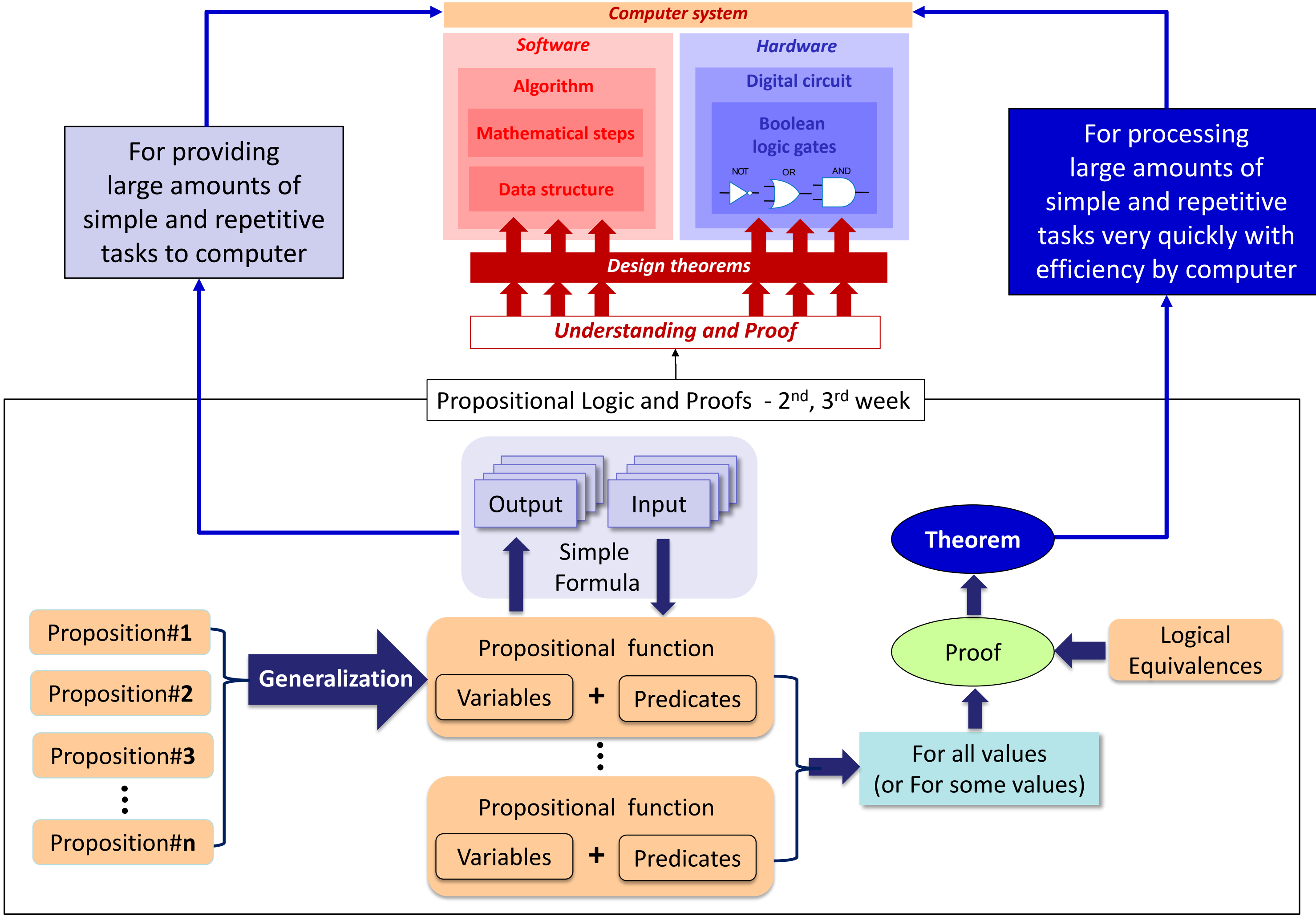
- For what values of x is $P(x)$ true?

Propositional Functions

Functions with multiple variables:

- $P(x,y) = x + y == 0$
  - $P(1,2)$ is false, $P(1,-1)$ is true.
- $P(x,y,z) = x + y == z$
  - $P(3,4,5)$ is false, $P(1,2,3)$ is true.
- $P(x_1,x_2,x_3 \ldots x_n) = \ldots$

Meaning of Propostional Function



Proof Methods

Terminology

Conjecture: a statement that is being proposed to be a true statement.

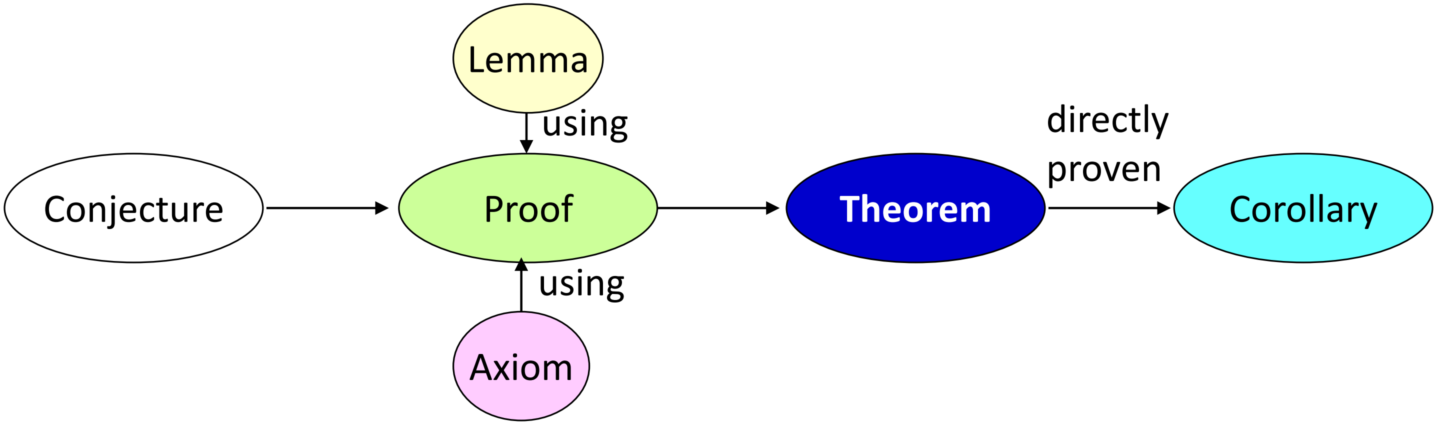
Proof: demonstration that a conjecture is true.

Axiom: a statement that is assumed to be true.

Lemma: a fact that is useful to prove a conjecture.

Theorem: a statement that can be shown true. Sometimes called facts.

Corollary: a theorem that can be proven directly from a theorem that has been proved.



Proof Methods

We will discuss seven proof methods:

1. Direct proofs
2. Indirect proofs
3. Proof by contradiction
4. Proof by cases
5. Proofs of equivalence
6. Existence proofs
7. Uniqueness proofs

Direct Proofs

Consider proving that an implication ($p \to q$) is true.

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

To perform a direct proof

- Assume that p is true, and show that q must be true.

Example

Show that the square of an even number is an even number.

- Rephrased: if n is even, then n^2 is even.

(Proof)

- Assume n is even.
- Thus, $n = 2k$, for integer k (definition of even numbers)
- $n^2 = (2k)^2 = 4k^2 = 2(2k^2)$
- As n^2 is 2 times an integer, n^2 is thus even.

Indirect Proofs

Consider proving that an implication ($p \to q$) is true.

- It's contrapositive is $\neg q \to \neg p$
  - Is logically equivalent to the original implication!
- To perform indirect proof
  - Assume that $\neg q$ is true, and show that $\neg p$ must be true.

				Conditional	Contrapositive
p	q	$\neg p$	$\neg q$	$p \to q$	$\neg q \to \neg p$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

Indirect Proof Example

If n^2 is an odd number then n is an odd number.

- Prove the contrapositive: If n is even, then n^2 is even.
  - You need to specify the domain (or universe) of n.
  - In this case, the domain is integer.

(Proof)

- $n=2k$ for integer k (definition of even numbers)
- $n^2 = (2k)^2 = 4k^2 = 2(2k^2)$
- Since n^2 is 2 times an integer, it is even.

Example of Which to Use

Prove that if n^3+5 is odd, then n is even.

Via direct proof

- $n^3+5 = 2k+1$ for integer k (definition of odd numbers)
- $n^3 = 2k-4$
- Umm… $n = \sqrt[3]{2k-4}$???

So direct proof didn't work out. So: indirect proof

- Contrapositive: If n is odd, then n^3+5 is even.
  - You need to specify the domain (or universe) of n.
  - In this case, the domain is integer.
- Assume n is odd, and show that n^3+5 is even.
- $n=2k+1$ for integer k (definition of odd numbers)
- $n^3+5 = (2k+1)^3+5 = 8k^3+12k^2+6k+6 = 2(4k^3+6k^2+3k+3)$
- As $2(4k^3+6k^2+3k+3)$ is 2 times an integer, it is even.

Proof by Contradiction

Consider proving that an implication (p→q) is true.

To perform a proof by contradiction

- Assume that p is true and q is false. Then show that the assumption is contradiction.
- It means that p→q is true!

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Proof by Contradiction Example

Prove that if n^3+5 is odd, then n is even.

(Proof)

- Assume p is true and q is false - n^3+5 is odd, and n is odd.
  - You need to specify the domain (or universe) of n.
  - In this case, the domain is integer.
- $n=2k+1$ for integer k (definition of odd numbers)
- $n^3+5 = (2k+1)^3+5 = 8k^3+12k^2+6k+6 = 2(4k^3+6k^2+3k+3)$
- As $2(4k^3+6k^2+3k+3)$ is 2 times an integer, it must be even.
- Contradiction!
- Therefore, "If n^3+5 is odd, then n is even." is true.

Proof Methods for Implication

	p	q	$p \to q$	
Direct Proof	T	T	T	
Proof by Contradiction	T	F	F	
?	F	T	T	**It depends on the given domain (universe).**
Indirect Proof	F	F	T	

Example1) If n is even, then n^2 is even.

Example2) If n^2 is odd then n is odd.

Example3) If if n^3+5 is odd, then n is even.

We specified that the domain (or universe) of n is integer.

Proof by Cases

Show a statement is true by showing all possible cases are true.

Make sure you get ALL the cases

- The biggest mistake is to leave out some of the cases.

Proof by Cases Example

Prove that $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$

- Note that b ≠ 0

(Proof)

- Case 1: a ≥ 0 and b > 0
  - Then |a| = a, |b| = b, and

$$\left|\frac{a}{b}\right| = \frac{a}{b} = \frac{|a|}{|b|}$$

- Case 2: a ≥ 0 and b < 0
  - Then |a| = a, |b| = -b, and

$$\left|\frac{a}{b}\right| = -\frac{a}{b} = \frac{a}{-b} = \frac{|a|}{|b|}$$

- Case 3: a < 0 and b > 0
  - Then |a| = -a, |b| = b, and

$$\left|\frac{a}{b}\right| = -\frac{a}{b} = \frac{-a}{b} = \frac{|a|}{|b|}$$

- Case 4: a < 0 and b < 0
  - Then |a| = -a, |b| = -b, and

$$\left|\frac{a}{b}\right| = \frac{a}{b} = \frac{-a}{-b} = \frac{|a|}{|b|}$$

Proofs of Equivalences

This is showing the definition of a bi-conditional.

Given a statement of the form “p if and only if q”.

- Show it is true by showing $(p \to q) \wedge (q \to p)$ is true.

Proofs of Equivalence Example

Show that $m^2=n^2$ if and only if $m=n$ or $m=-n$

Rephrased: $(m^2=n^2) \leftrightarrow [(m=n)\vee(m=-n)]$

(Proof)

- $[(m=n)\vee(m=-n)] \rightarrow (m^2=n^2)$
  - Proof by cases!
    - Case 1: $(m=n) \rightarrow (m^2=n^2)$
      - » $(m)^2 = m^2$, and $(n)^2 = n^2$, so this case is proven
    - Case 2: $(m=-n) \rightarrow (m^2=n^2)$
      - » $(m)^2 = m^2$, and $(-n)^2 = n^2$, so this case is proven
- $(m^2=n^2) \rightarrow [(m=n)\vee(m=-n)]$
  - Direct proof!
    - Subtract n^2 from both sides to get $m^2-n^2=0$
    - Factor to get $(m+n)(m-n) = 0$
    - Since that equals zero, one of the factors must be zero
    - Thus, either $m+n=0$ (which means $m=-n$)
    - Or $m-n=0$ (which means $m=n$)

Existence Proofs

We only have to show that (P(c) = true) exists for some value of c.

Two types:

- Constructive: Find a specific value of c for which P(c) is true.
- Nonconstructive: Show that such a c exists, but don't actually find it.
  - It means : Assume it does not exist, and show a contradiction.

Constructive Existence Proof Example

Show that a square exists that is the sum of two other squares.

(Proof) : $3^2 + 4^2 = 5^2$

Show that a cube exists that is the sum of three other cubes.

(Proof) : $3^3 + 4^3 + 5^3 = 6^3$

Uniqueness Proofs

A conjecture may state that only one such value exists.

To prove this, you need to show:

- Existence: that such a value does indeed exist.
  - Either via a constructive or non-constructive existence proof
- Uniqueness: that there is only one such value.

Uniqueness Proof Example

If the real number equation 5x+3=a has a solution then it is unique.

Existence –constructive proof

- We can manipulate 5x+3=a to yield x=(a-3)/5

Uniqueness

- Proof by contradiction!
  - If there are two such numbers, then they would fulfill the following: a = 5x+3 = 5y+3
  - We can manipulate this to yield that x = y
  - Thus, the one solution is unique!