• (X,Y)의 결합확률밀도함수가 $f(x,y) = Cxy^2$, $0<x<1$, $0<y<1$로 주어져 있을 때

1) C의 값을 구하시오.

$$\int_0^1\int_0^1 Cxy^2\,dx\,dy = 1$$

$$\int_0^1\int_0^1 Cxy^2\,dx\,dy = \int_0^1 \left[\frac{C}{2}x^2y^2\right]_0^1 dy = \int_0^1 \frac{C}{2}y^2\,dy = \left[\frac{C}{6}y^3\right]_0^1 = \frac{C}{6} = 1$$

$\therefore\ C = 6$

2) 확률변수 X의 주변확률밀도함수를 구하시오.

$$f_X(x) = \int_0^1 f(x,y)\,dy = \int_0^1 6xy^2\,dy = \left[\frac{6}{3}xy^3\right]_0^1 = 2x$$

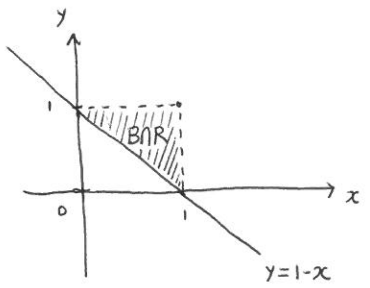
$\therefore\ f_X(x) = 2x$, $0<x<1$

3) 확률변수 Y의 주변확률밀도함수를 구하시오.

$$f_Y(y) = \int_0^1 f(x,y)\,dx = \int_0^1 6xy^2\,dx = \left[\frac{6}{2}x^2y^2\right]_0^1 = 3y^2$$

$\therefore\ f_Y(y) = 3y^2$, $0<y<1$

4) P(X+Y>=1)의 확률을 구하시오.



$R = \{(x,y) : f_{X,Y}(x,y) > 0\}$

$B = \{(x,y) : x+y \geq 1\}$

$B\cap R = \{(x,y) : 0<y<1,\ 1-y \leq x < 1\}$

$$P(X+Y\geq 1)$$

$$= P((X,Y)\in B)$$

$$= \iint_B f_{X,Y}(x,y)\,dx\,dy$$

$$= \iint_{B\cap R} f_{X,Y}(x,y)\,dx\,dy + \iint_{B\cap R^c} f_{X,Y}(x,y)\,dx\,dy$$

$$= \iint_{B\cap R} f_{X,Y}(x,y)\,dx\,dy$$

$$= \int_0^1\int_{1-y}^1 6xy^2\,dx\,dy$$

$$= \int_0^1 \left[\frac{6}{2}x^2y^2\right]_{1-y}^1 dy = \int_0^1 (3y^2-3(1-y)^2y^2)\,dy = \int_0^1 (3y^2-3y^2+6y^3-3y^4)\,dy = \int_0^1 (6y^3-3y^4)\,dy = \left[\frac{6}{4}y^4 - \frac{3}{5}y^5\right]_0^1 = \frac{3}{2} - \frac{3}{5} = \frac{9}{10}$$

$\therefore\ P(X+Y\geq 1) = \frac{9}{10}$

• (X,Y)의 결합확률밀도함수가 $f(x,y)=8xy$, $0<x<y<1$로 주어져 있을때 변환 $U=X/Y$, $V=Y$에 의해 새롭게 정의된 확률벡터 (U,V)에 대해

1) (U,V)의 결합확률 밀도함수를 구하시오.

$$U=\frac{X}{Y} \qquad V=Y \qquad \Rightarrow X=UV \quad Y=V$$

$$\frac{\partial(x,y)}{\partial(u,v)}=\begin{pmatrix} v & u \\ 0 & 1 \end{pmatrix} \qquad \det\frac{\partial(x,y)}{\partial(u,v)}=\det\begin{pmatrix} v & u \\ 0 & 1 \end{pmatrix}=v \qquad \left|\det\frac{\partial(x,y)}{\partial(u,v)}\right|=v$$

$$f_{U,V}(u,v)=f_{X,Y}(uv,v)\left|\det\frac{\partial(x,y)}{\partial(u,v)}\right|$$

$$=8uv\cdot v\cdot|v|$$

$$=8uv^3$$

범위 : $0<x<y<1$

$0<uv<v<1$

$0<u<1$, $0<v<1$

$\therefore f_{U,V}(u,v)=8uv^3$, $0<u<1$, $0<v<1$

2) 확률변수 U의 주변확률밀도함수를 구하시오.

$$f_U(u)=\int_0^1 f_{U,V}(u,v)\,dv=\int_0^1 8uv^3\,dv=\left[2uv^4\right]_0^1=2u$$

$\therefore f_U(u)=2u$, $0<u<1$

• 확률변수 X와 Y의 결합확률밀도함수가 $f(x,y)=10xy^2$, $0<x<y<1$로 주어져 있을때 변환 $U=X/Y$, $V=Y$에 의해 정의된 새로운 확률변수 U,V의 결합확률밀도함수를 구하시오.

$$U=\frac{X}{Y} \;,\; V=Y \quad \Rightarrow X=UV \quad Y=V$$

범위 : $0<x<y<1$

$0<uv<v<1$

$0<u<1$, $0<v<1$

$$\frac{\partial(x,y)}{\partial(u,v)}=\begin{pmatrix} v & u \\ 0 & 1 \end{pmatrix}=v \qquad \left|\det\frac{\partial(x,y)}{\partial(u,v)}\right|=\left|\det\begin{pmatrix} v & u \\ 0 & 1 \end{pmatrix}\right|=|v|=v$$

$$f_{U,V}(u,v)=f_{X,Y}(uv,v)\left|\det\frac{\partial(x,y)}{\partial(u,v)}\right|$$

$$=10uv\cdot v^2\cdot v$$

$$=10uv^4$$

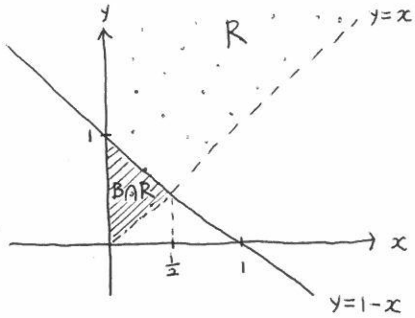
$\therefore f_{U,V}(u,v)=10uv^4$, $0<u<1$, $0<v<1$

- 확률변수 X와 Y의 결합확률밀도함수가 $f(x,y) = 2e^{-x}e^{-y}$, $0<x<y<\infty$ 로 주어져 있을 때 X의 주변확률밀도함수를 구하시오.

$$f_X(x) = \int_x^\infty (2e^{-x}e^{-y})\,dy = \left[-2e^{-x}e^{-y}\right]_x^\infty = 2e^{-2x}, \quad 0<x<\infty$$

$$\therefore f_X(x) = 2e^{-2x}, \quad 0<x<\infty$$

- 확률 변수 X와 Y의 결합확률밀도함수가 $f(x,y) = e^{-y}$, $0<x<y<\infty$ 로 주어져 있을때 $P(X+Y\le 1)$을 구하시오.



$$R = \{(x,y) : f_{X,Y}(x,y) > 0\}$$

$$B = \{(x,y) : x+y \le 1\}$$

$$B\cap R = \{(x,y) : 0<x<\tfrac{1}{2},\ x<y\le 1-x\}$$

$$P(X+Y\le 1)$$
$$= P((x,y)\in B)$$
$$= \iint_B f_{X,Y}(x,y)\,dy\,dx$$
$$= \iint_{B\cap R} f_{X,Y}(x,y)\,dy\,dx + \iint_{B\cap R^c} f_{X,Y}(x,y)\,dy\,dx$$
$$= \iint_{B\cap R} f_{X,Y}(x,y)\,dy\,dx$$
$$= \int_0^{\frac{1}{2}}\int_x^{1-x} e^{-y}\,dy\,dx$$
$$= \int_0^{\frac{1}{2}}\left[-e^{-y}\right]_x^{1-x} dx = \int_0^{\frac{1}{2}}(-e^{x-1}+e^{-x})\,dx = \left[-e^{x-1}-e^{-x}\right]_0^{\frac{1}{2}} = -e^{-\frac{1}{2}}-e^{-\frac{1}{2}}+e^{-1}+1 = -2e^{-\frac{1}{2}}+e^{-1}+1$$

$$\therefore P(X+Y\le 1) = -2e^{-\frac{1}{2}}+e^{-1}+1$$