STA 200B HW7 Solution

9.14 a. Recall the Bonferroni Inequality (1.2.9), $P(A_1 \cap A_2) \geq P(A_1) + P(A_2) - 1$. Let $A_1 = P(\text{interval covers } \mu)$ and $A_2 = P(\text{interval covers } \sigma^2)$. Use the interval (9.2.14), with $t_{n-1,\alpha/4}$ to get a $1-\alpha/2$ confidence interval for μ. Use the interval after (9.2.14) with $b = \chi^2_{n-1,\alpha/4}$ and $a = \chi^2_{n-1,1-\alpha/4}$ to get a $1-\alpha/2$ confidence interval for σ. Then the natural simultaneous set is

$$\begin{aligned} C_a(x) &= \left\{ (\mu,\sigma^2) : \bar{x} - t_{n-1,\alpha/4}\frac{s}{\sqrt{n}} \leq \mu \leq \bar{x} + t_{n-1,\alpha/4}\frac{s}{\sqrt{n}} \right. \\ &\qquad \left. \text{and } \frac{(n-1)s^2}{\chi^2_{n-1,\alpha/4}} \leq \sigma^2 \leq \frac{(n-1)s^2}{\chi^2_{n-1,1-\alpha/4}} \right\} \end{aligned}$$

and $P\left(C_a(X) \text{ covers } (\mu,\sigma^2)\right) = P(A_1 \cap A_2) \geq P(A_1) + P(A_2) - 1 = 2(1-\alpha/2) - 1 = 1-\alpha$.

b. If we replace the μ interval in a) by $\left\{\mu : \bar{x} - \frac{k\sigma}{\sqrt{n}} \leq \mu \leq \bar{x} + \frac{k\sigma}{\sqrt{n}}\right\}$ then $\frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \sim \mathrm{n}(0,1)$, so we use $z_{\alpha/4}$ and

$$C_b(x) = \left\{ (\mu,\sigma^2) : \bar{x} - z_{\alpha/4}\frac{\sigma}{\sqrt{n}} \leq \mu \leq \bar{x} + z_{\alpha/4}\frac{\sigma}{\sqrt{n}} \text{ and } \frac{(n-1)s^2}{\chi^2_{n-1,\alpha/4}} \leq \sigma^2 \leq \frac{(n-1)s^2}{\chi^2_{n-1,1-\alpha/4}} \right\}$$

and $P\left(C_b(X) \text{ covers } (\mu,\sigma^2)\right) \geq 2(1-\alpha/2) - 1 = 1-\alpha$.

c. The sets can be compared graphically in the (μ,σ) plane: C_a is a rectangle, since μ and σ^2 are treated independently, while C_b is a trapezoid, with larger σ^2 giving a longer interval. Their areas can also be calculated

$$\begin{aligned} \text{Area of } C_a &= \left[2t_{n-1,\alpha/4}\frac{s}{\sqrt{n}}\right]\left\{\sqrt{(n-1)s^2}\left(\frac{1}{\chi^2_{n-1,1-\alpha/4}} - \frac{1}{\chi^2_{n-1,\alpha/4}}\right)\right\} \\ \text{Area of } C_b &= \left[z_{\alpha/4}\frac{s}{\sqrt{n}}\left(\sqrt{\frac{n-1}{\chi^2_{n-1,1-\alpha/4}}} + \sqrt{\frac{n-1}{\chi^2_{n-1,\alpha/4}}}\right)\right] \\ &\quad \times \left\{\sqrt{(n-1)s^2}\left(\frac{1}{\chi^2_{n-1,1-\alpha/4}} - \frac{1}{\chi^2_{n-1,\alpha/4}}\right)\right\} \end{aligned}$$

and compared numerically.

HW5.1. $\forall \varepsilon > 0$. $\exists M_1 > 0$. s.t $P(|Z| > M_1) < \frac{\varepsilon}{4}$.

Take $M_2 > M_1$. then $P(|Z| > M_2) \leq P(|Z| > M_1) < \frac{\varepsilon}{4}$.

By the triangular inequality, $|Z_n| \leq |Z_n - Z| + |Z|$.

Then $\{|Z_n - Z| \leq M_2 - M_1\} \cap \{|Z| \leq M_1\} \subset \{|Z_n| \leq M_2\}$

So $P(|Z_n| > M_2) \leq P(\{|Z_n - Z| > M_2 - M_1\} \cup \{|Z| > M_1\})$

$\qquad \leq P(|Z_n - Z| > M_2 - M_1) + P(|Z| > M_1)$

Since $Z_n \xrightarrow{P} Z$, it implies that $P(|Z_n - Z| > M_2 - M_1) < \frac{\varepsilon}{4}$ for large enough n.

So $P(|Z_n| > M_2) \leq \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}$

Note that g is continuous, so g is uniformly continuous on $[-M_2, M_2]$.

That is, $\forall \eta > 0$. $\exists \delta > 0$. s.t.

$$\{|Z_n - Z| < \delta\} \cap \{|Z_n| \le M_2\} \cap \{|Z| \le M_2\} \subset \{|g(Z_n) - g(Z)| \le \eta\}$$

so
$$P(|g(Z_n) - g(Z)| > \eta) \le P(|Z_n - Z| \ge \delta) + P(|Z_n| > M_2) + P(|Z| > M_2)$$
$$\le P(|Z_n - Z| \ge \delta) + \frac{\varepsilon}{4} + \frac{\varepsilon}{4}$$

Again, since $Z_n \xrightarrow{P} Z$, then $P(|Z_n - Z| \ge \delta) \le \frac{\varepsilon}{4}$ for large enough n.

It follows that $P(|g(Z_n) - g(Z)| > \eta) < \frac{\varepsilon}{4} + \frac{\varepsilon}{2} + \frac{\varepsilon}{4} = \varepsilon$ for large enough n, $\forall \eta > 0$.

Thus, $g(Z_n) \xrightarrow{P} g(Z)$.

1. (a) It's easy to know that $\frac{\sum (X_i - \mu_1)^2}{\sigma_1^2} = \sum_i \left(\frac{X_i - \mu_1}{\sigma_1}\right)^2 \sim \chi^2_n$, $\frac{\sum (Y_i - \mu_2)^2}{\sigma_2^2} \sim \chi^2_m$

and these two random variables (statistics) are independent

so
$$\frac{\sum (X_i - \mu_1)^2 / (\sigma_1^2 n)}{\sum (Y_i - \mu_2)^2 / (\sigma_2^2 m)} \sim F_{n,m}$$

Then
$$P\left(F_{n,m,r_1} \le \frac{\sum (X_i - \mu_1)^2 / (\sigma_1^2 n)}{\sum (Y_i - \mu_2)^2 / (\sigma_2^2 m)} \le F_{n,m,r_2} \right) = 0.95$$

where $F_{n,m,r}$ is the rth quantile of $F_{n,m}$. and $r_2 - r_1 = 0.95$.

We can rewrite it as
$$P\left(\frac{\sum (X_i - \mu_1)^2}{\sum (Y_i - \mu_2)^2} \cdot \frac{m}{n} \cdot \frac{1}{F_{n,m,r_2}} \le \frac{\sigma_1^2}{\sigma_2^2} \le \frac{\sum (X_i - \mu_1)^2}{\sum (Y_i - \mu_2)^2} \cdot \frac{m}{n} \cdot \frac{1}{F_{n,m,r_1}} \right) = 0.95. \quad (*)$$

So
$$\left[\frac{\sum (X_i - \mu_1)^2 m}{\sum (Y_i - \mu_2)^2 n \cdot F_{n,m,r_2}}, \frac{\sum (X_i - \mu_1)^2 m}{\sum (Y_i - \mu_2)^2 n F_{n,m,r_1}} \right]$$
is a 95% CI for $\frac{\sigma_1^2}{\sigma_2^2}$.

(b) It's known that $\frac{\sum (X_i - \bar{X})^2}{\sigma_1^2} \sim \chi^2_{n-1}$, $\frac{\sum (Y_i - \bar{Y})^2}{\sigma_2^2} \sim \chi^2_{m-1}$, and they are independent.

So
$$\frac{\sum (X_i - \bar{X})^2 / (\sigma_1^2 (n-1))}{\sum (Y_i - \bar{Y})^2 / (\sigma_2^2 (m-1))} \sim F_{n-1,m-1}.$$

By the similar argument as (a). a 95% CI for $\frac{\sigma_1^2}{\sigma_2^2}$ is
$$\left[\frac{\sum (X_i - \bar{X})^2 (m-1)}{\sum (Y_i - \bar{Y})^2 (n-1) F_{n-1,m-1,r_2}}, \frac{\sum (X_i - \bar{X})^2 (m-1)}{\sum (Y_i - \bar{Y})^2 (n-1) F_{n-1,m-1,r_1}} \right].$$

(c) Like in (a), we can rewrite the analogous $(*)$ in (c). i.e.
$$P\left(\sqrt{\frac{\sum (X_i - \bar{X})^2 (m-1)}{\sum (Y_i - \bar{Y})^2 (n-1) F_{n-1,m-1,r_2}}} \le \frac{\sigma_1}{\sigma_2} \le \sqrt{\frac{\sum (X_i - \bar{X})^2 (m-1)}{\sum (Y_i - \bar{Y})^2 (n-1) F_{n-1,m-1,r_1}}} \right) = 0.95.$$

So a 95% CI is
$$\left[\sqrt{\frac{\sum (X_i - \bar{X})^2 (m-1)}{\sum (Y_i - \bar{Y})^2 (n-1) F_{n-1,m-1,r_2}}}, \sqrt{\frac{\sum (X_i - \bar{X})^2 (m-1)}{\sum (Y_i - \bar{Y})^2 (n-1) F_{n-1,m-1,r_1}}} \right].$$

2. (a) The shortest 90% CI for μ is

$$\bar{X} \pm t_{19,0.95} \cdot \frac{S}{\sqrt{n}} = 156.85 \pm 1.73 \times \frac{22.64}{\sqrt{20}} = [148.10, 165.60]$$

(b) Since 150 lies in this CI, we fail to reject H_0.

(c). The test statistic is $t = \frac{\bar{X} - \mu_0}{S/\sqrt{n}} = \frac{156.85 - 150}{22.64/\sqrt{20}} = 1.35$

So the pvalue $= P(|t_{n-1}| > |t|) = P(|t_{19}| > 1.35) = 0.192$

(d) The hypothesis is $H_0: \mu \le 150$ v.s. $H_1: \mu > 150$.

The 95% confidence set for μ is $[\bar{X} - t_{19,0.95}\frac{S}{\sqrt{n}}, \infty) = [156.85 - 1.73 \times \frac{22.64}{\sqrt{20}}, \infty) = (148.10, \infty)$

Since 150 lies in this confidence set, we fail to reject H_0.

(e). The p-value $= P(t_{n-1} > t) = P(t_{19} > 1.35) = 0.096$.

(f) The type II error might be committed. The truth could be the average calorie level for hot dogs exceeds 150, but the calorie levels for these 20 brands are controled better than the others, resulting in no rejection the null hypothesis.

3. (a) The 90% CI for μ is $[\bar{X} - z_{0.95} \cdot \frac{\sigma}{\sqrt{n}}, \bar{X} + z_{0.95} \cdot \frac{\sigma}{\sqrt{n}}]$

The length of the CI is $2 \cdot z_{0.95} \cdot \frac{\sigma}{\sqrt{n}}$.

Consider $2 \cdot z_{0.95} \cdot \frac{\sigma}{\sqrt{n}} \le \frac{\sigma}{3}$, we have $n \ge (2 \cdot z_{0.95} \cdot 3)^2 = (2 \times 1.64 \times 3)^2 = 97.4$

So we need $n = 98$ at the least.

(b) The 90% CI for μ is $[\bar{X} - t_{n-1,0.95}\frac{S}{\sqrt{n}}, \bar{X} + t_{n-1,0.95}\frac{S}{\sqrt{n}}]$ when σ is unknown

So the length of CI $2 \cdot t_{n-1,0.95}\frac{S}{\sqrt{n}} \le \frac{\sigma}{3}$.

(c). From (b), we have $P(2t_{n-1,0.95} \cdot \frac{S}{\sqrt{n}} \le \frac{\sigma}{3}) \ge 0.95$

$\Rightarrow P\left(\frac{(n-1)S^2}{\sigma^2} \le \frac{(n-1)n}{36 t_{n-1,0.95}}\right) \ge 0.95$.

Note that $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$, we need to find the smallest n through numerical methods such that $\frac{(n-1)n}{36 t_{n-1,0.95}} \ge \chi^2_{n-1,0.95}$

4. (a) It's shown that $[\frac{X_{(n)}}{b}, \frac{X_{(n)}}{a}]$ is a level r CI for θ,

where $0<a<b\le 1$, and $b^n - a^n = r$ (*)

The length of the CI is $\frac{X_{(n)}}{a} - \frac{X_{(n)}}{b} = X_{(n)}(\frac{1}{a}-\frac{1}{b})$

It follows from (*) that $\frac{da}{db} = (\frac{b}{a})^{n-1}$

let $L(a,b) = X_{(n)}(\frac{1}{a}-\frac{1}{b})$. then $\frac{\partial L}{\partial b} = X_{(n)}(\frac{1}{b^2} - \frac{1}{a^2}\cdot\frac{da}{db}) = X_{(n)}\cdot\frac{a^{n+1}-b^{n+1}}{b^2 a^{n+1}} < 0$.

So the minimum is achieved at $b=1$. Accordingly, $a=(1-r)^{\frac{1}{n}}$.

That is, the shortest CI is $[X_{(n)}, \frac{X_{(n)}}{(1-r)^{1/n}}]$.

(b) The joint pdf of $\underset{\sim}{X}=(X_1,\dots X_n)$ is $f_n(\underset{\sim}{X}|\theta) = \prod_{i=1}^{n}\frac{1}{\theta}1_{\{x_i<\theta\}} = \frac{1}{\theta^n}1_{\{X_{(n)}<\theta\}}$

The posterior pdf of θ is

$\pi(\theta|\underset{\sim}{X}) \propto f_n(\underset{\sim}{X}|\theta)\cdot\pi(\theta) = \frac{1}{\theta^n}1_{\{X_{(n)}<\theta\}}\cdot\frac{\beta\alpha^\beta}{\theta^{\beta+1}}1_{\{\alpha<\theta\}} \propto \frac{1}{\theta^{n+\beta+1}}1_{\{\max(X_{(n)},\alpha)<\theta\}}$

So the posterior distribution of θ is Pareto$(\max(X_{(n)},\alpha), n+\beta)$

It can be easily derived that the cdf of Pareto(α,β) is

$F(x) = \int_\alpha^x f(y)dy = \int_\alpha^x \frac{\beta\alpha^\beta}{y^{\beta+1}}dy = \beta\alpha^\beta\int_\alpha^x y^{-(\beta+1)}dy = \beta\alpha^\beta\cdot\frac{1}{-\beta}\cdot y^{-\beta}\Big|_\alpha^x = 1-(\frac{\alpha}{x})^\beta$.

We can find a r-level Bayes credible interval (a,b) such that

$P(a<\theta<b|\underset{\sim}{x}) = r$.

and $\max(X_{(n)},\alpha)\le a<b<\infty$.

Let $\lambda = \max(X_{(n)},\alpha)$, $K = n+\beta$. that is

$r = F(b|\underset{\sim}{x}) - F(a|\underset{\sim}{x}) = [1-(\frac{\lambda}{b})^K] - [1-(\frac{\lambda}{a})^K] = \lambda^K(a^{-K}-b^{-K})$ (*)

It follows from (*) that $\frac{db}{da} = (\frac{b}{a})^{K+1}$

The length of the Bayes credible interval (a,b) is

$L(a,b) := b-a$.

Then $\frac{\partial L}{\partial a} = \frac{db}{da} - 1 = (\frac{b}{a})^{K+1} - 1 = \frac{b^{K+1}-a^{K+1}}{a^{K+1}} > 0$.

So the minimal is achieved at $a=\lambda=\max(X_{(n)},\alpha)$,

Accordingly, $b = \frac{\lambda}{(1-r)^{1/K}} = \frac{\max(X_{(n)},\alpha)}{(1-r)^{1/(n+\beta)}}$

That is, the shortest Bayes credible interval is $[\max(X_{(n)},\alpha), \frac{\max(X_{(n)},\alpha)}{(1-r)^{1/(n+\beta)}}]$.

5.(a) The likelihood for λ is $L(\lambda|\underline{x}) = \prod_{i=1}^{n} f(x_i|\lambda) = \prod_{i=1}^{n} \lambda e^{-\lambda x_i} = \lambda^n e^{-\lambda \Sigma X_i}$

The log likelihood is $\ell(\lambda|\underline{x}) = \log L(\lambda|\underline{x}) = n\log\lambda - \lambda \Sigma X_i$

Let $\frac{\partial \ell}{\partial \lambda} = \frac{n}{\lambda} - \Sigma X_i = 0.$ $\Rightarrow \hat{\lambda} = \frac{n}{\Sigma X_i} = \frac{1}{\bar{X}}$.

and $\frac{\partial^2 \ell}{\partial \lambda^2} = -\frac{n}{\lambda^2} < 0$.

So $\hat{\lambda} = \frac{1}{\bar{X}}$ is the MLE of λ.

The median m is the number such that $P(X_1 > m) = \frac{1}{2}$.

i.e $\frac{1}{2} = P(X_1 > m) = \int_m^\infty \lambda e^{-\lambda x} dx = e^{-\lambda m}$

It implies that $m = \frac{\log 2}{\lambda}$

By the invariance of MLE, the MLE of the median is $\hat{m} = \frac{\log 2}{\hat{\lambda}} = \bar{X} \cdot \log 2$.

By the Law of Large Numbers, $\bar{X} \xrightarrow{P} E(X_1) = \frac{1}{\lambda}$ as $n \to \infty$.

Then $\hat{m} \xrightarrow{P} \frac{\log 2}{\lambda} = m$.

Thus, $\hat{m}$ is a consistent estimator of m.

(b) The posterior pdf of λ is $\pi(\lambda|\underline{x}) \propto f_n(\underline{x}|\lambda) \cdot \pi(\theta) = \lambda^n e^{-\lambda \Sigma X_i} \cdot \lambda^{\alpha-1} e^{-\beta\lambda} = \lambda^{n+\alpha-1} \cdot e^{-\lambda(\Sigma X_i + \beta)}$

It corresponds to Gamma$(n+\alpha, \Sigma X_i + \beta)$

Then the Bayes estimator w.r.t. the square error loss is $\tilde{\lambda} = \frac{n+\alpha}{\Sigma X_i + \beta}$

And $\tilde{\lambda} = \frac{1+\frac{\alpha}{n}}{\bar{X} + \frac{\beta}{n}} \xrightarrow{P} \frac{1+0}{\frac{1}{\lambda}+0} = \lambda$ as $n \to \infty$

Thus, $\hat{\lambda}$ is a consistent estimator of λ.