연습문제 7.2

15-b)

55. $\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+2}\right) = \left(1 - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{4} - \frac{1}{6}\right) + \cdots = 1 + \frac{1}{2} = \frac{3}{2}$, converges

15-c)

57. $\sum_{n=1}^{\infty} \frac{3n-1}{2n+1}$

$$\lim_{n\to\infty} \frac{3n-1}{2n+1} = \frac{3}{2} \neq 0$$

Diverges by Theorem 7.9

15-f)

63. Since $n > \ln(n)$, the terms $a_n = \frac{n}{\ln(n)}$ do not approach 0 as $n \to \infty$. Hence, the series $\sum_{n=2}^{\infty} \frac{n}{\ln(n)}$ diverges.

15-h)

67. $\lim_{n\to\infty} \arctan n = \frac{\pi}{2} \neq 0$

Hence, $\sum_{n=1}^{\infty} \arctan n$ diverges.

17-a)

75. $\sum_{n=1}^{\infty} \frac{x^n}{2^n} = \sum_{n=1}^{\infty}\left(\frac{x}{2}\right)^n = \frac{x}{2}\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^n$

Geometric series: converges for $\left|\frac{x}{2}\right| < 1$ or $|x| < 2$

$$f(x) = \frac{x}{2}\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^n$$
$$= \frac{x}{2}\frac{1}{1-(x/2)} = \frac{x}{2}\frac{2}{2-x} = \frac{x}{2-x}, \quad |x| < 2$$

17-d)

81. $\sum_{n=0}^{\infty}\left(\frac{1}{x}\right)^n$

Geometric series: converges if $\left|\frac{1}{x}\right| < 1$

$\Rightarrow |x| > 1 \Rightarrow x < -1$ or $x > 1$

$$f(x) = \sum_{n=0}^{\infty}\left(\frac{1}{x}\right)^n = \frac{1}{1-(1/x)} = \frac{x}{x-1}, \quad x > 1 \text{ or } x < -1$$

연습문제7.3

2

15. $\sum_{n=1}^{\infty} \frac{n^{k-1}}{n^k + c}$

Let $f(x) = \frac{x^{k-1}}{x^k + c}$.

f is positive, continuous, and decreasing for $x > \sqrt[k]{c(k-1)}$ since

$$f'(x) = \frac{x^{k-2}[c(k-1) - x^k]}{(x^k + c)^2} < 0$$

for $x > \sqrt[k]{c(k-1)}$.

$$\int_1^{\infty} \frac{x^{k-1}}{x^k + c}\,dx = \left[\frac{1}{k}\ln(x^k + c)\right]_1^{\infty} = \infty$$

Diverges by Theorem 7.10

6

27. $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$

If $p = 1$, then the series diverges by the Integral Test.
If $p \neq 1$,

$$\int_2^{\infty} \frac{1}{x(\ln x)^p}\,dx = \int_2^{\infty} (\ln x)^{-p}\frac{1}{x}\,dx = \left[\frac{(\ln x)^{-p+1}}{-p+1}\right]_2^{\infty}.$$

Converges for $-p + 1 < 0$ or $p > 1$

10-b)

35. $S_4 = \frac{1}{e} + \frac{2}{e^4} + \frac{3}{e^9} + \frac{4}{e^{16}} \approx 0.4049$

$$R_4 \le \int_4^{\infty} xe^{-x^2}\,dx = \left[-\frac{1}{2}e^{-x^2}\right]_4^{\infty} = \frac{e^{-16}}{2} \approx 5.6 \times 10^{-8}$$

$$0.4049 \le \sum_{n=1}^{\infty} ne^{-n^2} \le 0.4049 + 5.6 \times 10^{-8}$$

12-b)

43. $0 < \dfrac{1}{3^n + 1} < \dfrac{1}{3^n}$

Therefore,

$$\sum_{n=0}^{\infty} \frac{1}{3^n + 1}$$

converges by comparison with the convergent geometric series

$$\sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n.$$

12-c)

45. For $n \geq 3$, $\dfrac{\ln n}{n + 1} > \dfrac{1}{n + 1} > 0$.

Therefore,

$$\sum_{n=1}^{\infty} \frac{\ln n}{n + 1}$$

diverges by comparison with the divergent series

$$\sum_{n=1}^{\infty} \frac{1}{n + 1}.$$

Note: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n + 1}$ diverges by the Integral Test.

13-e)

59. $\displaystyle\lim_{n\to\infty} \frac{(n^{k-1})/(n^k + 1)}{1/n} = \lim_{n\to\infty} \frac{n^k}{n^k + 1} = 1$

Therefore,

$$\sum_{n=1}^{\infty} \frac{n^{k-1}}{n^k + 1}$$

diverges by a limit comparison with the divergent *p*-series

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

13-f)

61. $$\lim_{n\to\infty} \frac{\sin(1/n)}{1/n} = \lim_{n\to\infty} \frac{(-1/n^2)\cos(1/n)}{-1/n^2}$$

$$= \lim_{n\to\infty} \cos\left(\frac{1}{n}\right) = 1$$

Therefore,

$$\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$$

diverges by a limit comparison with the divergent *p*-series

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

연습문제 7.4

1-b)

3. $\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{n^2+1}$

$\lim_{n\to\infty} \frac{n^2}{n^2+1} = 1$. Thus, $\lim_{n\to\infty} a_n \neq 0$.

Diverges by the nth-Term Test

1-d)

7. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n+1)}{\ln(n+1)}$

$\lim_{n\to\infty} \frac{n+1}{\ln(n+1)} = \lim_{n\to\infty} \frac{1}{1/(n+1)} = \lim_{n\to\infty} (n+1) = \infty$

Diverges by the nth-Term Test

1-f)

11. $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$

$a_{n+1} = \frac{1}{(n+1)!} < \frac{1}{n!} = a_n$

$\lim_{n\to\infty} \frac{1}{n!} = 0$

Converges by Theorem 7.14

1-h)

15. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n!}{1 \cdot 3 \cdot 5 \cdots (2n-1)}$

$a_{n+1} = \frac{(n+1)!}{1 \cdot 3 \cdot 5 \cdots (2n-1)(2n+1)} = \frac{n!}{1 \cdot 3 \cdot 5 \cdots (2n-1)} \cdot \frac{n+1}{2n+1} = a_n\left(\frac{n+1}{2n+1}\right) < a_n$

$$\begin{aligned}\lim_{n\to\infty} a_n &= \lim_{n\to\infty} \frac{n!}{1 \cdot 3 \cdot 5 \cdots (2n-1)} \\ &= \lim_{n\to\infty} \frac{1 \cdot 2 \cdot 3 \cdots n}{1 \cdot 3 \cdot 5 \cdots (2n-1)} \\ &= \lim_{n\to\infty} 2\left[\frac{3}{3} \cdot \frac{4}{5} \cdot \frac{5}{7} \cdots \frac{n}{2n-3}\right] \cdot \frac{1}{2n-1} = 0\end{aligned}$$

Converges by Theorem 7.14

6-b)

29. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n^3-1}$

By Theorem 7.15,

$|R_N| \leq a_{N+1} = \frac{1}{2(N+1)^3 - 1} < 0.001.$

This inequality is valid when $N = 7$. Use 7 terms. N은 7이상

7-a)

31. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(n+1)^2}$

$\sum_{n=1}^{\infty} \frac{1}{(n+1)^2}$ converges by comparison to the p-series

$\sum_{n=1}^{\infty} \frac{1}{n^2}$.

Therefore, the given series converges absolutely.

7-c)

35. $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$

The given series converges by the Alternating Series Test but does not converge absolutely since the series

$\sum_{n=2}^{\infty} \frac{1}{\ln n}$ diverges by comparison to the harmonic series

$\sum_{n=1}^{\infty} \frac{1}{n}$. Therefore, the series converges conditionally.

7-e)

39. $\sum_{n=0}^{\infty} \frac{\cos n\pi}{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$

The given series converges by the Alternating Series Test, but

$\sum_{n=0}^{\infty} \frac{|\cos n\pi|}{n+1} = \sum_{n=0}^{\infty} \frac{1}{n+1}$

diverges by a limit comparison to the divergent harmonic series,

$\sum_{n=1}^{\infty} \frac{1}{n}$.

$\lim_{n\to\infty} \frac{|\cos n\pi|/(n+1)}{1/n} = 1$, therefore the series converges conditionally.

8-a)

43. $\sum_{n=0}^{\infty} \frac{n!}{3^n}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!}{3^{n+1}}\cdot\frac{3^n}{n!}\right|$$

$$= \lim_{n\to\infty}\frac{n+1}{3} = \infty$$

Therefore, by the Ratio Test, the series diverges.

8-c)

47. $\sum_{n=1}^{\infty} \frac{2^n}{n^2}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{2^{n+1}}{(n+1)^2}\cdot\frac{n^2}{2^n}\right|$$

$$= \lim_{n\to\infty}\frac{2n^2}{(n+1)^2} = 2$$

Therefore, by the Ratio Test, the series diverges.

8-e)

51. $\sum_{n=1}^{\infty} \frac{n!}{n3^n}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!}{(n+1)3^{n+1}}\cdot\frac{n3^n}{n!}\right|$$

$$= \lim_{n\to\infty}\frac{n}{3} = \infty$$

Therefore, by the Ratio Test, the series diverges.

10-a)

63. $\sum_{n=1}^{\infty} \left(\frac{n}{2n+1}\right)^n$

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{\left(\frac{n}{2n+1}\right)^n}$$

$$= \lim_{n\to\infty}\frac{n}{2n+1} = \frac{1}{2}$$

Therefore, by the Root Test, the series converges.

10-c)

67. $\sum_{n=1}^{\infty} \left(2\sqrt[n]{n}+1\right)^n$

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{\left(2\sqrt[n]{n}+1\right)^n} = \lim_{n\to\infty}\left(2\sqrt[n]{n}+1\right)$$

To find $\lim_{n\to\infty}\sqrt[n]{n}$, let $y = \lim_{n\to\infty}\sqrt[n]{n}$. Then

$$\ln y = \lim_{n\to\infty}\left(\ln\sqrt[n]{n}\right) = \lim_{n\to\infty}\frac{1}{n}\ln n = \lim_{n\to\infty}\frac{\ln n}{n} = \lim_{n\to\infty}\frac{1/n}{1} = 0.$$

Thus, $\ln y = 0$, so $y = e^0 = 1$ and $\lim_{n\to\infty}\left(2\sqrt[n]{n}+1\right) = 2(1)+1 = 3$. Therefore, by the Root Test, the series diverges.

10-e)

71. $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n^2}\right)^n$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left(\frac{1}{n} - \frac{1}{n^2}\right)^n}$$

$$= \lim_{n\to\infty} \left(\frac{1}{n} - \frac{1}{n^2}\right) = 0 - 0 = 0 < 1$$

Hence, the series converges.

18

113. $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^p}$

If $p = 0$, then $\sum_{n=1}^{\infty} (-1)^n$ diverges.

If $p < 0$, then $\sum_{n=1}^{\infty} (-1)^n n^{-p}$ diverges.

If $p > 0$, then $\lim_{n\to\infty} \frac{1}{n^p} = 0$ and

$$a_{n+1} = \frac{1}{(n+1)^p} < \frac{1}{n^p} = a_n.$$

Therefore, the series converges for $p > 0$.