통계수학1 과제#4 풀이

*연습문제 6.6 – 문제 4~7 (극한값만 제시), 문제 13

답:

4(b) $\lim_{x\to\infty} x \ln = (\infty)(\infty) = \infty$

5(b) 1

5 (b)

$$\lim_{x\to\infty} x\sin\frac{1}{x} = \lim_{x\to\infty}\frac{\sin(1/x)}{1/x} = \lim_{x\to\infty}\frac{(-1/x^2)\cos(1/x)}{-1/x^2} = \lim_{x\to\infty}\cos\left(\frac{1}{x}\right) = 1$$

6(b) 1

6 (b)

$$\lim_{x\to\infty} \ln\left(x^{\frac{1}{x}}\right) = \lim_{x\to\infty}\frac{\ln x}{x} = \lim_{x\to\infty}\left(\frac{1/x}{1}\right) = 0$$

로피탈의 정리

Therefore,

$$\lim_{x\to\infty} x^{1/x} = 1.$$

7(b) e

7 (b)

$$\lim_{x\to 0^+} \ln\left((1+x)^{\frac{1}{x}}\right) = \lim_{x\to 0^+}\frac{\ln(1+x)}{x} = \lim_{x\to 0^+}\left(\frac{1/(1+x)}{1}\right) = 1$$

로피탈의 정리

Therefore, $\lim_{x\to 0^+}(1+x)^{1/x} = e.$

13 (a) $\lim_{x\to\infty}\frac{x^2}{e^{5x}} = \lim_{x\to\infty}\frac{2x}{5e^{5x}} = \lim_{x\to\infty}\frac{2}{25e^{5x}} = 0$

(b) 0

$$\lim_{x\to\infty}\frac{(\ln x)^3}{x} = \lim_{x\to\infty}\frac{3(\ln x)^2(1/x)}{1} = \lim_{x\to\infty}\frac{3(\ln x)^2}{x} = \lim_{x\to\infty}\frac{6(\ln x)(1/x)}{1} = \lim_{x\to\infty}\frac{6(\ln x)}{x} = \lim_{x\to\infty}\frac{6}{x} = 0$$

(c) 0

$$\lim_{x\to\infty}\frac{(\ln x)^n}{x^m} = \lim_{x\to\infty}\frac{n(\ln x)^{n-1}/x}{mx^{m-1}} = \lim_{x\to\infty}\frac{n(\ln x)^{n-1}}{mx^m} = \lim_{x\to\infty}\frac{n(n-1)(\ln x)^{n-2}}{m^2x^m} = \cdots = \lim_{x\to\infty}\frac{n!}{m^n x^m} = 0$$

*연습문제 3.3 – 문제 3(k)(l) (극값만 구할 것)

답: (k) (−3,−8), (1,0)

$$f(x) = \frac{x^2 - 2x + 1}{x + 1}$$

$$f'(x) = \frac{(x+1)(2x-2) - (x^2 - 2x + 1)(1)}{(x+1)^2} = \frac{x^2 + 2x - 3}{(x+1)^2} = \frac{(x+3)(x-1)}{(x+1)^2}$$

Critical numbers: $x = -3, 1$

Discontinuity: $x = -1$

Test intervals	$-\infty < x < -3$	$-3 < x < -1$	$-1 < x < 1$	$1 < x < \infty$
Sign of $f'(x)$	$f' > 0$	$f' < 0$	$f' < 0$	$f' > 0$
Conclusion	Increasing	Decreasing	Decreasing	Increasing

Increasing on: $(-\infty, -3), (1, \infty)$

Decreasing on: $(-3, -1), (-1, 1)$

Relative maximum: $(-3, -8)$

Relative minimum: $(1, 0)$

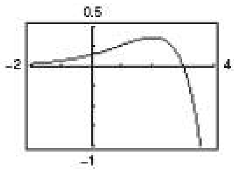
(l) $(2, e^{-1})$

$f(x) = (3 - x)e^{x-3}$

$f'(x) = (3 - x)e^{x-3} - e^{x-3}$

$= e^{x-3}(2 - x)$

Critical number: $x = 2$



Test intervals	$-\infty < x < 2$	$2 < x < \infty$
Sign of $f'(x)$	$f' > 0$	$f' < 0$
Conclusion	Increasing	Decreasing

Increasing on: $(-\infty, 2)$

Decreasing on: $(2, \infty)$

Relative maximum: $(2, e^{-1})$

*연습문제 3.4 - 문제 (f)

(f) (e,e)

$$y = \frac{x}{\ln x}$$

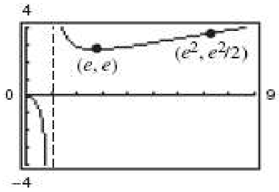
Domain: $0 < x < 1,\ x > 1$

$$y' = \frac{(\ln x)(1) - (x)(1/x)}{(\ln x)^2} = \frac{\ln x - 1}{(\ln x)^2} = 0 \text{ when } x = e.$$

$$y'' = \frac{2 - \ln x}{x(\ln x)^3} = 0 \text{ when } x = e^2.$$

Relative minimum: (e, e)

Point of inflection: $\left(e^2, \frac{e^2}{2}\right)$



*연습문제 4.4 – 문제 16

답:

(a) 8

$$F(x) = \int_x^{x+2} (4t+1)\,dt$$
$$= \left[2t^2 + t\right]_x^{x+2}$$
$$= [2(x+2)^2 + (x+2)] - [2x^2 + x]$$
$$= 8x + 10$$
$$F'(x) = 8$$

Alternate Solution:

$$F(x) = \int_x^{x+2} (4t+1)\,dt$$
$$= \int_x^0 (4t+1)\,dt + \int_0^{x+2} (4t+1)\,dt$$
$$= -\int_0^x (4t+1)\,dt + \int_0^{x+2} (4t+1)\,dt$$
$$F'(x) = -(4x+1) + 4(x+2) + 1 = 8$$

(b) $\sqrt{\cos x}\,(\sin x)$ or $\sqrt{\sin x}\,(\cos x)$

$$F(x) = \int_0^{\sin x} \sqrt{t}\,dt = \left[\frac{2}{3}t^{3/2}\right]_0^{\sin x} = \frac{2}{3}(\sin x)^{3/2}$$
$$F'(x) = (\sin x)^{1/2}\cos x = \cos x\sqrt{\sin x}$$

Alternate Solution:

$$F(x) = \int_0^{\sin x} \sqrt{t}\,dt$$
$$F'(x) = \sqrt{\sin x}\,\frac{d}{dx}(\sin x) = \sqrt{\sin x}(\cos x)$$

(c) $3x^2\sin x^6$

$$F(x) = \int_0^{x^3} \sin t^2\,dt$$
$$F'(x) = \sin(x^3)^2 \cdot 3x^2 = 3x^2\sin x^6$$

*연습문제 4.5 – 문제 6 (d)(j)

답: (d) $\frac{1}{4}\sin^2(2x) + C$ (j) $-\tan(e^{-x}) + C$

*연습문제 4.7 – 문제 2, 문제 5(b)(정적분만 구할 것)

답:

2(a) $\sqrt{2x} - \ln(1+\sqrt{2x}) + C$

$$u = 1 + \sqrt{2x},\ du = \frac{1}{\sqrt{2x}}\,dx \Rightarrow (u-1)\,du = dx$$
$$\int \frac{1}{1+\sqrt{2x}}\,dx = \int \frac{(u-1)}{u}\,du = \int\left(1 - \frac{1}{u}\right)du$$
$$= u - \ln|u| + C_1$$
$$= \left(1+\sqrt{2x}\right) - \ln\left|1+\sqrt{2x}\right| + C_1$$
$$= \sqrt{2x} - \ln\left(1+\sqrt{2x}\right) + C$$

where $C = C_1 + 1$.

(b) $x + 6\sqrt{x} + 18\ln|\sqrt{x} - 3| + C$

$$u = \sqrt{x} - 3,\ du = \frac{1}{2\sqrt{x}}\,dx \implies 2(u+3)\,du = dx$$

$$\begin{aligned}
\int \frac{\sqrt{x}}{\sqrt{x}-3}\,dx &= 2\int \frac{(u+3)^2}{u}\,du \\
&= 2\int \frac{u^2+6u+9}{u}\,du = 2\int\left(u + 6 + \frac{9}{u}\right)du \\
&= 2\left[\frac{u^2}{2} + 6u + 9\ln|u|\right] + C_1 \\
&= u^2 + 12u + 18\ln|u| + C_1 \\
&= \left(\sqrt{x}-3\right)^2 + 12\left(\sqrt{x}-3\right) + 18\ln\left|\sqrt{x}-3\right| + C_1 \\
&= x + 6\sqrt{x} + 18\ln\left|\sqrt{x}-3\right| + C
\end{aligned}$$

where $C = C_1 - 27$.

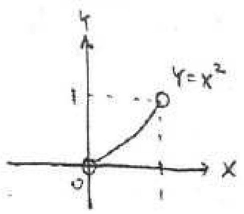
5(b) $\frac{7}{3}$

$$u = 1 + \ln x,\ du = \frac{1}{x}\,dx$$

$$\int_1^e \frac{(1+\ln x)^2}{x}\,dx = \left[\frac{1}{3}(1+\ln x)^3\right]_1^e = \frac{7}{3}$$

***확률변수 X의 확률밀도함수가 $f_X(x) = 1,\ 0 < x < 1$로 주어져 있다고 하자. 새로운 확률변수 Y를 $Y = X^2$로 정의하였을 때 확률변수 Y의 pdf를 구하시오.**

답:



$$F_Y(y) = P(Y \le y) = P(X^2 \le y)$$

$$= \begin{cases} 0 & , y \le 0 \\ P(0 \le X \le \sqrt{y}) & , 0 < y < 1 \\ 1 & , y \ge 1 \end{cases}$$

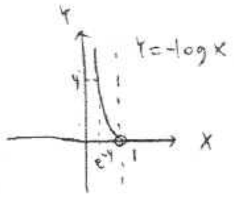
$$P(0 \le X \le \sqrt{y}) = \int_0^{\sqrt{y}} 1\,dx = x \Big|_0^{\sqrt{y}} = \sqrt{y}$$

$$= \begin{cases} 0 & , y \le 0 \\ \sqrt{y} & , 0 < y < 1 \\ 1 & , y \ge 1 \end{cases}$$

$$\Rightarrow f_Y(y) = \begin{cases} \frac{1}{2\sqrt{y}} & , 0 < y < 1 \\ 0 & , \text{그외의 경우} \end{cases}$$

*확률변수 X의 확률밀도함수가 $f_X(x)=1,\ 0<x<1$로 주어져 있다고 하자. 새로운 확률변수 Y를 $Y=-\log X$로 정의하였을 때 확률변수 Y의 pdf를 구하시오

답:



$$
\begin{aligned}
F_Y(y) &= P(Y \le y) \\
&= P(-\log X \le y) \\
&= P(\log X \ge -y) \\
&= P(X \ge e^{-y}) = \begin{cases} 0, & y \le 0 \\ P(e^{-y} \le X < 1), & 0 < y < \infty \end{cases}
\end{aligned}
$$

$$P(e^{-y} \le X < 1) = \int_{e^{-y}}^{1} 1\,dx = x\Big|_{e^{-y}}^{1} = 1 - e^{-y}$$

$$= \begin{cases} 0, & y \le 0 \\ 1 - e^{-y}, & 0 < y < \infty \end{cases}$$

$$\Rightarrow f_Y(y) = \begin{cases} e^{-y}, & 0 < y < \infty \\ 0, & \text{그 외의 경우} \end{cases}$$

*연습문제 6.1 – 문제 11(a)

답:

$u = \sqrt{x} \implies u^2 = x \implies 2u\,du = dx$

$$\int \sin \sqrt{x}\,dx = \int \sin u(2u\,du) = 2\int u \sin u\,du$$

Integration by parts: $w = u,\ dw = du,\ dv = \sin u\,du,$
$v = -\cos u$

$$
\begin{aligned}
2\int u \sin u\,du &= 2\left(-u\cos u + \int \cos u\,du\right) \\
&= 2(-u\cos u + \sin u) + C \\
&= 2\left(-\sqrt{x}\cos\sqrt{x} + \sin\sqrt{x}\right) + C
\end{aligned}
$$

*연습문제 6.7 – 문제 2(a)(b) (수렴,발산을 판정하고 수렴하면 수렴하는 값을 제시), 문제 4(b)(c)(f), 문제 6(b), 문제 20(a)(b)(c), 문제 21(b)

답:

2(a) 4로 수렴

Infinite discontinuity at $x = 0$.

$$\int_0^4 \frac{1}{\sqrt{x}}\,dx = \lim_{b \to 0^+} \int_b^4 \frac{1}{\sqrt{x}}\,dx$$
$$= \lim_{b \to 0^+} \left[2\sqrt{x}\right]_b^4$$
$$= \lim_{b \to 0^+} \left(4 - 2\sqrt{b}\right) = 4$$

Converges

(b) 발산

Infinite discontinuity at $x = 1$.

$$\int_0^2 \frac{1}{(x-1)^2}\,dx = \int_0^1 \frac{1}{(x-1)^2}\,dx + \int_1^2 \frac{1}{(x-1)^2}\,dx$$
$$= \lim_{b\to 1^-} \int_0^b \frac{1}{(x-1)^2}\,dx + \lim_{c\to 1^+} \int_c^2 \frac{1}{(x-1)^2}\,dx$$
$$= \lim_{b\to 1^-} \left[-\frac{1}{x-1}\right]_0^b + \lim_{c\to 1^+} \left[-\frac{1}{x-1}\right]_c^2 = (\infty - 1) + (-1 + \infty)$$

Diverges

4(b) 발산

15. $\displaystyle\int_{-\infty}^0 xe^{-2x}\,dx = \lim_{b\to-\infty} \int_b^0 xe^{-2x}\,dx = \lim_{b\to-\infty} \frac{1}{4}\left[(-2x-1)e^{-2x}\right]_b^0$

Diverges

$$= \lim_{b\to-\infty} \frac{1}{4}\left[-1 + (2b+1)e^{-2b}\right] = -\infty \quad \text{(Integration by parts)}$$

(c) 2로 수렴

17. $\displaystyle\int_0^\infty x^2e^{-x}\,dx = \lim_{b\to\infty} \int_0^b x^2e^{-x}\,dx = \lim_{b\to\infty} \left[-e^{-x}(x^2+2x+2)\right]_0^b = \lim_{b\to\infty} \left(-\frac{b^2+2b+2}{e^b} + 2\right) = 2$

Since $\displaystyle\lim_{b\to\infty} \left(-\frac{b^2+2b+2}{e^b}\right) = 0$ by L'Hôpital's Rule.

(f) π로 수렴

23. $\displaystyle\int_{-\infty}^\infty \frac{2}{4+x^2}\,dx = \int_{-\infty}^0 \frac{2}{4+x^2}\,dx + \int_0^\infty \frac{2}{4+x^2}\,dx$

$$= \lim_{b\to-\infty} \int_b^0 \frac{2}{4+x^2}\,dx + \lim_{c\to\infty} \int_0^c \frac{2}{4+x^2}\,dx$$
$$= \lim_{b\to-\infty} \left[\arctan\left(\frac{x}{2}\right)\right]_b^0 + \lim_{c\to\infty} \left[\arctan\left(\frac{x}{2}\right)\right]_0^c$$
$$= \left(0 - \left(-\frac{\pi}{2}\right)\right) + \left(\frac{\pi}{2} - 0\right) = \pi$$

6(b) $p<1$일때 수렴

42. If $p=1$, $\int_0^1 \frac{1}{x}\,dx = \lim_{a\to 0^+} \ln x\Big]_a^1 = \lim_{a\to 0^+} -\ln a = \infty$.

Diverges. If $p \neq 1$,

$$\int_0^1 \frac{1}{x^p}\,dx = \lim_{a\to 0^+}\left[\frac{x^{1-p}}{1-p}\right]_a^1 = \lim_{a\to 0^+}\left[\frac{1}{1-p} - \frac{a^{1-p}}{1-p}\right].$$

This converges to $\frac{1}{1-p}$ if $1-p>0$ or $p<1$.

20 $\Gamma(n) = \int_0^\infty x^{n-1}e^{-x}dx$

(a) $\Gamma(1) = \int_0^\infty e^{-x}dx = \lim_{b\to\infty}[-e^{-x}]_0^b = 1,$

$\Gamma(2) = \int_0^\infty xe^{-x}dx = \lim_{b\to\infty}[-e^{-x}(x+1)]_0^b = 1,$

$\Gamma(3) = \int_0^\infty x^2e^{-x}dx = \lim_{b\to\infty}[-x^2e^{-x} - 2xe^{-x} - 2e^{-x}]_0^b = 2$

(b) $\Gamma(n+1) = \int_0^\infty x^ne^{-x}dx = \lim_{b\to\infty}[-x^ne^{-x}]_0^b + \lim_{b\to\infty} n\int_0^b x^{n-1}e^{-x}dx = 0 + n\Gamma(n)$

$(u = x^n,\ dv = e^{-x}dx)$

(c) $\Gamma(n) = (n-1)!$

21(b)

$f(t) = t^2$

$$F(s) = \int_0^\infty t^2e^{-st}\,dt = \lim_{b\to\infty}\left[\frac{1}{s^3}(-s^2t^2 - 2st - 2)e^{-st}\right]_0^b$$

$$= \frac{2}{s^3},\ s>0$$