8.6절

#1 $H_0 : \mu = 25$ vs $H_1 : \mu < 25$ $\quad C : \bar{x} \le 22.5$

$X \sim N(\mu, 9)$ $\quad n=4$ $\Rightarrow$ $\bar{X} \sim N(4, \frac{9}{4})$

(a) $K(\mu)$, $\alpha = K(25)$?

$K(\mu) = P(\bar{X} \le 22.5 ; \mu) = P\left(Z \le \frac{22.5-\mu}{3/2}\right) = \Phi\left(\frac{22.5-\mu}{1.5}\right)$

$\alpha = K(25) = \Phi\left(\frac{22.5-25}{1.5}\right) = \Phi(-1.667) = 0.0478$

답 : $K(\mu) = \Phi\left(\frac{22.5-\mu}{1.5}\right)$ $\quad \alpha = K(25) = 0.0478$

(b) $x_1 = 21.24$ $\quad x_2 = 24.81$ $\quad x_3 = 23.62$ $\quad x_4 = 26.82$

$\bar{x} = \frac{1}{4}\sum_{i=1}^{4} x_i = 24.1225 > 22.5$ 이므로 H_0 기각 못한다.

(c) (b)에서의 p값

p값 $= P(\bar{X} \le 24.1225 ; \mu) = \Phi\left(\frac{24.1225-25}{1.5}\right) = \Phi(-0.585) = 0.2793$ $\quad$ 답 : 0.2793

8.6-3

$X \sim N(1.5,\ 0.09)$

$$\begin{cases} H_0 : \mu = 1.5 \\ H_1 : \mu > 1.5 \end{cases}$$

$$K(\mu) = P(\overline{X} \geq c\ ;\ \mu) = 1 - \Phi\left(\frac{c-\mu}{0.3/\sqrt{n}}\right)$$

$$K(1.5) = 1 - \Phi\left(\frac{c-1.5}{0.3/\sqrt{n}}\right) = 0.05$$

$$\Phi\left(\frac{c-1.5}{0.3/\sqrt{n}}\right) = 0.95 \rightarrow \frac{c-1.5}{0.3/\sqrt{n}} = 1.645 \qquad \sqrt{n}(c-1.5) = 0.4935 \cdots ①$$

$$K(1.7) = 1 - \Phi\left(\frac{c-1.7}{0.3/\sqrt{n}}\right) = 0.95$$

$$\Phi\left(\frac{c-1.7}{0.3/\sqrt{n}}\right) = 0.05 \rightarrow \frac{c-1.7}{0.3/\sqrt{n}} = -1.645 \qquad \sqrt{n}(c-1.7) = -0.4935 \cdots ②$$

$0.2\sqrt{n} = 0.987$

$\sqrt{n} = 4.935$

$n = 24.35 \rightarrow$ n은 정수이므로 25

$$c = \frac{(-0.4935)}{5} + 1.7 = 1.6013$$

#8.6-4

$p=0.4$

$n=25$

$$\begin{cases} H_0: p=0.4 \\ H_1: p>0.4 \end{cases}$$

Y: 첫번째 서브성공 횟수

$C=\{Y: Y\geq 14\}$

(a)

$$K(p)=\sum_{y=14}^{25}\binom{25}{y}p^y(1-p)^{25-y} \quad , \ 0.4\leq p\leq 1$$

(b)

$$\text{유의수준 } \alpha = K(0.40)=\sum_{y=14}^{25}\binom{25}{y}(0.4)^y(0.6)^{25-y}$$
$$=1-\sum_{y=0}^{13}\binom{25}{y}(0.4)^y(0.6)^{25-y}$$
$$=1-(0.9222)$$
$$=0.0778$$

(c)

i) $$K(0.45)=\sum_{y=14}^{25}\binom{25}{y}(0.45)^y(0.55)^{25-y}$$
$$=1-\sum_{y=0}^{13}\binom{25}{y}(0.45)^y(0.55)^{25-y}$$
$$=1-0.8173$$
$$=0.1827$$

ii) $$K(0.5)=\sum_{y=14}^{25}\binom{25}{y}(0.5)^y(0.5)^{25-y}$$
$$=1-\sum_{y=0}^{13}\binom{25}{y}(0.5)^y(0.5)^{25-y}$$
$$=1-0.6550$$
$$=0.345$$

iii) $$K(0.6)=\sum_{y=14}^{25}\binom{25}{y}(0.6)^y(0.4)^{25-y}$$
$$=1-\sum_{y=0}^{13}\binom{25}{y}(0.6)^y(0.4)^{25-y}$$
$$\Rightarrow \quad 25-y=x$$
$$=1-\sum_{x=12}^{25}\binom{25}{x}(0.6)^{25-x}(0.4)^x$$
$$=1-\left\{1-\sum_{x=0}^{11}\binom{25}{x}(0.6)^{25-x}(0.4)^x\right\}$$
$$=\sum_{x=0}^{11}\binom{25}{x}(0.6)^{25-x}(0.4)^x$$
$$=0.7323$$

iv) $$K(0.7)=\sum_{y=14}^{25}\binom{25}{y}(0.7)^y(0.3)^{25-y}$$
$$=1-\sum_{y=0}^{13}\binom{25}{y}(0.7)^y(0.3)^{25-y}$$
$$=1-\sum_{x=12}^{25}\binom{25}{x}(0.7)^{25-x}(0.3)^x$$
$$=\sum_{x=0}^{11}\binom{25}{x}(0.7)^{25-x}(0.3)^x$$
$$=0.9558$$

v) $K(0.8) = \sum_{y=14}^{25} \binom{25}{y} (0.8)^y (0.2)^{25-y}$

$= 1 - \sum_{y=0}^{13} \binom{25}{y} (0.8)^y (0.2)^{25-y}$

$= 1 - \sum_{x=12}^{25} \binom{25}{x} (0.8)^{25-x} (0.2)^x$

$= 1 - \left\{ 1 - \sum_{x=0}^{11} \binom{25}{x} (0.8)^{25-x} (0.2)^x \right\}$

$= 0.9985$

vi) $K(0.9) = \sum_{y=14}^{25} \binom{25}{y} (0.9)^y (0.1)^{25-y}$

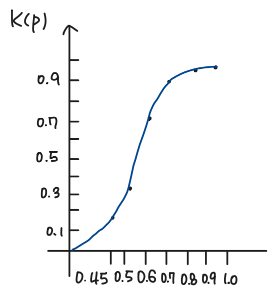
$= 1 - \sum_{y=0}^{13} \binom{25}{y} (0.9)^y (0.1)^{25-y}$

$= 1 - \sum_{x=12}^{25} \binom{25}{x} (0.9)^{25-x} (0.1)^x$

$= \sum_{x=0}^{11} \binom{25}{x} (0.9)^{25-x} (0.1)^x$

$= 1$

(d) $K(0.45) = 0.1827$ $K(0.5) = 0.345$ $K(0.6) = 0.7323$ $K(0.7) = 0.9558$ $K(0.8) = 0.9985$ $K(0.9) = 1$



(e) $Y=15$이면, 기각역 $C = \{y : y \geq 14\}$에 포함되므로 H_0 기각된다.

(f) $P(Y \geq 15 ; p = 0.4) = \sum_{y=15}^{25} \binom{25}{y} (0.4)^y (0.6)^{25-y}$

$= 1 - \sum_{y=0}^{14} \binom{25}{y} (0.4)^y (0.6)^{25-y}$

$= 1 - 0.9656$

$= 0.0344$

8.1절

#2 X의 pdf $f(x;\theta) = \frac{1}{\theta} e^{-\frac{x}{\theta}}$ $0 < x < \infty$ $X_1, X_2, \cdots, X_n$

(a) $H_0 : \theta = 3$ vs $H_1 : \theta = 5$

$$L(\theta) = \prod_{i=1}^{n} \frac{1}{\theta} e^{-\frac{x_i}{\theta}} = \left(\frac{1}{\theta}\right)^n e^{-\frac{\Sigma x_i}{\theta}}$$

$$\frac{L(3)}{L(5)} = \frac{\left(\frac{1}{3}\right)^n e^{-\frac{\Sigma x_i}{3}}}{\left(\frac{1}{5}\right)^n e^{-\frac{\Sigma x_i}{5}}} = \left(\frac{5}{3}\right)^n e^{-\frac{2}{15}\sum_{i}^{n} x_i} \le k$$

$$\Rightarrow e^{-\frac{2}{15}\sum_{i}^{n} x_i} \le \left(\frac{3}{5}\right)^n k$$

$$\Rightarrow -\frac{2}{15}\sum_{i}^{n} x_i \le \log\left[\left(\frac{3}{5}\right)^n k\right]$$

$$\Rightarrow \sum_{i=1}^{n} x_i \ge -\frac{15}{2}\log\left[\left(\frac{3}{5}\right)^n k\right]$$

$$\Rightarrow \sum_{i=1}^{n} x_i \ge -\frac{15}{2}\left[\log k - \log\left(\frac{3}{5}\right)^n\right] = c.$$

(b) $n=12$ $\frac{2}{\theta}\sum_{i=1}^{12} X_i \sim \chi^2(24)$ $\alpha = 0.1$ 인 최량기각역

$$0.1 = P\left(\sum_{i=1}^{12} X_i \ge -\frac{15}{2}\left(\log k - \log\left(\frac{5}{3}\right)^{12}\right) ; \theta = 3\right)$$

$$= P\left(\frac{8}{12}\sum_{i=1}^{12} X_i \ge -5\left(\log k - \log\left(\frac{5}{3}\right)^{12}\right) ; \theta = 3\right)$$

$\hookrightarrow \chi^2_{0.1}(24) = 33.2$

$$= P(8\bar{X} \ge 33.2)$$

$$\Rightarrow \bar{x} \ge 4.15$$

(c) $n=12$ $H_0 : \theta = 3$ vs $H_1 : \theta = 7$ $\alpha = 0.1$ 인 최량기각역.

기각역을 구할 때 대립가설에 주어진 값을 사용하지 않으므로 (b)에서 구한 기각역과 동일하다. 답: $\bar{x} \ge 4.15$

(d) 답: Yes.

#8.7-3

$X \sim N(\mu, 36)$

(a)
$$\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

$$\begin{cases} H_0: \mu = 50 \\ H_1: \mu < 50 \end{cases}$$

$$\frac{L(50)}{L(\mu_1)} = \frac{(2\pi \cdot 36)^{-\frac{n}{2}} \exp\left\{-\frac{\sum_{i=1}^{n}(x_i-50)^2}{2 \cdot 36}\right\}}{(2\pi \cdot 36)^{-\frac{n}{2}} \exp\left\{-\frac{\sum_{i=1}^{n}(x_i-\mu_1)^2}{2 \cdot 36}\right\}}$$

$$= \exp\left\{-\frac{1}{72}\left(\sum_{i=1}^{n}(x_i-50)^2 - \sum_{i=1}^{n}(x_i-\mu_1)^2\right)\right\}$$

$$= \exp\left\{-\frac{1}{72}\left(2(\mu_1-50)\sum_{i=1}^{n}x_i + n(50^2-\mu_1^2)\right)\right\} \leq k$$

$$-\frac{1}{72}\left(2(\mu_1-50)\sum_{i=1}^{n}x_i + n(50^2-\mu_1^2)\right) \leq \ln k$$

$$2(\mu_1-50)\sum_{i=1}^{n}x_i \geq -72\ln k - n(50^2-\mu_1^2)$$

$\mu_1 < 50$
$\mu_1 - 50 < 0$

$$\bar{x} \leq \frac{-72\ln k - n(50^2-\mu_1^2)}{2n(\mu_1-50)}$$

$$\therefore C_2 = \left\{\bar{x} : \bar{x} \leq \frac{-72\ln k - n(50^2-\mu_1^2)}{2n(\mu_1-50)}\right\}$$

(b)

예제4.7의 결과: $C = \left\{\bar{x} : \bar{x} \geq \frac{-72\ln k - n(50^2-\mu_1^2)}{2n(\mu_1-50)}\right\} \Leftarrow \begin{cases} H_0: \mu = 50 \\ H_1: \mu > 50 \end{cases}$

(a)의 결과: $C_2 = \left\{\bar{x} : \bar{x} \leq \frac{-72\ln k - n(50^2-\mu_1^2)}{2n(\mu_1-50)}\right\} \Leftarrow \begin{cases} H_0: \mu = 50 \\ H_1: \mu < 50 \end{cases}$

이때, 기각역 C와 C_2는 대립가설의 부등호 방향에 따라 정해지기 때문에
복합대립가설이 양측일때 ($H_1 = \mu \neq 50$), 균일 최강력 검정은 항상 존재하지 않는다.

#8.7-5

$p(x;\theta) = \theta^{x}(1-\theta)^{1-x}, \quad x=0,1$

$$\begin{cases} H_0 : \theta = \frac{1}{2} \\ H_1 : \theta < \frac{1}{2} \end{cases}$$

$$\frac{L(\theta=\frac{1}{2})}{L(\theta_1)} = \frac{\left(\frac{1}{2}\right)^{\sum_{i=1}^{5} x_i}\left(\frac{1}{2}\right)^{1-\sum_{i=1}^{5} x_i}}{\theta_1^{\sum_{i=1}^{5} x_i}(1-\theta_1)^{1-\sum_{i=1}^{5} x_i}} \le K$$

$$= \frac{\left(\frac{1}{2}\right)}{\theta_1^{\sum_{i=1}^{5} x_i}(1-\theta_1)^{1-\sum_{i=1}^{5} x_i}} \le K$$

$$\frac{1}{\theta_1^{\sum_{i=1}^{5} x_i}(1-\theta_1)^{1-\sum_{i=1}^{5} x_i}} \le 2K$$

$$-\sum_{i=1}^{5} x_i \ln\theta_1 - \left(1-\sum_{i=1}^{5} x_i\right)\ln(1-\theta_1) \le \ln 2K$$

$$\left(-\sum_{i=1}^{5} x_i\right)(\ln\theta_1) - \ln(1-\theta_1) + \left(\sum_{i=1}^{5} x_i\right)\ln(1-\theta_1) \le \ln 2K$$

$$\sum_{i=1}^{5} x_i\left(\ln(1-\theta_1) - \ln\theta_1\right) \le \ln 2K + \ln(1-\theta_1)$$

$$\sum_{i=1}^{5} x_i \le \frac{\ln 2K + \ln(1-\theta_1)}{\ln(1-\theta_1) - \ln\theta_1}$$

네이만-피어슨 보조정리에 따라 최량기각역은 $C = \left\{ (x_1, \cdots, x_5) : \sum_{i=1}^{5} x_i \le \frac{\ln 2K + \ln(1-\theta_1)}{\ln(1-\theta_1) - \ln\theta_1} \right\}$

⇒ 이 최량기각역을 통해 most powerful test를 한다.

검정력 함수 : $K(\theta) = P((x_1, \cdots, x_5) \in C \mid \theta)$

$$= P\left(\sum_{i=1}^{5} x_i \le c\right) \quad c=1$$

$$= P\left(\sum_{i=1}^{5} x_i \le 1\right)$$

$$= P\left(\sum_{i=1}^{5} x_i = 0\right) + P\left(\sum_{i=1}^{5} x_i = 1\right)$$

$$= \binom{5}{0}\theta^0(1-\theta)^5 + \binom{5}{1}\theta^1(1-\theta)^4$$

$$= (1-\theta)^5 + 5\theta(1-\theta)^4 = (1-\theta)^4(5\theta+1-\theta) = (1-\theta)^4(4\theta+1), \quad 0 \le \theta \le \frac{1}{2}$$

8.8절

#3 $H_0: \mu=30$ $H_1: \mu\neq 30$

μ와 σ^2 unknown $n=9$ $\bar{x}=32.8$ $s=4$ $\alpha=0.05$

$$\lambda = \frac{L(\hat{\omega})}{L(\hat{\Omega})} \le k$$

i) $L(\hat{\omega})$

$\hat{\mu}=\bar{x}$, $\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2$

$$L(\hat{\Omega}) = \left(\frac{1}{2\pi\cdot\frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2}\right)^{\frac{n}{2}} \exp\left[-\frac{\Sigma(x_i-\bar{x})^2}{2\cdot\frac{1}{n}\Sigma(x_i-\bar{x})^2}\right] = \left(\frac{ne^{-1}}{2\pi\sum_{i=1}^{n}(x_i-\bar{x})^2}\right)^{\frac{n}{2}}$$

ii) $L(\hat{\omega})$

$\hat{\mu}=\mu_0$ $\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i-\mu_0)^2$

$$L(\hat{\omega}) = \left(\frac{1}{2\pi\cdot\frac{1}{n}\sum_{i=1}^{n}(x_i-\mu_0)^2}\right)^{\frac{n}{2}} \exp\left[-\frac{\Sigma(x_i-\mu_0)^2}{2\cdot\frac{1}{n}\Sigma(x_i-\mu_0)^2}\right] = \left(\frac{ne^{-1}}{2\pi\sum_{i=1}^{n}(x_i-\mu_0)^2}\right)^{\frac{n}{2}}$$

$$\lambda = \left(\frac{\Sigma(x_i-\bar{x})^2}{\Sigma(x_i-\mu_0)^2}\right)^{\frac{n}{2}} = \left(\frac{1}{1+\frac{\Sigma(x_i-\mu_0)^2}{\Sigma(x_i-\bar{x})^2}}\right)^{\frac{n}{2}} \le k$$

$$\Rightarrow \frac{\Sigma(x_i-\mu_0)^2}{\Sigma(x_i-\bar{x})^2} \ge k'$$

$$\Rightarrow \frac{n\,\Sigma(x_i-\mu_0)^2}{\frac{1}{n-1}\Sigma(x_i-\bar{x})^2} \ge k''$$

$$\therefore \frac{\Sigma(x_i-\mu_0)^2}{s^2/n} \ge k''$$

$$C: |T| = \left|\frac{\bar{X}-\mu_0}{S/\sqrt{n}}\right| \ge t_{\alpha/2}(n-1)$$

$|T| = \left|\frac{32.8-30}{4/3}\right| = 2.1 < t_{0.025}(8) = 2.306$ 이므로 H_0 기각 할 수 없다.

p-value $= 2\cdot P(t \ge 2.1) = 2\times 0.0345 = 0.069$

8.8 절

#4 $X \sim N(\mu, \sigma^2)$ $\quad \alpha = 0.1$

$H_0: \mu = 1.8$ vs $H_1: \mu > 1.8$ $\quad n=121 \quad \bar{x} = 1.84 \quad s = 0.2$

$$\lambda = \frac{L(\hat{\omega})}{L(\hat{\Omega})} \le k$$

i) $L(\hat{\Omega})$

$\hat{\mu} = \bar{x}$, $\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2$

$$L(\hat{\Omega}) = \left(\frac{1}{2\pi \cdot \frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2} \right)^{\frac{n}{2}} \exp\left[- \frac{\sum(x_i-\bar{x})^2}{2 \cdot \frac{1}{n}\sum(x_i-\bar{x})^2} \right] = \left(\frac{ne^{-1}}{2\pi \sum_{i=1}^{n}(x_i-\bar{x})^2} \right)^{\frac{n}{2}}$$

ii) $L(\hat{\omega})$

$\hat{\mu} = \mu_0 \quad \hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i-\mu_0)^2$

$$L(\hat{\omega}) = \left(\frac{1}{2\pi \cdot \frac{1}{n}\sum_{i=1}^{n}(x_i-\mu_0)^2} \right)^{\frac{n}{2}} \exp\left[- \frac{\sum(x_i-\mu_0)^2}{2 \cdot \frac{1}{n}\sum(x_i-\mu_0)^2} \right] = \left(\frac{ne^{-1}}{2\pi \sum_{i=1}^{n}(x_i-\mu_0)^2} \right)^{\frac{n}{2}}$$

$$\lambda = \left(\frac{\sum(x_i-\bar{x})^2}{\sum(x_i-\mu_0)^2} \right)^{\frac{n}{2}} = \left(\frac{1}{1 + \frac{\sum(x_i-\mu_0)^2}{\sum(x_i-\bar{x})^2}} \right)^{\frac{n}{2}} \le k$$

$$\Rightarrow \frac{\sum(x_i-\mu_0)^2}{\sum(x_i-\bar{x})^2} \ge k'$$

$$\Rightarrow \frac{n\sum(x_i-\mu_0)^2}{\frac{1}{n-1}\sum(x_i-\bar{x})^2} \ge k''$$

$$\therefore \frac{\sum(x_i-\mu_0)^2}{s^2/n} \ge k''$$

$$C: \quad T = \frac{\bar{X}-\mu_0}{S/\sqrt{n}} \ge t_\alpha(n-1)$$

$$T = \frac{1.84-1.8}{0.2/11} = 2.2 > t_{0.1,120} = 1.29$$ 이므로 H_0 기각할 수 있다.

p-value $= P(t \ge 2.2) \approx P(Z \ge 2.2) = 0.0139$ (n이 크므로)

8.8절

#5

(a) $\sigma_X^2 = \sigma_Y^2$, $H_0 : \mu_X = \mu_Y$ vs $H_1 : \mu_X \neq \mu_Y$

$\mu_X = \mu_Y = \mu$, $\sigma_X^2 = \sigma_Y^2 = \sigma^2$ 이면

$$\hat{\mu} = \frac{\sum_{i=1}^{n} x_i + \sum_{j=1}^{m} y_j}{n+m} , \quad \hat{\sigma}^2 = \frac{\sum_{i=1}^{n}(x_i - \hat{\mu})^2 + \sum_{j=1}^{m}(y_j - \hat{\mu})^2}{n+m}$$

$\mu_X \neq \mu_Y$, $\sigma_X^2 = \sigma_Y^2 = \sigma^2$ 이면

$$\hat{\mu}_X = \bar{x} , \quad \hat{\mu}_Y = \bar{y} \qquad \hat{\sigma}^2 = \frac{\sum_{i=1}^{n}(x_i - \bar{x})^2 + \sum_{j=1}^{m}(y_j - \bar{y})^2}{n+m}$$

$$\lambda = \frac{1}{\left\{ 1 + (\bar{x} - \bar{y})^2 / \left[\sum_{i=1}^{n}(x_i - \bar{x})^2 + \sum_{j=1}^{m}(y_i - \bar{y})^2 \right] \right\}^{\frac{n+m}{2}}}$$

자유도가 $(n+m-2)$인 t 통계량의 함수.

$$t = c \cdot \frac{\bar{x} - \bar{y}}{\sqrt{\sum_{i=1}^{n}(x_i - \bar{x})^2 + \sum_{j=1}^{m}(y_i - \bar{y})^2}}$$

c : x, y 와 관련없는 부분

(b)

$\sigma_X^2 = \sigma_Y^2$ 이면 $$\hat{\sigma}^2 = \frac{\sum_{i=1}^{n}(x_i - \bar{x})^2 + \sum_{j=1}^{m}(y_i - \bar{y})^2}{n+m}$$

$\sigma_X^2 \neq \sigma_Y^2$ 이면 $\hat{\sigma}_X^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2$ $\quad \hat{\sigma}_Y^2 = \frac{1}{m}\sum_{j=1}^{m}(y_i - \bar{y})^2$

$$\lambda = \frac{(n+m)^{\frac{n+m}{2}}}{n^{\frac{n}{2}} m^{\frac{m}{2}}} \frac{\left[\sum_{j=1}^{m}(y_j - \bar{y})^2 / \sum_{i=1}^{n}(x_i - \bar{x})^2 \right]^{\frac{m}{2}}}{\left[1 + \sum_{j=1}^{m}(y_j - \bar{y})^2 / \sum_{i=1}^{n}(x_i - \bar{x})^2 \right]^{\frac{n+m}{2}}}$$

자유도가 $m-1$, $n-1$ 인 F 통계량의 함수

$$F = \frac{\sum_{j=1}^{m}(y_i - \bar{y})^2 / (m-1)}{\sum_{i=1}^{n}(x_i - \bar{x})^2 / (n-1)}$$