#7.1-8

(a) μ와 σ에 대한 점추정치

$$\hat{\mu} = \frac{\sum_{i=1}^{20} x_i}{n} = \frac{(28.8 + \cdots + 24.4)}{20} = 25.475$$

$$\hat{\sigma} = \sqrt{\frac{\sum_{i=1}^{20}(x_i - \bar{x})^2}{n-1}} = \sqrt{\frac{(28.8 - 25.475)^2 + \cdots + (24.4 - 25.475)^2}{19}} = 2.494$$

(b) μ에 대해 하한을 가진 99% 단측신뢰구간

$$t_{0.01}(19) = 2.539$$

$$\left[\, 25.475 - (2.539)\frac{2.494}{\sqrt{20}},\ \infty \right) = [24.059,\ \infty)$$

7.1절

#9 $X_1, X_2, \cdots, X_n$ $N(\mu, \sigma^2)$ $n=5$

(a) 분산을 알 때,

95% CI $\Rightarrow$ $\bar{x} \pm 1.96 \frac{\sigma}{\sqrt{n}}$

$E(2 \cdot 1.96 \frac{\sigma}{\sqrt{5}}) = E(1.753\sigma) = 1.753\sigma$

답 : 1.753σ

(b) 분산을 모를 때,

95% CI $\Rightarrow$ $\bar{x} \pm t_{0.025, 4} \frac{S}{\sqrt{n}}$

$E(2 \times 2.776 \times \frac{S}{\sqrt{5}}) = E(2.483S) = 2.483E(S)$

$W = \frac{4S^2}{\sigma^2} \sim \chi^2(4)$

$E(W^{1/2}) = \int_0^\infty \frac{w e^{-\frac{w}{2}}}{\Gamma(2)2^2} \cdot w^{\frac{1}{2}} dw = \frac{\Gamma(\frac{5}{2})\sqrt{2}}{\Gamma(2)} \underbrace{\int_0^\infty \frac{w^{\frac{3}{2}} e^{-\frac{w}{2}}}{\Gamma(\frac{5}{2})2^{\frac{5}{2}}} dw}_{\text{gamma}(\frac{5}{2}, 2)}$

$= \sqrt{2}\,\Gamma(\frac{5}{2})$

$E(W^{1/2}) = E(\frac{2S}{\sigma}) = \frac{2}{\sigma}E(S) = \sqrt{2}\,\Gamma(\frac{5}{2})$ $\Rightarrow$ $E(S) = \frac{\sqrt{2}}{2}\Gamma(\frac{5}{2})\sigma = \frac{1}{\sqrt{2}} \times 1.329 \times \sigma$

$E(2 \times 2.776 \times \frac{S}{\sqrt{5}}) = 2.483 \times \frac{1}{\sqrt{2}} \times 1.329 \times \sigma = 2.333\sigma$

답 : 2.333σ

#7.2-4

(a) 새로운 제품에 대한 20개 인장력이 고객들이 요구하는 최소 인장력 값인 20파운드보다 작기 때문에 새로운 제품은 고객의 요구에 부합하지 않는다.

(b)

$\begin{cases} X: \text{기존 제품의 인장력} \\ Y: \text{새로운 제품의 인장력} \end{cases}$

$$\bar{x} = \frac{\sum_{i=1}^{20} x_i}{20} = 25.475 \qquad \bar{y} = \frac{\sum_{i=1}^{20} y_i}{20} = 12.88$$

$$S_x^2 = \frac{\sum_{i=1}^{20} (x_i - 25.475)}{19} = 6.218 \qquad S_y^2 = \frac{\sum_{i=1}^{20} (y_i - 12.88)}{19} = 2.759$$

$$\alpha = 0.1 \rightarrow t_{0.05}(33) = 1.692$$

$$r = \frac{\left(\frac{6.218}{20} + \frac{2.759}{20}\right)^2}{\frac{1}{19}\left(\frac{6.218}{20}\right)^2 + \frac{1}{19}\left(\frac{2.759}{20}\right)^2} = 33.088 \approx 33$$

평균 인장력 차이에 대한 90% 신뢰구간:

$$(\bar{x} - \bar{y}) \pm (1.692) \cdot \sqrt{\frac{S_x^2}{20} + \frac{S_y^2}{20}} = (25.475 - 12.88) \pm (1.692) \cdot \sqrt{\frac{6.218}{20} + \frac{2.759}{20}}$$
$$= (11.461,\ 13.729)$$

(C)

$x_{(1)} = 20.8,\quad x_{(20)} = 30.1$

$k = 19 \cdot (0.25) + 1 = 5.75 \rightarrow Q_1 = (0.75) \cdot x_{(5)} + (0.25) \cdot x_{(6)} = (0.75)(23.9) + (0.25)(24.4) = 24.025$

$k = 19 \cdot (0.5) + 1 = 10.5 \rightarrow Q_2 = (0.5) \cdot x_{(10)} + (0.5) \cdot x_{(11)} = (0.5)(25.1) + (0.5)(25.6) = 25.35$

$k = 19 \cdot (0.75) + 1 = 15.25 \rightarrow Q_3 = (0.25) \cdot x_{(15)} + (0.75) \cdot x_{(16)} = (0.25)(27.6) + (0.75)(27.7) = 27.675$

IQR = 3.65

$L = Q_1 - 1.5(3.65) = 18.55$

$U = Q_3 + 1.5(3.65) = 33.15$

$y_{(1)} = 8.5 \quad y_{(20)} = 14.8$

$Q_1 = (0.75)(11.9) + (0.25)(12.1) = 11.95$

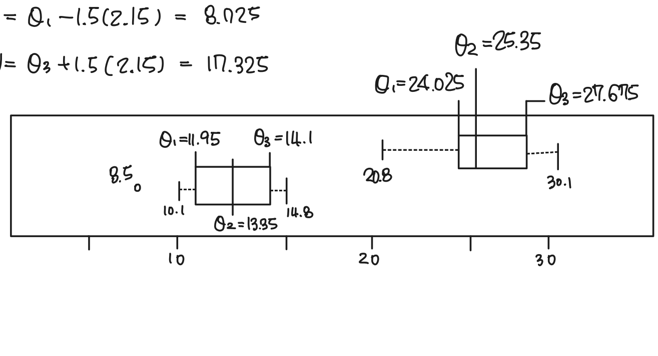
$Q_2 = (0.5)(13.2) + (0.5)(13.5) = 13.35$

$Q_3 = (0.25)(14.1) + (0.75)(14.1) = 14.1$

IQR = 2.15

$L = Q_1 - 1.5(2.15) = 8.025$

$U = Q_3 + 1.5(2.15) = 17.325$



(d)

Y : 새로운 제품의 인장력

$$\bar{y} = \frac{\sum_{i=1}^{20} Y_i}{20} = 12.88$$

$$S_Y^2 = \frac{\sum_{i=1}^{20} (Y_i - 12.88)}{19} = 2.759 \rightarrow S_Y = \sqrt{2.759} = 1.661$$

$$t_{0.05}(19) = 1.729$$

$\alpha = 0.05$에서 새로운 제품의 인장력에 대한 단측신뢰구간의 하한값이 $12.88 - (1.729)\frac{1.661}{\sqrt{20}} = 12.238$이기 때문에

고객의 요구(최소 인장력이 20파운드)에 부합하지 않으므로 기존 제품을 사용한다.

7.2절

#7 $\sigma_X^2 / \sigma_Y^2 = d$ (known)

(a) 진분

$\bar{X} \sim N(\mu_X, \frac{\sigma_X^2}{n_X})$ $\quad$ $\bar{Y} \sim N(\mu_Y, \frac{\sigma_Y^2}{n_Y})$

$\bar{X} - \bar{Y} \sim N(\mu_X - \mu_Y, \frac{\sigma_X^2}{n_X} + \frac{\sigma_Y^2}{n_Y})$

표준화

$$\frac{\bar{X}-\bar{Y} - (\mu_X - \mu_Y)}{\sqrt{\frac{\sigma_X^2}{n_X} + \frac{\sigma_Y^2}{n_Y}}} = \frac{\bar{X}-\bar{Y} - (\mu_X - \mu_Y)}{\sqrt{\frac{d\sigma_Y^2}{n_X} + \frac{\sigma_Y^2}{n_Y}}} \sim N(0,1)$$

$\sigma_X^2 = d\sigma_Y^2$ 이므로.

(b) 진분

$$\frac{(n_X-1)S_X^2}{\sigma_X^2} = \frac{(n_X-1)S_X^2}{d\sigma_Y^2} \sim \chi^2(n_X-1)$$

$$\frac{(n_Y-1)S_Y^2}{\sigma_Y^2} \sim \chi^2(n_Y-1)$$

$\frac{(n_X-1)S_X^2}{d\sigma_Y^2} = V$, $\frac{(n_Y-1)S_Y^2}{\sigma_Y^2} = W$ 라고 하면

$V+W$의 mgf $= M_{V+W}(t) = E(e^{Vt+Wt}) = E(e^{Vt})E(e^{Wt}) = M_V(t)M_W(t)$

$= (1-2t)^{-\frac{n_X-1}{2}}(1-2t)^{-\frac{n_Y-1}{2}} = (1-2t)^{-\frac{n_X+n_Y-2}{2}}$ ← $\chi^2(n_X+n_Y-2)$의 mgf

$$\therefore \frac{(n_X-1)S_X^2}{d\sigma_Y^2} + \frac{(n_Y-1)S_Y^2}{\sigma_Y^2} \sim \chi^2(n_X+n_Y-2)$$

(c) 진분

표본 평균과 표본분산은 독립이므로,
$\bar{X}$와 S_X^2 은 독립이고, $\bar{Y}$와 S_Y^2 은 독립이다. 따라서

$$\frac{(\bar{X}-\bar{Y}) - (\mu_X - \mu_Y)}{\sqrt{d\sigma_Y^2/n_X + \sigma_Y^2/n_Y}}$$ 와 $$\frac{(n_X-1)S_X^2}{d\sigma_Y^2} + \frac{(n_Y-1)S_Y^2}{\sigma_Y^2}$$ 는 독립이다.

(d)

(c) 를 바탕으로 다음과 같이 표현할 수 있다.

$$T = \frac{\dfrac{(\bar{X}-\bar{Y})-(\mu_X-\mu_Y)}{\sqrt{d\sigma_Y^2/n_X + \sigma_Y^2/n_Y}}}{\sqrt{\left(\dfrac{(n_X-1)S_X^2}{d\sigma_Y^2} + \dfrac{(n_Y-1)S_Y^2}{\sigma_Y^2}\right)\Big/(n_X+n_Y-2)}} \sim t(n_X+n_Y-2)$$

σ_Y^2 에 의존하지 않는 확률변수로 만들기 위해 분모, 분자에 $\sqrt{\dfrac{d\sigma_Y^2}{(n_X-1)(n_Y-1)}}$ 곱해준다.

$$T = \frac{[(\bar{X}-\bar{Y})-(\mu_X-\mu_Y)]\sqrt{\dfrac{\dfrac{d\cancel{\sigma_Y^2}\,1}{(n_X-1)(n_Y-1)}}{\dfrac{d\cancel{\sigma_Y^2}\,1}{n_X}+\dfrac{\cancel{\sigma_Y^2}}{n_Y\,d}}}}{\sqrt{\dfrac{d\sigma_Y^2}{(n_X-1)(n_Y-1)}\left[\dfrac{(n_X-1)S_X^2}{d\sigma_Y^2}+\dfrac{(n_Y-1)S_Y^2}{\sigma_Y^2}\right]\times\dfrac{1}{(n_X+n_Y-2)}}}$$

$$= \frac{[(\bar{X}-\bar{Y})-(\mu_X-\mu_Y)]\sqrt{\dfrac{d\,n_X n_Y}{(d n_Y+n_X)\cancel{(n_X-1)(n_Y-1)}}}}{\sqrt{\dfrac{(n_X-1)S_X^2+d(n_Y-1)S_Y^2}{\cancel{(n_X-1)(n_Y-1)}}\times\dfrac{1}{(n_X+n_Y-2)}}}$$

$$= \frac{(\bar{X}-\bar{Y})-(\mu_X-\mu_Y)}{\sqrt{\dfrac{(n_X-1)S_X^2+d(n_Y-1)S_Y^2}{(n_X+n_Y-2)}}\sqrt{\dfrac{d n_Y+n_X}{d n_X n_Y}}}$$

$$= \frac{(\bar{X}-\bar{Y})-(\mu_X-\mu_Y)}{\sqrt{\dfrac{(n_X-1)S_X^2+d(n_Y-1)S_Y^2}{(n_X+n_Y-2)}}\sqrt{\dfrac{1}{n_X}+\dfrac{1}{d n_Y}}} \qquad \blacksquare$$

T는 σ_Y^2에 의존하지 않으면서 자유도가 $(n+m-2)$인 t 분포를 따른다.

#7.3-3

(a)

$$\hat{p} \pm z_{\alpha/2} \cdot \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \qquad \alpha=0.05 \quad n=537 \quad \hat{p} = \frac{460}{537} = 0.857$$

$$0.857 \pm 1.96\sqrt{\frac{(0.857)(0.143)}{537}} = [0.827,\ 0.887]$$

(b)

$$1+\frac{z_0^2}{n} = 1+\frac{(1.96)^2}{537} = 1.007$$

$$\hat{p}+\frac{z_0^2}{2n} = 0.857 + \frac{(1.96)^2}{2(537)} = 0.861$$

$$(1.96)\sqrt{\frac{(0.857)(0.143)}{537} + \frac{(1.96)^2}{4(537)^2}} = 0.030$$

$$\frac{0.861 \pm 0.030}{1.007} = [0.825,\ 0.885]$$

(c)

$$\tilde{p} = \frac{460 + (1.96)^2/2}{537 + (1.96)^2} = 0.854$$

$$0.854 \pm (1.96)\sqrt{\frac{(0.854)(1-0.854)}{537+(1.96)^2}} = [0.824,\ 0.884]$$

(d) $z_{0.95} = 1.645$

95% 단측 신뢰구간: $\hat{p} + (1.645) \cdot \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = 0.857 + (1.645)\sqrt{\frac{(0.857)(0.143)}{537}}$

$= 0.882$

$\Rightarrow [0,\ 0.882]$

∴ 공공보건관리는 면역프로그램을 실시해야 한다.

7.3절

#5

(a) $\hat{p}_1 = \frac{1009}{1230}$ $\quad$ $\hat{p}_2 = \frac{207}{340}$ $\quad$ $\hat{p}_1 - \hat{p}_2 = 0.2115$ $\quad$ 답: 0.2115

(b)

$p_1 - p_2$의 근사적 95% CI

$$\Rightarrow \hat{p}_1 - \hat{p}_2 \pm 1.96 \sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n_1} + \frac{\hat{p}_2(1-\hat{p}_2)}{n_2}}$$

$$0.2115 \pm 1.96 \sqrt{\frac{\frac{1009}{1230} \times \left(1-\frac{1009}{1230}\right)}{1230} + \frac{\frac{207}{340}\left(1-\frac{207}{340}\right)}{340}}$$

$$= 0.2115 \pm 1.96 \times 0.0286$$

$[0.1554, 0.2676]$ $\quad$ 답: $[0.1554, 0.2676]$