Propositional Logic

Propositions (명제 = 이진논리)

A proposition is a statement that can be either true or false.

- “Yoonjin has an Apple laptop.”
- “Yoonjin is a professor.”
- “3 == 2 + 1” T
- “3 == 2 + 2” F

Not propositions:

- “Are you Bob?” 의문문
- “x == 7” 미지수
- “I am heavy.” 기준이 모호

구체화되면 명제가 되기도 함

Propositional variables 명제변수

명제값을 지닐 수 있는 변수

We use propositional variables to refer to propositions.

- Usually are lower case letters starting with p (i.e. *p, q, r,* etc.).
- A propositional variable can have one of two values: true (T) or false (F).

A proposition can be

- A single variable: *p*
- An operation of multiple variables: $p \wedge (q \vee \neg r)$ 명제 연산자

* 변수 → 연산자 → 함수 → …

: 일반화. 거기 무엇을 넣어도 y값을 알 수 있는 일반화 (패턴, 루틴)

일반화는 컴퓨터 그 자체이다. 단순작업 반복, 대량으로 빠르게

함수는 가장 기본적인 프로그래밍 단위. 코딩이란 이 틀을 만드는 것

Introduction to Logical Operators

논리연산자

About a dozen logical operators (이진논리에 국한된, 함수를 기술하기위해 필요한)

- Similar to algebraic operators + * - /

In the following examples,

- p = “Today is Friday.”
- q = “Today is my birthday.”

Logical operators: Not

A "not" operation switches (negates) the truth value.

Symbol: $\neg$ or ~ (tilda)

$\neg p \equiv$ "Today is not Friday."

p	$\neg p$
T	F
F	T

Logical operators: And

An “and” operation is true if both operands are true.

Symbol: $\wedge$

- It’s like the ‘A’ in And.

$p \wedge q \equiv$ “Today is Friday and today is my birthday.”

and 연산에 대한 일반화
즉 p와 q에 어떤것이 들어가도 연산할수 있게됨
일종의 함수 (≠ propositional function)
혼동할 여지가 있으므로 함수라고 지칭하지는 x

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Logical operators: Or

An "or" operation is true if either operands are true.

Symbol: $\vee$

$p \vee q \equiv$ "Today is Friday or today is my birthday (or possibly both)."

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

☆ 의미를 부여하는 것은 인간이다.
컴퓨터는 그냥 연산만 함
프로그래밍을 할 때 이것이 타당한지

공식없이 사람의 판단으로 고민해야함.

Logical operators: Exclusive Or

XOR

An exclusive or operation is true if one of the operands are true, but false if both are true.

다르면 T
같으면 F

Symbol: $\oplus$

+) 컴퓨터 시스템에 필수적인 논리회로 (덧셈기)
T=1, F=0 이라고 했을때 비트덧셈
cf) NOR : 저장장치

Often called XOR

$p \oplus q \equiv (p \vee q) \wedge \neg(p \wedge q)$

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

$p \oplus q \equiv$ "Today is Sunday or today is a weekday, but not both."

- weekday: Monday ~ Saturday

둘 중에 하나만 T. 아무 명제나 올 수 X

p / q

XOR에 적합한 명제

Logical operators: Conditional

A conditional means "if p then q" 조건문 중첩 → 논리적인 depth

Symbol: $\rightarrow$ p⊂q로 생각하자

$p \rightarrow q \equiv$ "If today is Sunday, then today is a holiday."

$p \rightarrow q \equiv \neg p \vee q$ ~(p 아q)

the antecedent or premise (전제)

the consequence or conclusion (결론)

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

일요일이 아닌 휴일 ←

일요일도 휴일도 아닌 날 ←

can be true

묶어서 생각하기

Such a conditional statement is also called an implication.

Note that if p is false, then the conditional can be true regardless of whether q is true or false.

Logical operators: Conditional

CAUTION!

Be careful when translating English sentences into Propositions.

For example, **unreasonable example**

잘못된 (비합리적인) 예제

- Let p = “Today is Friday.” and q = “Today is my birthday.” 상관이 없는 두 명제. 의미부여가 되지않음
- I state: $p \rightarrow q \equiv$ “If today is Friday, then today is my birthday.”
- Consider all possibilities

p	q	$p \rightarrow q$
T	T	?
T	F	?
F	T	?
F	F	?

Logical operators: Conditional

Reasonable example

Let p = “he is 5 years old.” and q = “he is a child.”

I state: $p \rightarrow q \equiv$ “If he is 5 years old, then he is a child.”

Consider all possibilities

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

2살

Logical operators: Conditional

Alternate ways of stating a conditional:

- *p* implies *q*
- If *p*, *q*
- *p* only if *q*
- *p* is sufficient for *q*
- *q* if *p*
- *q* whenever *p*
- *q* is necessary for *p*

p 와 q의관계가 반드시 reasonable 하다면
모두사용가능한표현

Logical operators: Conditional

				Conditional	Inverse	Converse	Contrapositive
p	q	$\neg p$	$\neg q$	$p \to q$	$\neg p \to \neg q$	$q \to p$	$\neg q \to \neg p$
T	T	F	F	T	T	T	T
T	F	F	T	F	T	T	F
F	T	T	F	T	F	F	T
F	F	T	T	T	T	T	T

$$\boxed{p \to q = \neg p \vee q}$$

Logical operators: Bi-conditional

A bi-conditional means "*p* if and only if *q*" = XNOR

Symbol: $\leftrightarrow$ 필요충분조건

Alternatively, it means
"(if *p* then *q*) and
(if *q* then *p*)"

같으면 T
cf. XOR은 다르면 T

Note that a bi-conditional
has the opposite truth values
of the Exclusive Or

Also called Exclusive Nor

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Logical operators: Bi-conditional

Let p = "Today is Saturday" and q = "Tomorrow is Sunday."

Then $p \leftrightarrow q$ means 뗄레야 뗄수 없는관계!

"Today is Saturday if and only if tomorrow is Sunday"

Alternatively, it means "If today is Saturday, then tomorrow is Sunday and if tomorrow is Sunday then today is Saturday"

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Boolean operators summary

의미부여 필수

		not	not	and	or	xor	conditional	Bi-conditional
p	q	$\neg p$	$\neg q$	$p \wedge q$	$p \vee q$	$p \oplus q$	$p \rightarrow q$	$p \leftrightarrow q$
T	T	F	F	T	T	F	T	T
T	F	F	T	F	T	T	F	F
F	T	T	F	F	T	T	T	F
F	F	T	T	F	F	F	T	T

Learn what they mean, don't just memorize the table!

Precedence of operators 우선순위

Just as in algebra, operators have precedence.

- 4+3*2 = 4+(3*2), not (4+3)*2

Precedence order (from highest to lowest):

$\neg \ \wedge \ \vee \ \rightarrow \ \leftrightarrow$

- The first three are the most important.

This means that $p \vee q \wedge \neg r \rightarrow s \leftrightarrow t$ (주석처리 같은. clarity)
yields: $((p \vee (q \wedge (\neg r))) \rightarrow s) \leftrightarrow (t)$ 괄호사용을 잘하자. 전달성에 있어서

$\neg$ (Not) is *always* performed before any other operation.

Propositional Equivalence

명제 등가

(증명에서 유용)

논리적인 depth가 이미 깊어진 것을 이해하기 위한 tool

Tautology, Contradiction, Equivalence

↖ 등가의 가장 기본적인 두 체계 ↗

negation laws : 부정의 법칙

① **Tautology**: a statement that's always true.

- $p \vee \neg p$ will always be true

  p or ~p

② **Contradiction**: a statement that's always false.

- $p \wedge \neg p$ will always be false

  p and ~p

③ A logical **equivalence** means that the two sides always have the same truth values.

- Symbol is $\equiv$

Logical Equivalences

12가지

$p \wedge T \equiv p$ $p \vee F \equiv p$	Identity Laws	$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative Laws
$p \vee T \equiv T$ $p \wedge F \equiv F$	Domination Laws	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	**Distributive Laws** 논리적인 step의 증명
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent Laws	$\neg (p \wedge q) \equiv \neg p \vee \neg q$ $\neg (p \vee q) \equiv \neg p \wedge \neg q$	**De Morgan's Laws**
$\neg(\neg p) \equiv p$	Double negation law	$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption Laws
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative Laws	$p \vee \neg p \equiv T$: Tautology $p \wedge \neg p \equiv F$: Contradiction	Negation Laws
$p \rightarrow q \equiv \neg p \vee q$	Definition of Implication	$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$	Definition of Bi-conditional

How to prove equivalence?

For example, Let's prove the below proposition.

- $(p \rightarrow r) \vee (q \rightarrow r) \equiv (p \wedge q) \rightarrow r$

Two methods:

- Using truth tables
  - Not good for long formula
  - In this course, only allowed if specifically stated!
- Using the **logical equivalences**
  - The preferred method

Truth Table Solution

$$(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r$$

p	q	r	$p \to r$	$q \to r$	$(p \to r) \vee (q \to r)$	$p \wedge q$	$(p \wedge q) \to r$
T	T	T	T	T	T	T	T
T	T	F	F	F	F	T	F
T	F	T	T	T	T	F	T
T	F	F	F	T	T	F	T
F	T	T	T	T	T	F	T
F	T	F	T	F	T	F	T
F	F	T	T	T	T	F	T
F	F	F	T	T	T	F	T

Proof using Logical Equivalence

(Proof)

$(p \rightarrow r) \vee (q \rightarrow r)$

$\equiv (\neg p \vee r) \vee (\neg q \vee r)$ Definition of implication

$\equiv \neg p \vee (r \vee \neg q) \vee r$ Associative 결합

$\equiv \neg p \vee (\neg q \vee r) \vee r$ Commutative 교환

$\equiv (\neg p \vee \neg q) \vee (r \vee r)$ Associative

$\equiv \neg (p \wedge q) \vee r$ De Morgan, Idempotent

$\equiv (p \wedge q) \rightarrow r$ Definition of implication

Example

Show that $(p \wedge q) \rightarrow (p \vee q)$ is a Tautology.

항상 T

(Proof)

$(p \wedge q) \rightarrow (p \vee q) \equiv T$

$\equiv \neg (p \wedge q) \vee (p \vee q)$	Definition of implication
$\equiv (\neg p \vee \neg q) \vee (p \vee q)$	De Morgan
$\equiv \neg p \vee (\neg q \vee p) \vee q$	Associative
$\equiv \neg p \vee (p \vee \neg q) \vee q$	Commutative
$\equiv (\neg p \vee p) \vee (\neg q \vee q)$	Associative
$\equiv T \vee T$	Negation
$\equiv T$	

p는 p끼리
q는 q끼리 최대한 모으기