STA 200B HW6 Solution

9.2. $p = P(\bar{X} - \frac{1.96}{\sqrt{n}} < X_{n+1} < \bar{X} + \frac{1.96}{\sqrt{n}}) = P(-\frac{1.96}{\sqrt{n}} < X_{n+1} - \bar{X} < \frac{1.96}{\sqrt{n}}) = P(|X_{n+1} - \bar{X}| < \frac{1.96}{\sqrt{n}})$

Note that $X_{n+1} - \bar{X} = X_{n+1} - \frac{X_1 + \cdots + X_n}{n} = X_{n+1} + (-\frac{1}{n})X_1 + \cdots + (-\frac{1}{n})X_n \sim N(0, \frac{n+1}{n})$

So $Z := \frac{X_{n+1} - \bar{X}}{\sqrt{(n+1)/n}} \sim N(0,1)$. Then

$p = P(|\frac{X_{n+1} - \bar{X}}{\sqrt{(n+1)/n}}| < \frac{1.96}{\sqrt{n+1}}) = P(|Z| < \frac{1.96}{\sqrt{n+1}}) < P(|Z| < 1.96) = 0.95$.

i.e. p is less than 0.95.

9.17 Method 1

(a) Note that $X_i - \theta \sim U(-\frac{1}{2}, \frac{1}{2})$.

Then $\bar{X} - \theta = \frac{\Sigma(X_i - \theta)}{n}$ is a pivot as it's distribution does not depend on θ.

Let $F(\cdot)$ be the cdf of $\bar{X} - \theta$, we can choose the r_1-th quantile and r_2-th quantile of F

Such that $P(F_{r_1} < \bar{X} - \theta < F_{r_2}) = 1 - \alpha$ and $r_2 - r_1 = 1 - \alpha$.

So the $1-\alpha$ CI for θ is $(\bar{X} - F_{r_2}, \bar{X} - F_{r_1})$.

(b) Note that $\frac{X_i}{\theta}$ has the pdf $f(y) = 2y$, $0 \le y \le 1$.

Then $\frac{\bar{X}}{\theta} = \frac{\Sigma X_i}{n\theta} = \frac{1}{n}\Sigma(\frac{X_i}{\theta})$ is a pivot as it's distribution does not depend on θ.

Let $G(\cdot)$ be the cdf of $\frac{\bar{X}}{\theta}$. we can choose the r_1th quantile and r_2-th quantile of G

such that $P(G_{r_1} < \frac{\bar{X}}{\theta} < G_{r_2}) = 1 - \alpha$ and $r_2 - r_1 = 1 - \alpha$

So the $1-\alpha$ CI for θ is $(\frac{\bar{X}}{G_{r_2}}, \frac{\bar{X}}{G_{r_1}})$

Method 2.

(a) Note that $Y_i = X_i - \theta + \frac{1}{2} \sim U(0,1)$

Then $Y_{(n)} = \max_{1 \le i \le n} Y_i = \max_{1 \le i \le n} X_i - \theta + \frac{1}{2} = X_{(n)} - \theta + \frac{1}{2}$ has the cdf $F(y) = y^n$ for $0 < y < 1$.

The reason is that $P(Y_{(n)} \le y) = P(Y_1 \le y, \ldots, Y_n \le y) = \prod_{i=1}^{n} P(Y_i \le y) = y^n$.

So we can find constants a and b such that $P(a < Y_{(n)} < b) = b^n - a^n = 1 - \alpha$

Namely, $P(a < X_{(n)} - \theta + \frac{1}{2} < b) = 1 - \alpha$ and $P(X_{(n)} + \frac{1}{2} - b < \theta < X_{(n)} + \frac{1}{2} - a) = 1 - \alpha$

Thus, a $1-\alpha$ CI for θ is $(X_{(n)} + \frac{1}{2} - b, X_{(n)} + \frac{1}{2} - a)$, where $b^n - a^n = 1 - \alpha$.

(b) It's known that $\frac{X_i}{\theta}$ has the pdf $f(y)=2y$, $0<y<1$

Then $Y_{(n)} = \max_{1\le i\le n}\left(\frac{X_i}{\theta}\right) = \frac{X_{(n)}}{\theta}$ has the cdf $G(y)=y^{2n}$

The reason is that $P(Y_{(n)}\le y) = P\left(\frac{X_{(n)}}{\theta}\le y\right) = \prod_{i=1}^{n} P\left(\frac{X_i}{\theta}\le y\right) = \prod_{i=1}^{n} y^2 = y^{2n}$

So we can find constants c and d such that $P(c<Y_{(n)}<d) = d^{2n}-c^{2n}=1-\alpha$

Namely, $P\left(c<\frac{X_{(n)}}{\theta}<d\right)=1-\alpha$ and $P\left(\frac{X_{(n)}}{d}<\theta<\frac{X_{(n)}}{c}\right)=1-\alpha$

Thus, a $1-\alpha$ CI for θ is $\left(\frac{X_{(n)}}{d}, \frac{X_{(n)}}{c}\right)$, where $d^{2n}-c^{2n}=1-\alpha$

9.30 (a) Using the conclusions (ii) & (iii) of Example 4.2.3 in the lecture notes

(but note that $Gamma(\alpha,\beta)$ in the lecture notes corresponds to $Gamma(\alpha,\frac{1}{\beta})$ in the text)

$$\frac{2(nb+1)}{b}\lambda \sim Gamma\left(a+\sum X_i, [n+(1/b)]\cdot[2(nb+1)/b]^{-1}\right) = Gamma\left(a+\sum X_i, \tfrac{1}{2}\right) \sim \chi^2_{2(a+\sum X_i)}$$

(c). $$P\left\{\frac{1}{2n}\chi^2_{2\sum X_i,1-\alpha/2} \le \lambda \le \frac{1}{2n}\chi^2_{2(\sum X_i+1),\alpha/2}\right\} = P\left\{\frac{1}{2n}\chi^2_{2\sum X_i,1-\alpha/2}\cdot\frac{2(nb+1)}{b} \le \frac{2(nb+1)}{b}\lambda \le \frac{1}{2n}\chi^2_{2(\sum X_i+1),\alpha/2}\cdot\frac{2(nb+1)}{b}\right\}$$

$$= P\left\{\frac{nb+1}{nb}\chi^2_{2\sum X_i,1-\alpha/2} \le T \le \frac{nb+1}{nb}\chi^2_{2(\sum X_i+1),\alpha/2}\right\} = P\left\{\frac{\frac{nb+1}{nb}\chi^2_{2\sum X_i,1-\alpha/2}-ET}{\sqrt{2(a+\sum X_i)}} \le \frac{T-ET}{\sqrt{Var(T)}} \le \frac{\frac{nb+1}{nb}\chi^2_{2(\sum X_i+1),\alpha/2}-ET}{\sqrt{2(a+\sum X_i)}}\right\}$$

$$= P\left\{\frac{\frac{nb+1}{nb}\chi^2_{2\sum X_i,1-\alpha/2} - 2(a+\sum X_i)}{\sqrt{2(a+\sum X_i)}} \le \frac{T-ET}{\sqrt{Var(T)}} \le \frac{\frac{nb+1}{nb}\chi^2_{2(\sum X_i+1),\alpha/2} - 2(a+\sum X_i)}{\sqrt{2(a+\sum X_i)}}\right\}$$

Using (b), we have $\gamma = P(\chi^2_\nu \le \chi^2_{\nu,\gamma}) = P\left(\frac{\chi^2_\nu-\nu}{\sqrt{2\nu}} \le \frac{\chi^2_{\nu,\gamma}-\nu}{\sqrt{2\nu}}\right) \approx \Phi\left(\frac{\chi^2_{\nu,\gamma}-\nu}{\sqrt{2\nu}}\right)$ as $\nu\to\infty$.

where $\Phi(\cdot)$ is the cdf of $N(0,1)$.

so $\chi^2_{\nu,\gamma} \approx \nu+\sqrt{2\nu}\cdot\Phi^{-1}(\gamma)$, as $\nu\to\infty$.

The standardized lower cutoff point is then (as $\sum X_i\to\infty$)

$$\frac{\frac{nb+1}{nb}\chi^2_{2\sum X_i,1-\alpha/2} - 2(a+\sum X_i)}{\sqrt{2(a+\sum X_i)}} \approx \frac{\frac{nb+1}{nb}\cdot\left[(2\sum X_i)+\sqrt{2\sum X_i}\,\Phi^{-1}(1-\alpha/2)\right] - 2(a+\sum X_i)}{\sqrt{2(a+\sum X_i)}}$$

$$= \frac{\frac{2}{nb}\sum X_i - 2a + \frac{nb+1}{nb}\sqrt{2\sum X_i}\cdot\Phi^{-1}(1-\alpha/2)}{\sqrt{2(a+\sum X_i)}} \to \infty, \text{ as } \sum X_i\to\infty.$$

The desired result is obtained.

9.58 (a) $P(m \in (-\infty, X_{(n)})) = P(X_{(n)} \geq m) = 1 - P(X_{(n)} < m) = 1 - P(X_1 < m, \dots, X_n < m)$

$$= 1 - \prod_{i=1}^{n} P(X_i < m) = 1 - \prod_{i=1}^{n} (\tfrac{1}{2}) = 1 - (\tfrac{1}{2})^n$$

$P(m \in [X_{(1)}, \infty)) = P(X_{(1)} \leq m) = 1 - P(X_{(1)} > m) = 1 - P(X_1 > m, \dots, X_n > m)$

$$= 1 - \prod_{i=1}^{n} P(X_i > m) = 1 - (\tfrac{1}{2})^n.$$

$P(m \in [X_{(1)}, X_{(n)}]) = P(m \in (-\infty, X_{(n)})) + P(m \in [X_{(1)}, \infty)) - 1$

$$= 1 - (\tfrac{1}{2})^n + 1 - (\tfrac{1}{2})^n - 1 = 1 - 2(\tfrac{1}{2})^n.$$

1. (a) $Var(X_1) = E(X_1) = \lambda$. By the central limit theorem

$$\frac{\bar{X} - \lambda}{\sqrt{\lambda / n}} \Rightarrow N(0,1).$$

Then an approximate C.I at level 99% can be found through the following equations for λ

i.e. $\left| \frac{\bar{X} - \lambda}{\sqrt{\lambda / n}} \right| = z_{0.995}$, where $z_{0.995}$ is the 0.995 quantile of $N(0,1)$

Equivalently, we ~~s~~ solve the equation $(\bar{X} - \lambda)^2 = z_{0.995}^2 \cdot \frac{\lambda}{n}$ and obtain that

$$\lambda = \frac{(2\bar{X} + \frac{z_{0.995}^2}{n}) \pm \sqrt{\frac{z_{0.995}^4}{n^2} + \frac{4\bar{X} z_{0.995}^2}{n}}}{2}$$

And the two solutions serve as the end points of the approximate C.I.

2. (a) Clearly, $\bar{X} - \mu_1 \sim N(0, \frac{\sigma_1^2}{n})$, $\bar{Y} - \mu_2 \sim N(0, \frac{\sigma_2^2}{m})$

Since $\underset{\sim}{X} = (X_1, \dots, X_n)$ and $\underset{\sim}{Y} = (Y_1, \dots, Y_m)$ are independent,

$$(\bar{X} - \bar{Y}) - (\mu_1 - \mu_2) = (\bar{X} - \mu_1) - (\bar{Y} - \mu_2) \sim N(0, \frac{\sigma_1^2}{n} + \frac{\sigma_2^2}{m})$$

So the best 90% C.I for $\mu_1 - \mu_2$ is $(\bar{X} - \bar{Y}) \pm z_{0.95} \cdot \sqrt{\frac{\sigma_1^2}{n} + \frac{\sigma_2^2}{m}}$, where $z_{0.95}$ is the 0.95 quantile of $N(0,1)$

(b) Let $\sigma_1 = \sigma_2 = \sigma$. then $(\bar{X} - \bar{Y}) - (\mu_1 - \mu_2) \sim N(0, \sigma^2(\frac{1}{n} + \frac{1}{m}))$, i.e. $\frac{\bar{X} - \bar{Y} - (\mu_1 - \mu_2)}{\sigma\sqrt{\frac{1}{n} + \frac{1}{m}}} \sim N(0,1)$.

It's known that $\frac{(n-1)S_X^2}{\sigma^2} \sim \chi_{n-1}^2$, $\frac{(m-1)S_Y^2}{\sigma^2} \sim \chi_{m-1}^2$ and S_X^2 is independent of S_Y^2.

where $S_X^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2$. $S_Y^2 = \frac{1}{m-1} \sum_{j=1}^{m} (Y_j - \bar{Y})^2$.

Then $\frac{(n-1)S_X^2}{\sigma^2} + \frac{(m-1)S_Y^2}{\sigma^2} \sim \chi_{n+m-2}^2$.

It's also known that $\bar{X}$ and S_X^2 are independent, $\bar{Y}$ and S_Y^2 are independent

so $\frac{\bar{X} - \bar{Y} - (\mu_1 - \mu_2)}{\sigma\sqrt{\frac{1}{n} + \frac{1}{m}}}$ and $\frac{(n-1)S_X^2}{\sigma^2} + \frac{(m-1)S_Y^2}{\sigma^2}$ are independent.

It implies that $\dfrac{[\bar{X}-\bar{Y}-(\mu_1-\mu_2)]/\sqrt{\frac{1}{n}+\frac{1}{m}}}{\sqrt{[(n-1)S_X^2+(m-1)S_Y^2]/(n+m-2)}} = \dfrac{[\bar{X}-\bar{Y}-(\mu_1-\mu_2)]/(\sigma\sqrt{\frac{1}{n}+\frac{1}{m}})}{\sqrt{[(n-1)S_X^2+(m-1)S_Y^2]/(\sigma^2(n+m-2))}} \sim t_{n+m-2}$

Thus. the best 90% CI for $\mu_1-\mu_2$ is

$(\bar{X}-\bar{Y}) \pm t_{n+m-2,0.95}\cdot\sqrt{\frac{1}{n}+\frac{1}{m}}\cdot\sqrt{\frac{(n-1)S_X^2+(m-1)S_Y^2}{n+m-2}}$, where $t_{n+m-2,0.95}$ is the 0.95 quantile of t_{n+m-2}

(c) Once the two samples are obtained, calculate the above C.I.

If 0 lies in the interval, then we fail to reject H_0.

otherwise, we reject H_0.

(d) It's a one-sided test, so we use $t_{n+m-2,0.9}$, the 0.9 quantile of t_{n+m-2}.

Then the acceptance region of $\mu_1-\mu_2$ is $\left(\bar{X}-\bar{Y}-t_{n+m-2,0.9}\cdot\sqrt{\frac{1}{n}+\frac{1}{m}}\cdot\sqrt{\frac{(n-1)S_X^2+(m-1)S_Y^2}{n+m-2}},\ \infty\right)$

If 0 lies in this region, we fail to reject H_0.

Otherwise, we reject H_0.

3. (a) For $0<x<1$.

$P(X_{(n)}/\theta \le x) = P(X_{(n)} \le \theta x) = P(X_1 \le \theta x, \dots, X_n \le \theta x) = \prod_{i=1}^{n} P(X_i \le \theta x) = \prod_{i=1}^{n}\left(\frac{\theta x}{\theta}\right) = x^n$

By taking the derivative w.r.t x, we obtain the pdf of $X_{(n)}/\theta$ is

$f(x|\theta) = n x^{n-1}, \quad 0<x<1$

The distribution of $X_{(n)}/\theta$ does not depend on θ, so it's a pivot.

(b) For any $a<b\le 1$, we have $P(a<\frac{X_{(n)}}{\theta}<b) = b^n - a^n$.

Thus, choose a and b such that $b^n-a^n=r$, for example, $a=\left(\frac{1-r}{2}\right)^{1/n}$, $b=\left(\frac{1+r}{2}\right)^{1/n}$.

Then $P\left(\frac{X_{(n)}}{b}<\theta<\frac{X_{(n)}}{a}\right) = P\left(a<\frac{X_{(n)}}{\theta}<b\right) = r$.

i.e. $\left(\frac{X_{(n)}}{b}, \frac{X_{(n)}}{a}\right)$ is a level r CI for θ.

(c). Now $r=0.95$. take $a=\left(\frac{1-r}{2}\right)^{1/n}=(0.025)^{1/n}$, $b=\left(\frac{1+r}{2}\right)^{1/n}=(0.975)^{1/n}$

Then the 95% CI for θ is $\left(\frac{X_{(n)}}{(0.975)^{1/n}}, \frac{X_{(n)}}{(0.025)^{1/n}}\right)$.

9.28. (a) The joint pdf of $\underset{\sim}{X} = (X_1, \dots, X_n)$ is

$$f_n(\underset{\sim}{X} \mid \theta, \sigma^2) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(X_i-\theta)^2}{2\sigma^2}} = \frac{1}{(2\pi\sigma^2)^{n/2}} \cdot e^{-\frac{\sum(X_i-\theta)^2}{2\sigma^2}}$$

The posterior joint pdf $\pi(\theta, \sigma^2 \mid \underset{\sim}{X})$ is

$$\pi(\theta, \sigma^2 \mid \underset{\sim}{X}) \propto f_n(\underset{\sim}{X} \mid \theta, \sigma^2) \cdot \pi(\theta, \sigma^2 \mid \mu, \tau^2, a, b)$$

$$= \frac{1}{(2\pi\sigma^2)^{n/2}} e^{-\frac{\sum(X_i-\theta)^2}{2\sigma^2}} \cdot \frac{1}{\sqrt{2\pi\tau^2\sigma^2}} e^{-\frac{(\theta-\mu)^2}{2\tau^2\sigma^2}} \cdot \frac{1}{\Gamma(a) b^a} \cdot \left(\frac{1}{\sigma^2}\right)^{a+1} e^{-\frac{1}{b\sigma^2}}$$

$$= \frac{1}{(2\pi\sigma^2)^{n/2}} e^{-\frac{1}{2\tau^2\sigma^2}\left[(n\tau^2+1)\theta^2 - 2(\tau^2\sum X_i + \mu)\theta + (\tau^2\sum X_i^2 + \mu^2)\right]} \cdot \frac{1}{\sqrt{2\pi\tau^2\sigma^2}} \cdot \frac{1}{\Gamma(a) b^a} \cdot \left(\frac{1}{\sigma^2}\right)^{a+1} \cdot e^{-\frac{1}{b\sigma^2}}$$

$$= \frac{1}{(2\pi\sigma^2)^{n/2}} \cdot e^{-\frac{n\tau^2+1}{2\tau^2\sigma^2}\left(\theta - \frac{\tau^2\sum X_i + \mu}{n\tau^2+1}\right)^2} \cdot e^{-\frac{n\tau^2\sum(X_i-\bar{X})^2 + \sum(X_i-\mu)^2}{2\sigma^2(n\tau^2+1)}} \cdot \frac{1}{\sqrt{2\pi\tau^2\sigma^2}} \cdot \frac{1}{\Gamma(a) b^a} \cdot \left(\frac{1}{\sigma^2}\right)^{a+1} e^{-\frac{1}{b\sigma^2}}$$

$$\propto \frac{1}{\sqrt{2\pi\tau^2\sigma^2/(n\tau^2+1)}} e^{-\frac{\left(\theta - \frac{\tau^2\sum X_i + \mu}{n\tau^2+1}\right)^2}{2\tau^2\sigma^2/(n\tau^2+1)}} \cdot \left(\frac{1}{\sigma^2}\right)^{a+\frac{n}{2}+1} e^{-\frac{1}{\sigma^2}\left(\frac{n\tau^2\sum(X_i-\bar{X})^2 + \sum(X_i-\mu)^2}{2(n\tau^2+1)} + \frac{1}{b}\right)}$$

It is the pdf of $N(\mu_1, \tau_1^2\sigma^2)$ multiplied by the pdf of $IG(a_1, b_1)$

where $\mu_1 = \dfrac{\tau^2\sum X_i + \mu}{n\tau^2+1}$, $\tau_1^2 = \dfrac{\tau^2}{n\tau^2+1}$, $a_1 = a + \dfrac{n}{2}$, $b_1 = \left(\dfrac{n\tau^2\sum(X_i-\bar{X})^2 + \sum(X_i-\mu)^2}{2(n\tau^2+1)} + \dfrac{1}{b}\right)^{-1}$

Thus, this is a conjugate family.

(b). The posterior pdf of θ is

$$\pi(\theta \mid \underset{\sim}{X}) = \int_0^\infty \pi(\theta, \sigma^2 \mid \underset{\sim}{X}) d\sigma^2 = \int_0^\infty \frac{1}{\sqrt{2\pi\tau_1^2\sigma^2}} e^{-\frac{(\theta-\mu_1)^2}{2\tau_1^2\sigma^2}} \cdot \frac{1}{\Gamma(a_1) b_1^{a_1}} \left(\frac{1}{\sigma^2}\right)^{a_1+1} \cdot e^{-\frac{1}{b_1\sigma^2}} d\sigma^2$$

Let $y = \frac{1}{\sigma^2}$, it follows that

$$\pi(\theta \mid \underset{\sim}{X}) = \int_\infty^0 \frac{1}{\sqrt{2\pi\tau_1^2}} y^{1/2} e^{-\frac{(\theta-\mu_1)^2}{2\tau_1^2} y} \cdot \frac{1}{\Gamma(a_1) b_1^{a_1}} \cdot y^{a_1+1} \cdot e^{-\frac{1}{b_1} y} (-y^{-2}) dy$$

$$= \frac{1}{\sqrt{2\pi\tau_1^2}} \cdot \frac{1}{\Gamma(a_1) b_1^{a_1}} \cdot \int_0^\infty \underbrace{y^{a_1-\frac{1}{2}} e^{-\left\{\frac{(\theta-\mu_1)^2}{2\tau_1^2} + \frac{1}{b_1}\right\} y}} dy$$ (proportional to the pdf of Gamma distribution.)

$$= \frac{1}{\sqrt{2\pi\tau_1^2}} \cdot \frac{1}{\Gamma(a_1) b_1^{a_1}} \cdot \Gamma\left(a_1 + \frac{1}{2}\right) \cdot \left(\frac{(\theta-\mu_1)^2}{2\tau_1^2} + \frac{1}{b_1}\right)^{-(a_1+\frac{1}{2})}$$

$$\propto \left(1 + \frac{(\theta-\mu_1)^2}{2\tau_1^2/b_1}\right)^{-(a_1+\frac{1}{2})}$$

Then $\eta := \frac{\sqrt{a_1 b_1}}{\tau_1}(\theta - \mu_1)$ has the posterior pdf.

$$f(\eta \mid \underset{\sim}{X}) \propto \left(1 + \frac{\eta^2}{2a_1}\right)^{-(a_1+\frac{1}{2})}$$

One can find a γ_1-th quantile and γ_2-th quantile of the posterior distribution of η s.t $\gamma_2 - \gamma_1 = 1-\alpha$

Then $P\left(\gamma_1 < \frac{\sqrt{a_1 b_1}}{\tau_1}(\theta - \mu_1) < \gamma_2 \mid \underset{\sim}{X}\right) = 1-\alpha$

Thus, a $1-\alpha$ credible set for θ is $\left(\mu_1 + \frac{\gamma_1 \tau_1}{\sqrt{a_1 b_1}}, \mu_1 + \frac{\gamma_2 \tau_1}{\sqrt{a_1 b_1}}\right)$.

c). As $n \to \infty$, $a_1 = a + \frac{n}{2} \sim \frac{n}{2}$ for any fixed a.

Approximately, $\eta = \frac{\sqrt{a_1 b_1}}{\tau_1}(\theta - \mu_1)$ has the posterior t_n distribution

And the $1-\alpha$ credible set can be written as $\left|\frac{\sqrt{a_1 b_1}}{\tau_1}(\theta - \mu_1)\right| \le t_{n,\alpha/2}$

or equivalently. $|\theta - \mu_1|^2 \le t_{n,\alpha/2}^2 \cdot \frac{\tau_1^2}{a_1 b_1}$

Note that $t_n^2 = F_{1,n}$. and $t_{n,\alpha/2}^2 = F_{1,n,\alpha}$

One can choose $\tau^2 \to \infty$, $a \to 0$, $b \to \infty$.

Then $\mu_1 \to \bar{X}$, $a_1 \to \frac{n}{2}$, $b_1 \to \frac{2}{\sum(X_i - \bar{X})^2} = \frac{2}{nS^2}$ (or $\frac{2}{(n-1)S^2}$), $\tau_1^2 \to \frac{1}{n}$.

It follows that the credible set satifies

$$|\theta - \bar{X}|^2 \le F_{1,n,\alpha} \cdot \frac{S^2}{n} \quad . \quad \text{approximately}$$

The desired result is obtained.