Nonlinear Hyperbolic Scalar Partial Differential Equations of the Form:
$u_t + f(u)_x = 0$
Thoughts and Methods

Nicholas Potter

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1 Introduction

A variety of methods for analyzing the behavior of nonlinear hyperbolic equations exist. Here we examine a number of methods in analyzing the partial differential equation:

$$u_t + f(u)_x = 0 \tag{1}$$

In particular, we use Upwind, Lax-Friedrichs, and Lax-Wendroff methods to solve the PDE, comparing the results with exact solutions when possible, and focusing on solutions with shocks. A variety of flux functions f, including convex functions help to determine the relative strengths and weaknesses of each methods in a number of different settings. We use boundary conditions

$$u_x(\pm L, t) = 0$$

and begin with initial data

$$u(x,0) = u_L \; for \; x < 0$$
$$u(x,0) = u_R \; for \; x > 0$$

In analyzing the different schemes, we examine a number of different methods including: taking various flux functions, comparing schemes over continuous and discontinuous solutions, taking a number of different mesh sizes and performing a numerical convergence analysis for each scheme, examining variations in the CFL condition and mesh size for both shocks and rarefactions. We focus our analysis on the behavior of the schemes around shocks.

2 Method

We begin with our PDE as in (1). Since we are interested in the relative strengths and weaknesses of each scheme, we subject each scheme to a variety of initial data and flux equations. As a basis for comparison, we use Riemann initial data $u_L = -1, 0, 1$ and $u_R = -1, 0, 1$, with Burger's flux equation $f(u) = u^2/2$, giving us $u_t + uu_x = 0$ for our starting PDE.

If we evaluate our PDE generally, we see by the chain rule that $(f(u))_x = f(u)_x u_x$. Hence for those schemes that require it we can calculate f_x explicitly rather than taking an approximation. This also gives us our initial PDE as expressed in the previous paragraph.

Since we have Neumann conditions $u_x(\pm L, t) = 0$, we derive our boundary conditions using an extension of our grid. Doing so we achieve the following equations for our boundary values of U:

$$\begin{aligned} \frac{U_1 - U_{-1}}{2\Delta x} &= 0 \\ \frac{U_{J+1} - U_{J-1}}{2\Delta x} &= 0 \end{aligned} \tag{2}$$

which gives us the rather trivial boundary equations $U_{-1} = U_1$ and $U_{J+1} = U_{J-1}$ that are substituted into our difference schemes at the boundaries.

We want to ensure that the points used for calculation in our scheme lie within the domain of dependence of our PDE. Our characteristic lines are shown by $f(u)_x$, so our CFL condition is $|f(u)_x| \leq \frac{\Delta x}{\Delta t}$. Since our initial conditions are 1 and -1, $f(u)_x = u$ for Burger's equation, and so our CFL condition for Burger's equation is equivalent to $\nu \leq 1$, where $\nu = \frac{\Delta t}{\Delta x}$. Note that for $u(x, 0) = 0$, we have that $\frac{1}{\nu} \geq 0$, which is always true.

2.1 Upwind Differencing

In upwind differencing, we use a forward difference for time, and forward or backward differences depending on the value of $f(u)_x$. Thus we get

$$\begin{aligned} U_j^{n+1} &= U_j^n - \nu f(u)_x (U_{j+1} - U_j) \ for \ f(u)_x < 0 \\ U_j^{n+1} &= U_j^n - \nu f(u)_x (U_{j+1} - U_j) \ for \ f(u)_x > 0 \end{aligned} \tag{3}$$

Additionally, we substitute in our boundary equations from (2). This leaves us with

$$\begin{aligned} U_0^{n+1} &= U_0^n - \nu f(u)_x (U_0 - U_1) \\ U_J^{n+1} &= U_J^n - \nu f(u)_x (U_J - U_{J-1}) \end{aligned}$$

Notice that given our initial conditions, $U_0^0 = U_1^0$ and $U_J^0 = U_{J-1}^0$, so our scheme on the boundary reduces to $U_1^{n+1} = U_1^n$ and $U_J^{n+1} = U_J^n$ until some change occurs at U_1 or U_{J-1}. This means that our boundary is constant until affected by the rarefaction or shock.

2.2 Lax-Friedrichs

The Lax-Friedrichs scheme is uses forward differences in time, and a center difference in space. However, the scheme is modified by adding $\frac{U_{j+1}^n + U_{j-1}^n}{2} - U_j^n$ to the difference scheme, thereby explicitly including the point U_j^n and bridging the disconnect between our space and time discretizations. We are left with

$$U - j^{n+1} = U_j^n - \nu(F_{j+\frac{1}{2}}^n - F_{j-\frac{1}{2}}^n) \tag{4}$$

where

$$F_{j+\frac{1}{2}}^n = \frac{f_{j+1}^n + f_j^n}{2} - \frac{1}{2\nu}(U_{j+1}^n - U_j^n)$$

By substituting our boundary equations into our scheme, we are left with $U_1^{n+1} = U_1^n$ and $U_J^{n+1} = U_J^n$, so the boundaries are independent of time and space. This differs from the Upwind scheme, since our boundaries are constant for all time and space, rather than only until the shock or rarefaction reach the boundary.

2.3 Lax-Wendroff

We derive Lax-Wendroff scheme by taking the Taylor series of u_j^{n+1}. $(f(u))_x$ is explicitly differentiated to $A(u)u_x$, where $A(u) = f(u)_x$, and we plug the resulting PDE into our Taylor expansion. Our resulting scheme is

$$U_j^{n+1} = U_j^n - \nu(F_{j+\frac{1}{2}}^{n+\frac{1}{2}} - F_{j-\frac{1}{2}}^{n+\frac{1}{2}}) \tag{5}$$

where

$$F_{j+\frac{1}{2}}^{n+\frac{1}{2}} = \frac{f_{j+1}^n - f_{j-1}^n}{2} - \frac{1}{2\nu}\frac{A_{j+1}^n + A_j^n}{2}(f_{j+1}^n - f_j^n)$$

By pluggin our boundary equations into the scheme, we get

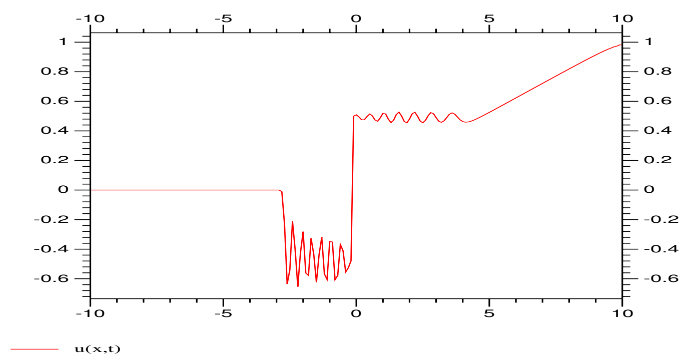
$$U_0^{n+1} = U_0^n - \nu(f_0 - f_1)$$
$$U_J^{n+1} = U_J^n - \nu(f_J - f_{J-1})$$

Here our scheme on the boundary is a little different in that it depends on f rather than U. Yet we are left with the same outcome, in that our initial conditions reduce the scheme on the boundary to $U_1^{n+1} = U_1^n$ and $U_J^{n+1} = U_J^n$, where it will remain until the values of f_1 or f_{J-1} change.

3 Results

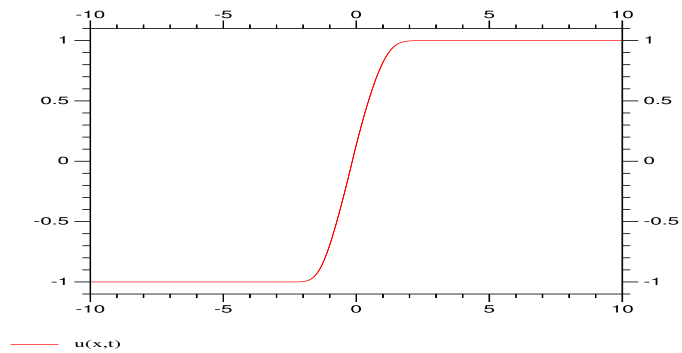
We are particularly interested in how our schemes handle shocks ($U_L > U_R$) and rarefactions ($U_L < U_R$). We expect that our various methods will look similar when dealing with rarefactions, while the inclusion of shocks will lead to interesting errors depending on the scheme. Here we will first present results for Burger's Equation with rarefactions and shocks. We follow this with an examination of violations of the CFL condition for both rarefactions and shocks.

Taking $U_L = 0$ and $U_R = 1$, we get similar behaviors for our Upwind and Lax-Friedrichs schemes, in which a wave moves outward from 0 to the right. However, for the Upwind scheme, there is no movement for $x < 0$, while Lax-Friedrichs gives movement to the right for U_R and to the left for U_L. However, the truly interesting result is obtained under the Lax-Wendroff scheme. We can see in Fig. 3 our Lax-Wendroff scheme result for $t = 10$.



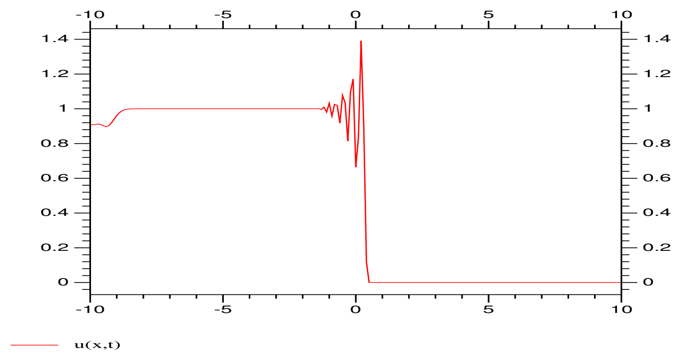
Our rarefaction should be moving to the right, which as we can see from the right end of Fig. 3 is occurring. However, the oscillations that plague the Lax-Wendroff scheme are quite visible and fairly large in magnitude with respect to the values of u itself.

If we let $U_L = -1$ and $U_R = 0$, we get similar results in the opposite direction (moving to the left). However, if we take $U_L = -1$ and $U_R = 1$, we get different results, again similar for our Upwind and Lax-Friedrichs schemes, with Lax-Wendroff standing apart. Here we get movement in both directions, as we can see in Fig. 3



Interestingly, there is no movement at all when using Lax-Wendroff. This is independent of how we seperate U_L and U_R, and, as we shall see, of the flux functions we consider.

Considering each scheme with regard to shocks, we take $U_L = 1$ and $U_R = 0$. Here each scheme responds differently. Our Upwind scheme propagates the shock to the right with a speed of u, which is what we would expect. For Lax-Friedrichs, we get a smoothing effect over the region of the shock with a slight propogation to the right. Lax-Wendroff gives the tell-tale "champaign glass" effect, shown in Fig. 3, while also propagating the discontinuity to the right, but at a larger speed.



Our results when taking $U_L = 1$ and $U_R = -1$ are again suprising. For both the Upwind and Lax-Wendroff Scheme we see no movement, while for Lax-Friedrichs we get similar smoothing across the shock.

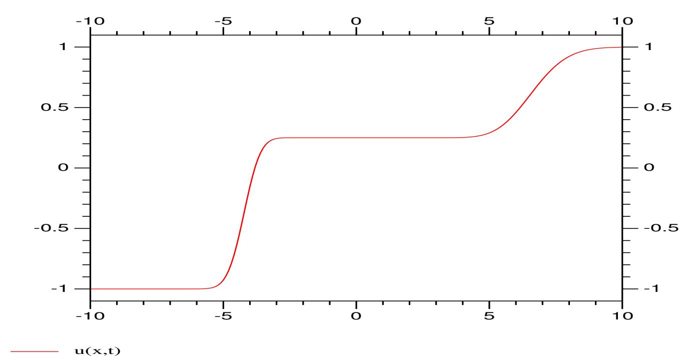
Insuring that the CFL condition is met is crucial for successful behavior in each of our schemes. Even with smooth solutions each scheme becomes unstable if the CFL condition is greiviously failed. Using $L = 10$, $U_L = 1$, and $U_R = 0$, we get unstable solutions for the various schemes beginning with the Lax-Wendroff scheme, which is unstable at $\Delta x = .093$ and $\Delta t = .1$. Lax-Friedrichs begins to be unstable with $\Delta x = .08$ and $\Delta t = .1$. Upwind Differences is unstable with $\Delta x \leq .24$. However, for Lax-Friedrich and Lax-Wendroff, there is a period in which the results of the schemes change, but the schemes are not unstable in the sense that their values approach infinity.

If we examine violations of the CFL condition with regard to shocks, there is a much smaller margin by which the CFL condition can be violated while maintaining a stable scheme. The Upwind scheme is unstable for any value of $\nu < 1$. Lax-Friedrich is unstable beyond $\Delta x \leq .24$. Lax-Wendroff is unstable with $\Delta x < .42$. Again we see patterns in which Lax-Friedrich and Lax-Wendroff shift their behavior in some interval before they become unstable.

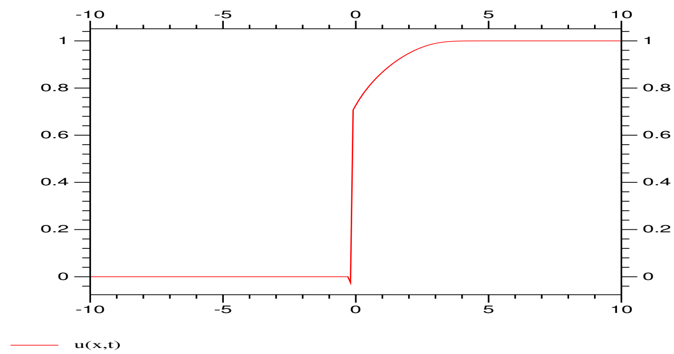
Shifting from Burger's Equation, we take the following values for $f(u)$

$$f(u) = u^4$$
$$f(u) = u^3 - u$$
$$f(u) = u^4 - u^2$$

For the first value shown, we notice a much faster propagation of shocks in each of the schemes. Lax-Friedrichs does not offer any suprises, but there are a couple of interesting behaviors in the others. Lax-Wendroff displays significantly reduced oscillations, both in the champaigne glass, and for the rarefactions as shown in Fig. 3. Additionally, if we take $U_L = -1$ and $U_R = 1$, the Upwind scheme leads to two disjoints, travelling in opposite directions (Fig. 3). Notice that the disjoint travelling to the right is smaller (from .25 to 1 compared with from .25 to -1) than the one travelling to the left, but that the one on the left travels more slowly and is maintains a higher slope.



For $f(u) = u^3 - u$, the Upwind and Lax-Wendroff schemes don't offer much in the way of analysis. While Lax-Friedrichs continues as it has with our other flux equations (albeit slightly differently), Upwind and Lax-Wendroff show no movement for shocks or rarefactions. There is one exception, for Upwind, with either $U_L = -1$ and $U_R = 1$ or vice-versa, which blends the right end of the disjoint. In all other cases, there is no response at all. This is true for $f(u) = u^4 - u^2$ as well, except when $U_L = 0$ and $U_R = 1$ with the Upwind scheme, in which case you get some blending as in the last case (Fig. 3). Notice however that in Fig. 3 there is a point on the rarefaction at approximately .7 where the blending that occurs stops. This was not the case with earlier blendings in the Upwind scheme.



4 Discussion

Having established a comparitive basis for our schemes, we can see significant differences between the results of each scheme. Our Lax-Friedrichs scheme is by far the most capable of handling a variety of different CFL numbers, flux equations, and shocks and rarefactions. Given any any initial data, the uniform response of Lax-Friedrichs is to smooth the jump discontinuity into a continuous curve. For both shocks and rarefactions, the discontinuity moves slowly to the right, and smooths toward the left and right.

The Upwind scheme does well representing rarefactions, smoothing the rarefaction to the right (for $u > 0$) rather than propogating the discontinuity. For shocks, the Upwind scheme propogates the shock, but with a variable speed. This is interesting, since our speed should be the value of u and we have all u the same. This would indicate a constant speed. However, what we see with the Upwind scheme is an oscillating propagation, in which the shock is propogated quickly, then comes to an almost complete stop, then speeds up again. The distance covered during the process is rather small compared with the propagation of shocks by the Lax-Friedrichs and Lax-Wendroff schemes. Additionally, if we take a shock with $U_L = 1$ and $U_R = -1$, the scheme shows no movement at all. However, if we take the rarefaction $U_L = -1$ and $U_R = 1$, the Upwind scheme looks very similar to Lax-Friedrichs.

The Lax-Wendroff scheme exhibits its trademark "champaigne glass" effect when dealing with shocks, with the oscillations occuring where $u \neq 0$. Like the Upwind scheme, Lax-Wendroff shows no movement when taking a shock with $U_L = 1$ and $U_R = -1$. However, Lax-Wendroff also shows no movement for the rarefaction $U_L = -1$ and $U_R = 1$. Lax-Wendroff exhibits a very unusual behavior, shown in Fig. 3 when dealing with rarefactions. It performs exactly as expected when dealing with shocks, but very differently when dealing with a rarefaction. The fact that the oscillatory behavior near the rarefaction seen in Fig. 3 is not a well discussed phenomena suggests that there is an error in the implimentation of the scheme that is leading to this behavior, but we have not been able to locate it. Additionally, Lax-Wendroff shows a drop in the value of U on the left boundary, as seen in Fig. 3, which may also be due to errors in implimentation.

While Lax-Friedrichs is the most versatile scheme, it does not do well with propagating discontinuities. The scheme essentially just pretends the discontinuity is not there, and so while it remains smooth for many different conditions, it does so by ignoring the discontinuity and approximating the connection between the two different values for u. To some extent this behavior is mitigated by taking smaller values for Δx, and hence also smaller values for Δt. However, this significantly increases computation time. Our Upwind scheme, on the other hand, does well with rarefactions. Shocks seem to represent a disconnect in the system where information from the left side of the system is not transferred to the right. Lax-Wendroff does well propagating a shock through the system, but at the expense of the oscillatory champaigne glass effect shown earlier. Additionally, our implimentation of Lax-Wendroff does model rarefactions, but contains oscillations that give the behavior of the rarefactions the strange characteristics discussed earlier and shown in Fig. 3.

Since no one scheme clearly dominates the others with respect to all configurations, our analysis indicates that it is extremely important to have a strong understanding of the kind of problem being modeled, whether we are modeling shocks and rarefactions, and what the initial conditions are. This enables one to choose a scheme with strengths that assist in the analysis.

Thus one might use Lax-Friedrichs where stability is difficult to achieve, Upwind Differences when looking at rarefactions, and Lax-Wendroff when looking at shocks.