수리통계학 I (002) 과제 4 : 소프트웨어학부 컴퓨터과학전공 2016133 이유진

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4.1-1

(a) $\sum\sum_{(x,y)\in S} f(x,y) = c\{(1+2\cdot1)+(1+2\cdot2)+(1+2\cdot3)+(2+2\cdot1)+(2+2\cdot2)+(2+2\cdot3)\}$

$= c\cdot 33 = 1 \rightarrow c=\frac{1}{33}$ ✓

(b) $\sum\sum_{(x,y)\in S} f(x,y) = c\{(1+1)+(2+1)+(2+2)+(3+1)+(3+2)+(3+3)\} = c\cdot 24 = 1$

$\rightarrow c=\frac{1}{24}$ ✓

(c) $6 \le x+y \le 8$, $0 \le y \le 5$ 를 만족하는 정수일때 (x,y) 의 조합은 $3\times6=18$개

$\therefore \sum\sum_{(x,y)\in S} f(x,y) = 18c=1 \rightarrow c=\frac{1}{18}$ ✓

(d) $\sum\sum_{(x,y)\in S} f(x,y) = c\{\frac{1}{4}\sum_{y=1}^{\infty}(\frac{1}{3})^y + (\frac{1}{4})^2\sum_{y=1}^{\infty}(\frac{1}{3})^y + \cdots\} = c\{\frac{1}{4}\cdot\frac{\frac{1}{3}}{1-\frac{1}{3}} + (\frac{1}{4})^2\cdot\frac{\frac{1}{3}}{1-\frac{1}{3}} + \cdots\}$

즉 $c\cdot\frac{\frac{1}{3}}{1-\frac{1}{3}}\cdot\frac{\frac{1}{4}}{1-\frac{1}{4}}$ 과 같다 (등비급수의 합)

$\therefore c\cdot\frac{1}{2}\cdot\frac{1}{3}=1 \rightarrow c=6$ ✓

4.1-5

(a) $f(x,y) = P(X=x, Y=y) = \frac{15!}{x!\,y!\,(15-x-y)!}(\frac{6}{10})^x(\frac{3}{10})^y(\frac{1}{10})^{15-x-y}$ ✓

상항분포를 따르며 x와 y는 $x+y\le15$를 만족하는 음이 아닌 정수

이때 $X\sim b(15, 0.6)$, $Y\sim b(15, 0.3)$ 이다.

(b)

−2

(c) $P(X=10, Y=4) = \frac{15!}{10!\,4!\,1!}(\frac{6}{10})^{10}(\frac{3}{10})^4(\frac{1}{10})^1 = 0.07354$ ✓

(d) X의 주변 pmf $f_X(x) = \sum_{y\in S_y} f(x,y) = \sum_{y=0}^{15-x}\frac{15!}{x!\,y!\,(15-x-y)!}(\frac{6}{10})^x(\frac{3}{10})^y(\frac{1}{10})^{15-x-y}$

$= \frac{15!}{x!\,(15-x)!}(\frac{6}{10})^x\cdot\sum_{y=0}^{15-x}\frac{(15-x)!}{y!\,(15-x-y)!}(\frac{3}{10})^y(\frac{1}{10})^{15-x-y}$

(이항정리 $(a+b)^m = \sum_{k=0}^{m}\binom{m}{k}a^k b^{m-k}$ 에 의해 정리하면)

$= \frac{15!}{x!\,(15-x)!}(\frac{6}{10})^x\cdot(1-\frac{6}{10})^{15-x} \rightarrow X\sim b(15, 0.6)$ ✓

(e) $P(X\le11) = \sum_{x=0}^{11}\binom{15}{x}0.6^x\,0.4^{15-x} = 0.90950$ ✓

4.2-3

−10

$K(a,b) = E[(Y-a-bX)^2] = E[Y^2 - (a+bX)Y + (a+bx)^2]$ $\therefore \rho = \frac{Cov(X,Y)}{\sigma_X\sigma_Y} = \frac{E(XY) - \mu_X\mu_Y}{\sigma_X\sigma_Y}$

$= E(Y^2) - E[(a+bX)Y] + E(a^2 + 2abX + b^2X^2)$

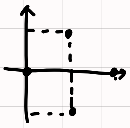
$= E(Y^2) - aE(Y) + bE(XY) + a^2 + 2ab\,E(X) + b^2E(X^2)$

$= \sigma_Y^2 + \mu_Y^2 - a\mu_Y + b(\mu_X\mu_Y + \rho\sigma_X\sigma_Y) + a^2 + 2ab\mu_X + b^2(\sigma_X^2 + \mu_X^2)$

① $\frac{\partial K}{\partial a} =$

4.2-4

(a) $f(x,y) = \frac{1}{4}$, $(x,y) = (0,0), (1,1), (1,-1), (2,0)$



공간이 '직사각형' 공간형이 아니므로 X와 Y는 독립이 아니다.

Y \ X	0	1	2	합
-1	0	$\frac{1}{4}$	0	$\frac{1}{4}$
0	$\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{2}$
1	0	$\frac{1}{4}$	0	$\frac{1}{4}$
합	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	1

$f_X(x) = \sum_{y \in S_Y} f(x,y) = \begin{cases} \sum_{y=0}^{0} f(x,y) & \text{when } x = 0, 2 \\ \sum_{y=-1}^{1} f(x,y) & \text{when } x = 1, y \neq 0 \end{cases}$

$f_Y(y) = \sum_{x \in S_X} f(x,y) = \begin{cases} \sum_{x=1}^{1} f(x,y) & \text{when } y = -1, 1 \\ \sum_{x=0}^{2} f(x,y) & \text{when } y = 0,\ x \neq 1 \end{cases}$

$f(1,1) \neq f_X(1) \cdot f_Y(1)$ 이므로 X와 Y는 독립이 아니다. ✓

$\downarrow \frac{1}{4}$ $\quad \downarrow \frac{1}{2}$ $\quad \downarrow \frac{1}{4}$

(b) $Cov(X,Y) = \sum_x\sum_y xy\,f(x,y) = 0 \cdot 0 \cdot \frac{1}{4} + 1 \cdot (-1) \cdot \frac{1}{4} + (-1) \cdot 1 \cdot \frac{1}{4} + 2 \cdot 0 \cdot \frac{1}{4} = 0$ ✓

$\rho = \frac{Cov(X,Y)}{\sigma_X\sigma_Y}$ 에서 분자가 0이므로 $\rho = 0$ ✓

분산을 계산해보면

$E(X) = \sum_x x f_X(x) = 0 \cdot \frac{1}{4} + 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4} = 1$

$E(X^2) = \sum_x x^2 f_X(x) = 0^2 \cdot \frac{1}{4} + 1^2 \cdot \frac{1}{2} + 2^2 \cdot \frac{1}{4} = \frac{3}{2}$ $\rightarrow Var(X) = E(X^2) - E(X)^2 = \frac{3}{2} - 1^2 = \frac{1}{2}$

$E(Y) = \sum_y y f_Y(y) = -1 \cdot \frac{1}{4} + 0 \cdot \frac{1}{2} + 1 \cdot \frac{1}{4} = 0$

$E(Y^2) = \sum_y y^2 f_Y(y) = (-1)^2 \cdot \frac{1}{4} + 0^2 \cdot \frac{1}{2} + 1^2 \cdot \frac{1}{4} = \frac{1}{2}$ $\rightarrow Var(Y) = E(Y^2) - E(Y)^2 = \frac{1}{2} - 0^2 = \frac{1}{2}$

$\rightarrow \rho = \frac{Cov(X,Y)}{\sigma_X\sigma_Y} = \frac{0}{\sqrt{\frac{1}{2}}\sqrt{\frac{1}{2}}} = 0$

#4.3-1

(a) $f_X(x) = \sum_{y \in S_Y} f(x,y) = \sum_{y=1}^{4} \frac{x+y}{32} = \frac{4x+10}{32}$ when $x=1,2$ ✓

$f_Y(y) = \sum_{x \in S_X} f(x,y) = \sum_{x=1}^{2} \frac{x+y}{32} = \frac{3+2y}{32}$ when $y=1,2,3,4$

(b) $g(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{\frac{x+y}{32}}{\frac{3+2y}{32}} = \frac{x+y}{3+2y}$ when $x=1,2$, $y=1,2,3,4$ ✓

(c) $h(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{\frac{x+y}{32}}{\frac{4x+10}{32}} = \frac{x+y}{4x+10}$ when $x=1,2$, $y=1,2,3,4$ ✓

(d) $P(1 \le Y \le 3 \mid X=1) = \sum_{y=1}^{3} \frac{y+1}{14} = \frac{9}{14}$ ✓

$P(Y \le 2 \mid X=2) = \sum_{y=1}^{2} \frac{y+2}{18} = \frac{7}{18}$ ✓

$P(X=2 \mid Y=3) = \frac{2+3}{3+2\cdot 3} = \frac{5}{9}$ ✓

(e) $E(Y|X=1) = \sum_{y=1}^{4} y\, h(y|1) = 1\cdot\frac{2}{14} + 2\cdot\frac{3}{14} + 3\cdot\frac{4}{14} + 4\cdot\frac{5}{14} = \frac{20}{7}$ ✓

$Var(Y|X=1) = \sum_{y=1}^{4} y^2 h(y|1) - E(Y|1)^2$

$= 1^2\cdot\frac{2}{14} + 2^2\cdot\frac{3}{14} + 3^2\cdot\frac{4}{14} + 4^2\cdot\frac{5}{14} - \left(\frac{20}{7}\right)^2 = \frac{55}{49}$ ✓

#4.4-2

$f(x,y) = 2e^{-x-y}$, $0 \le x \le y \le \infty$ 에서

$f_X(x) = \int_x^\infty f(x,y)\,dy = \int_x^\infty 2e^{-x-y}\,dy = 2e^{-2x}$, $0 \le x < \infty$ ✓

$f_Y(y) = \int_0^y f(x,y)\,dx = \int_0^y 2e^{-x-y}\,dx = 2e^{-2y}\cdot(e^y - 1)$, $0 \le y < \infty$ ✓

support가 서로 의존적이므로 X와 Y는 독립이 아니다. → $f(x,y) = f(x)\cdot f(y)$ 임을 확인

#4.4-7

(a) $f(x,y) = 2$, $0 \le y \le x \le 1$ 에서

$f_X(x) = \int_0^x f(x,y)\,dy = 2x$, $0 \le x \le 1$ ✓

$f_Y(y) = \int_y^1 f(x,y)\,dy = 2(1-y)$, $0 \le y \le 1$

(b) $E(X) = \int_0^1 x\cdot 2x\,dx = \frac{2}{3}$ ✓

$E(X^2) = \int_0^1 x^2\cdot 2x\,dx = \frac{1}{2}$

→ $Var(X) = E(X^2) - E(X)^2 = \frac{1}{2} - \left(\frac{2}{3}\right)^2 = \frac{1}{18}$ ✓

$E(Y) = \int_0^1 y\cdot 2(1-y)\,dy = \frac{1}{3}$ ✓

$E(Y^2) = \int_0^1 y^2\cdot 2(1-y)\,dy = \frac{1}{6}$

→ $Var(Y) = E(Y^2) - E(Y)^2 = \frac{1}{6} - \left(\frac{1}{3}\right)^2 = \frac{1}{18}$ ✓

$E(XY) = \int_0^1 \int_0^x xy\cdot 2\,dy\,dx = \int_0^1 x^3\,dx = \frac{1}{4}$

$\therefore Cov(X,Y) = E(XY) - E(X)E(Y) = \frac{1}{4} - \frac{2}{3}\cdot\frac{1}{3} = \frac{1}{36}$ ✓

$\rho_{XY} = \frac{Cov(X,Y)}{\sqrt{Var(X)}\sqrt{Var(Y)}} = \frac{\frac{1}{36}}{\sqrt{\frac{1}{18}}\sqrt{\frac{1}{18}}} = \frac{1}{2}$ ✓

※ $\mu_X = E(X)$, $\mu_Y = E(Y)$
$\sigma_X^2 = Var(X)$, $\sigma_Y^2 = Var(Y)$

(c) $y = \mu_Y + b(x - \mu_X)$ → 최소제곱회귀직선

$K(b) = E(d^2) = E[\{(Y-\mu_Y) - b(X-\mu_X)\}^2]$

$= E[(Y-\mu_Y)^2 - 2b(X-\mu_X)(Y-\mu_Y) + b^2(X-\mu_X)^2]$

$= \sigma_Y^2 - 2b\sigma_{XY} + b^2\sigma_X^2$

이때 $K''(b) = 2\sigma_X^2 > 0$ 이므로 $K(b)$를 b에 대해 미분하고 0으로 놓아 그 최솟값을 구하면

$\frac{\partial K(b)}{\partial b} = 2\sigma_{XY} + 2b\sigma_X^2 = -2\rho_{XY}\sigma_X\sigma_Y + 2b\sigma_X^2 = 0$

$\rightarrow b = \rho_{XY}\frac{\sigma_Y}{\sigma_X} = \frac{1}{2}\frac{\sqrt{\frac{1}{18}}}{\sqrt{\frac{1}{18}}} = \frac{1}{2}$

그림 −1

따라서 최소제곱회귀직선은 $y = \frac{1}{3} + \frac{1}{2}\left(x - \frac{2}{3}\right) = \frac{1}{2}x$ ✓

#4.4-10

$X \sim U(0,2)$, $Y|X=x \sim U(0, x^2)$ 일때

(a) $(X,Y) \sim U((0,2), (0,x^2))$ 이므로 $f(x,y) = \frac{1}{2x^2}$, $0<x<2$, $0<y<x^2$ ✓

(b) −2

(c) −3, (d) −3

#4.5-6	−16
#4.5-8	−10