연습문제 11.5

2)

11. $f(x, y) = x^2 + y^2$

$$\mathbf{u} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$$

$$\nabla f = 2x\mathbf{i} + 2y\mathbf{j}$$

$$D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u} = \frac{2}{\sqrt{2}}x + \frac{2}{\sqrt{2}}y = \sqrt{2}(x + y)$$

5)

19. $\overrightarrow{PQ} = 2\mathbf{i} + 4\mathbf{j}, \mathbf{u} = \frac{1}{\sqrt{5}}\mathbf{i} + \frac{2}{\sqrt{5}}\mathbf{j}$

$$\nabla g(x, y) = 2x\mathbf{i} + 2y\mathbf{j}, \nabla g(1, 2) = 2\mathbf{i} + 4\mathbf{j}$$

$$D_{\mathbf{u}} g = \nabla g \cdot \mathbf{u} = \frac{2}{\sqrt{5}} + \frac{8}{\sqrt{5}} = \frac{10}{\sqrt{5}} = 2\sqrt{5}$$

6-a)

21. $h(x, y) = x \tan y$

$$\nabla h(x, y) = \tan y\mathbf{i} + x \sec^2 y\mathbf{j}$$

$$\nabla h\left(2, \frac{\pi}{4}\right) = \mathbf{i} + 4\mathbf{j}$$

$$\left\| \nabla h\left(2, \frac{\pi}{4}\right) \right\| = \sqrt{17}$$

6-b)

23. $g(x, y) = \ln \sqrt[3]{x^2 + y^2} = \frac{1}{3}\ln(x^2 + y^2)$

$$\nabla g(x, y) = \frac{1}{3}\left[\frac{2x}{x^2 + y^2}\mathbf{i} + \frac{2y}{x^2 + y^2}\mathbf{j}\right]$$

$$\nabla g(1, 2) = \frac{1}{3}\left(\frac{2}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}\right) = \frac{2}{15}(\mathbf{i} + 2\mathbf{j})$$

$$\|\nabla g(1, 2)\| = \frac{2\sqrt{5}}{15}$$

연습문제 11.7

2-b)

9. $f(x, y) = -5x^2 + 4xy - y^2 + 16x + 10$

$$\left.\begin{array}{l} f_x = -10x + 4y + 16 = 0 \\ f_y = 4x - 2y = 0 \end{array}\right\} \quad \text{Solving simultaneously yields } x = 8 \text{ and } y = 16.$$

$f_{xx} = -10,\ f_{yy} = -2,\ f_{xy} = 4$

At the critical point $(8, 16)$, $f_{xx} < 0$ and $f_{xx} f_{yy} - (f_{xy})^2 > 0$. Therefore, $(8, 16, 74)$ is a relative maximum.

2-d)

13. $f(x, y) = 2\sqrt{x^2 + y^2} + 3$

$$\left.\begin{array}{l} f_x = \dfrac{2x}{\sqrt{x^2 + y^2}} = 0 \\ f_y = \dfrac{2y}{\sqrt{x^2 + y^2}} = 0 \end{array}\right\} \quad x = 0, y = 0$$

Since $f(x, y) \geq 3$ for all (x, y), $(0, 0, 3)$ is a relative minimum.

4-a)

21. $h(x, y) = x^2 - y^2 - 2x - 4y - 4$

$h_x = 2x - 2 = 2(x - 1) = 0$ when $x = 1$.

$h_y = -2y - 4 = -2(y + 2) = 0$ when $y = -2$.

$h_{xx} = 2,\ h_{yy} = -2,\ h_{xy} = 0$

At the critical point $(1, -2)$, $h_{xx} h_{yy} - (h_{xy})^2 < 0$. Therefore, $(1, -2, -1)$ is a saddle point.

4-d)

27. $f(x, y) = 2xy - \frac{1}{2}(x^4 + y^2) + 1$

$$\left.\begin{array}{l} f_x = 2y - 2x^3 \\ f_y = 2x - 2y^3 \end{array}\right\} \quad \begin{array}{l} \text{Solving by substitution yields 3 critical points:} \\ (0, 0), (1, 1), (-1, -1) \end{array}$$

$f_{xx} = -6x^2, f_{yy} = -6y^2, f_{xy} = 2$

At $(0, 0)$, $f_{xx}f_{yy} - (f_{xy})^2 < 0 \implies (0, 0, 1)$ saddle point.

At $(1, 1)$, $f_{xx}f_{yy} - (f_{xy})^2 > 0$ and $f_{xx} < 0 \implies (1, 1, 2)$ relative maximum.

At $(-1, -1)$, $f_{xx}f_{yy} - (f_{xy})^2 > 0$ and $f_{xx} < 0 \implies (-1, -1, 2)$ relative maximum.

10-a)

53. $f(x, y) = 12 - 3x - 2y$ has no critical points. On the line $y = x + 1$, $0 \le x \le 1$,

$$f(x, y) = f(x) = 12 - 3x - 2(x + 1) = -5x + 10$$

and the maximum is 10, the minimum is 5. On the line $y = -2x + 4$, $1 \le x \le 2$,

$$f(x, y) = f(x) = 12 - 3x - 2(-2x + 4) = x + 4$$

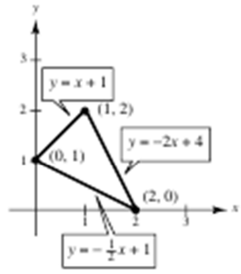
and the maximum is 6, the minimum is 5. On the line $y = -\frac{1}{2}x + 1$, $0 \le x \le 2$,

$$f(x, y) = f(x) = 12 - 3x - 2\left(-\tfrac{1}{2}x + 1\right) = -2x + 10$$

and the maximum is 10, the minimum is 6.

Absolute maximum: 10 at $(0, 1)$

Absolute minimum: 5 at $(1, 2)$



10-c)

57. $f(x, y) = x^2 + xy$, $R = \{(x, y): |x| \le 2, |y| \le 1\}$

$$\left.\begin{aligned} f_x &= 2x + y = 0 \\ f_y &= x = 0 \end{aligned}\right\} x = y = 0$$

$f(0, 0) = 0$

Along $y = 1$, $-2 \le x \le 2$, $f = x^2 + x$, $f' = 2x + 1 = 0 \Rightarrow x = -\frac{1}{2}$.

Thus, $f(-2, 1) = 2$, $f\left(-\frac{1}{2}, 1\right) = -\frac{1}{4}$ and $f(2, 1) = 6$.

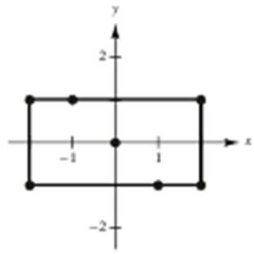
Along $y = -1$, $-2 \le x \le 2$, $f = x^2 - x$, $f' = 2x - 1 = 0 \Rightarrow x = \frac{1}{2}$.

Thus, $f(-2, -1) = 6$, $f\left(\frac{1}{2}, -1\right) = -\frac{1}{4}$, $f(2, -1) = 2$.

Along $x = 2$, $-1 \le y \le 1$, $f = 4 + 2y \Rightarrow f' = 2 \ne 0$.

Along $x = -2$, $-1 \le y \le 1$, $f = 4 - 2y \Rightarrow f' = -2 \ne 0$.

Thus, the maxima are $f(2, 1) = 6$ and $f(-2, -1) = 6$ and the minima are $f\left(-\frac{1}{2}, 1\right) = -\frac{1}{4}$ and $f\left(\frac{1}{2}, -1\right) = -\frac{1}{4}$.



20)

85. Let $S(a, b) = \sum_{i=1}^{n} (ax_i + b - y_i)^2$

The first partial derivatives of S are

$$S_a(a, b) = \sum_{i=1}^{n} 2x_i(ax_i + b - y_i) = 2a\sum_{i=1}^{n} x_i^2 + 2b\sum_{i=1}^{n} x_i - 2\sum_{i=1}^{n} x_i y_i$$

$$S_b(a, b) = \sum_{i=1}^{n} 2(ax_i + b - y_i) = 2a\sum_{i=1}^{n} x_i + 2nb - 2\sum_{i=1}^{n} y_i$$

Setting these equal to zero, you obtain

$$nb + \left(\sum_{i=1}^{n} x_i\right)a = \sum_{i=1}^{n} y_i$$

$$\left(\sum_{i=1}^{n}\right)b + \left(\sum_{i=1}^{n} x_i^2\right)a = \sum_{i=1}^{n} x_i y_i$$

Using the Second Partials Test, you can verify that this is a minimum.

연습문제 11.8

3-a)

11. Minimize $f(x, y, z) = x^2 + y^2 + z^2$.

Constraint: $x + y + z = 6$

$$\left.\begin{aligned} 2x &= \lambda \\ 2y &= \lambda \\ 2z &= \lambda \end{aligned}\right\} x = y = z$$

$$x + y + z = 6 \implies x = y = z = 2$$

$$f(2, 2, 2) = 12$$

4-b)

17. Maximize $f(x, y, z) = xy + yz$.

Constraints: $x + 2y = 6$

$$x - 3z = 0$$

$$\nabla f = \lambda \nabla g + \mu \nabla h$$

$$y\mathbf{i} + (x + z)\mathbf{j} + y\mathbf{k} = \lambda(\mathbf{i} + 2\mathbf{j}) + \mu(\mathbf{i} - 3\mathbf{k})$$

$$\left.\begin{aligned} y &= \lambda + \mu \\ x + z &= 2\lambda \\ y &= -3\mu \end{aligned}\right\} y = \frac{3}{4}\lambda \implies x + z = \frac{8}{3}y$$

$$x + 2y = 6 \implies y = 3 - \frac{x}{2}$$

$$x - 3z = 0 \implies z = \frac{x}{3}$$

$$x + \frac{x}{3} = \frac{8}{3}\left(3 - \frac{x}{2}\right)$$

$$x = 3, y = \frac{3}{2}, z = 1$$

$$f\left(3, \frac{3}{2}, 1\right) = 6$$