The Infinite Sum of $\frac{h}{n^h}$

Dan Lecocq

June 24, 2008

Abstract

In this, I will briefly examine the infinite sum: $\sum_{h=0}^{\infty} \frac{h}{n^h}$.

1 Guesswork

I was reading about heapsort, and they were claiming that the time complexity of building a heap was bounded:

$$Time \leq O\left(n \sum_{h=0}^{\infty} \frac{h}{2^h}\right) = O(n) \tag{1}$$

Curious about this, I began to take a look at its convergence. I would not at all have been surprised were it to converge on a constant, but what constant exactly? Initially, I attempted to intuit the convergence from a glance at the series expanded:

$$\sum_{h=0}^{\infty} \frac{h}{2^h} = 0 + \frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + ... \tag{2}$$

But, this doesn't look very attractive. There's not really much that can be paired right off the bat (like with the sum of consecutive numbers):

$$\sum_{i=0}^{k} i = 0+1+2+3+...+k = (k+1)+(k-1+2)+...(k-\frac{k}{2}+\frac{k}{2}+1) = \frac{k(k+1)}{2}$$

As such, I wrote a short Ruby scrpt to calculate the first k items in the sum, and give me a result of 2 after the first 100 terms were added. With that many terms the error was bound to be very small, but to actually get 2.0 made me wonder about the general pattern for $\sum_{h=0}^{\infty} \frac{h}{n^h}$. Plugging in for three, I got 0.75, and for 4, I got 0.444... or $\frac{4}{9}$. Seeing the pattern:

$$n = 2 \Rightarrow 2$$

$$n = 3 \Rightarrow \frac{3}{4}$$

$$n = 4 \Rightarrow \frac{4}{9}$$

$$n \Rightarrow \frac{n}{(n-1)^2}$$

2 Proof

I decided I'd start by multiplying the sum by $\frac{(n-1)^2}{(n-1)^2}$, and only to an arbitrary number of terms, k:

$$\frac{1}{(n-1)^2}\sum_{h=0}^{k}\frac{h(n-1)^2}{n^h}=\frac{1}{(n-1)^2}\sum_{h=0}^{k}\left[\frac{h}{n^{(h-2)}}-2\frac{h}{n^{(h-1)}}+\frac{h}{n^h}\right] \tag{3}$$

Focusing on the sum itself (and for now ignoring the constant) by expanding it out, we get some fortuitous cancellations:

$$\begin{array}{lllllll}
h=0 & 0n^2 & -0n & 0 & & & \\
h=1 & & +n & -2 & +\frac{1}{n} & & \\
h=2 & & & +2 & -4\frac{1}{n} & +2\frac{1}{n^2} & \\
h=3 & & & & +3\frac{1}{n} & -6\frac{1}{n^2} & +3\frac{1}{n^3} \\
h=4 & & & & & +4\frac{1}{n^2} & -8\frac{1}{n^3} \quad +4\frac{1}{n^4} \\
\ldots & & & & & & \ldots \qquad \ldots
\end{array} \tag{4}$$

We are left with:

$$\sum_{h=0}^{k}\left[\frac{h}{n^{(h-2)}}-2\frac{h}{n^{(h-1)}}+\frac{h}{n^h}\right]=n+\frac{-k-1}{n^{(k-1)}}+\frac{k}{n^k} \tag{5}$$

The limit as k approaches infinity is simply n. A more rigorous proof could be done using induction, but we are left with:

$$\sum_{h=0}^{\infty}\frac{h}{n^h}=\frac{n}{(n-1)^2} \tag{6}$$

2.1 Proof By Induction

From (4), we have

$$\sum_{h=0}^{2}\left[\frac{h}{n^{(h-2)}}-2\frac{h}{n^{(h-1)}}+\frac{h}{n^h}\right]=n-\frac{3}{n}+\frac{2}{n^2} \tag{7}$$

for which equation (5) holds true.

For k, we already have:

$$\sum_{h=0}^{k}\left[\frac{h}{n^{(h-2)}}-2\frac{h}{n^{(h-1)}}+\frac{h}{n^h}\right]=n+\frac{-k-1}{n^{(k-1)}}+\frac{k}{n^k}$$

So, for $k+1$,

$$\sum_{h=0}^{k+1}\ldots=n+\frac{-k-1}{n^{(k-1)}}+\frac{k}{n^k}+\frac{k+1}{n^{(k-1)}}-2\frac{k+1}{n^k}+\frac{k+1}{n^{(k+1)}}$$

$$\sum_{h=0}^{k+1}\ldots=n+\frac{-k-2}{n^k}+\frac{k+1}{n^{(k+1)}} \tag{8}$$

for which again equation (5) holds true when evaluated at $k+1$.