Representations up to Homotopy of Lie Groupoids

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Representation of Lie Groups

Recall: Representation of Lie Groups

A Representation of the Lie group G, acting on an n-dimensional vector space V, then an smooth group homomorphism

$$\Pi : G \to \mathrm{GL}(V).$$

i.e. $\Pi(g_1 g_2) = \Pi(g_1) \circ \Pi(g_2)\ \forall g_1, g_2 \in G$

If representation of G acts to its Lie algebra $\mathfrak{g}$, we call this representation $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$, $g \mapsto \mathrm{Ad}_g = d(\psi_g)$ as **Adjoint representation** of G.

Lie Groupoids and Algebroids

Definition 1.1 (groupoid)

Groupoid is a category in which all arrows are isomorphisms.

Definition 1.2 (Lie groupoid)

Lie groupoid G is a groupoid which has object set M and arrow set G, which M and G are smooth manifold and all of the following maps are smooth:

- source map $s : G \to M$
- target map $t : G \to M$
- copmosition map $m : G \times_M G \to G$
- inversion map $i : G \to G$
- identity map $\epsilon : M \to G$

Lie Groupoids and Algebroids

Definition 1.3 (Lie algebroid)

Lie algebroid over a manifold M is a vector bundle $\pi : A \to M$ together with

- a bundle map $\rho : A \to TM$ (called as **anchor map**)
- Lie bracket on section's space $\Gamma(A)$ s.t.

$$[\alpha, f\beta] = f[\alpha, \beta] + \rho(\alpha)(f)\beta \ \forall \alpha, \beta \in \Gamma(A), f \in C^\infty(M)$$

Remark. ρ can be considered as $\rho : \Gamma(A) \to \mathfrak{X}(M)$.

Definition 1.4 (Lie algebroid of G)

Lie algebroid A of a given Lie groupoid G is a vector bundle $\pi : A \to M$ s.t. $A_x = \ker(ds_x) \ \forall x \in M$.

Lie Groupoids and Algebroids

- Examples

1. Lie group G can be seen as a Lie groupoid s.t.

$$M \leftarrow \{G\},\ G \leftarrow G$$

2. If a Lie group G acts on a manifold M, there is an associated Lie groupoid s.t.

$$M \leftarrow M,\ G \leftarrow G \times M$$

Motivation of Representation of Lie Groupoids

- In the case of Lie group G, it is enough to bring a single vector space V to represent the action of G.
- However, Lie groupoids G do not connect every pair of point in M-so we need vector bundle over M to represent the action of G.

$$\lambda_g : E_{s(g)} \to E_{t(g)} \text{ for } g \in G$$

- But it is impossible to construct representation to satisfying associativity (i.e. $\lambda_{g_1}\lambda_{g_2} = \lambda_{g_1 g_2}$ for g_1, g_2 s.t. $s(g_1) = t(g_2)$).
- So we have to alleviate the associativity condition up to homotopy, with extending vector bundle E to the graded bundle $E^\bullet$.

Aim of This Paper

The main goal of this paper is generalize the representation theory of Lie groups to Lie groupoids.

1. Construct a **adjoint representation** of Lie groupoid G by using representation up to homotopy.
2. Compute a **differentiable cohomology** $H^{\bullet}_{\text{diff}}(G; E)$ by spectral sequence.
3. Generalize the **Bott's formula** about the cohomology of a classifying space BG to Lie algebroids G.

Graded Algebra

Definition 2.1 (graded algebra)

$A^\bullet = \bigoplus_{n\in\mathbb{Z}} A^n$ is called as a **graded algebra** if

- each A^n is R-module
- $a \in A^p, b \in A^q \Rightarrow ab \in A^{p+q}$ (graded multiplication)
- $a \in A^p, b \in A^q \Rightarrow ab = (-1)^{pq} ba$ (graded commutativity)

Definition 2.2 (differential graded algebra)

Graded algebra $A^\bullet$ is called **differential graded algebra** if it has a differential $d : A^\bullet \to A^{\bullet+1}$ s.t.

- $d^2 = 0$
- $d(ab) = d(a)b + (-1)^{|a|} a d(b)$ (Leibniz rule)

Classifying spaces

Definition 2.3 (space of strings)

For a Lie groupoid G, we denote by G_k the space of strings of k composable arrows $(g_1, \cdots, g_k)$ (i.e. $t(g_i) = s(g_{i-1})$) of G.

Remark. Since the source map $s : G \to M$ and target map $t : G \to M$ are submersions, all the G_k are manifolds.

Definition 2.4 (nerve of G)

The **nerve** of Lie groupoid G is the following simplicial manifold: the manifold of k-simplices is the G_k, with the simplicial structure given by the face maps $d_i(g_1, \cdots, g_k) = (g_1, \cdots, g_i g_i + 1, \cdots, g_k)$ and the degeneracy maps $s_i(g_1, \cdots, g_k) = (g_1, \cdots, g_i, 1, g_{i+1}, \cdots, g_k)$.

Classifying spaces

Definition 2.5 (classifying space BG)

The **classifying space** BG of G is the quotient space

$$BG = \left(\coprod_{k \geq 0} G_k \times \Delta^k \right)$$

obtained by identifying $d_i(p), v \in G_k \times \Delta^k$ with $(p, \delta_i(v)) \in G_{k+1} \times \Delta^{k+1}$.

Remark. $BG = EG/G$ for universal principal G-bundle of M.

Representation up to Homotopy

Notations

- $E = \bigoplus_{l \in \mathbb{Z}} E^l$: graded vector bundle over M.
- $C(G;E)^n := \bigoplus_{k+l=n} C^k(G;E^l) = \bigoplus_{k+l=n} \Gamma(G_k, t^* E^l)$
- $C^\bullet(G) := \bigoplus_{k \geq 0} C^k(G)$

  $C^k(G)$: set of smooth functions defined on G_k

Remark. $C^\bullet(G)$ is DGA and $C^\bullet(G;E)$ is a graded module over $C^\bullet(G)$.

Representation up to Homotopy

Definition 3.1 (representation up to homotopy)

A **representation up to homotopy** of G on a graded vector bundle E over M is a linear degree one operator $D : C(G;E)^\bullet \to C(G;E)^{\bullet+1}$ which satisfying $D^2 = 0$ and the Leibniz identity

$$D(\eta * f) = D(\eta) * f + (-1)^{|\eta|}\eta * \delta(f)$$

for every $\eta \in C^\bullet(G;E)$, $f \in C^\bullet(G)$.

This D is called the **structure operator** of the representation up to homotopy E.

Question. Why this cochain complex is called as representation *up to homotopy*?

Structure Equation

Notations

- $C_G(\mathrm{End}(E))$: Bigraded vector space
$$C_G^k(\mathrm{End}^l(E)) := \Gamma(G_k, \mathrm{Hom}(s^*(E^\bullet), t^*(E^{\bullet+l})))$$

Theorem 3.2

There is a bijective correspondence between following two concepts:

- representation up to homotopy of G on E
- sequences $\{R_k\}_{k\geq 0}$ of elements $R_k \in C^k(G; \mathrm{End}^{1-k}(E))$ satisfying

$$\sum_{j=1}^{k-1}(-1)^j R_{k-1}(g_1 \cdots g_j g_{j+1} \cdots g_k)$$
$$= \sum_{j=0}^{k} R_j(g_1 \cdots g_j) \circ R_{k-j}(g_j + 1 \cdots g_k) \text{ (\textit{structure equation})}$$

Structure Equation

Remark. $k = 0, 1, 2$ cases are important.

- $\partial := R_0 : E^\bullet \to E^{\bullet+1} \rightsquigarrow$ *coboundary map of* $E^\bullet$
- $\lambda := R_1 : E^\bullet \to E^\bullet \rightsquigarrow$ *graded quasi-action* s.t. $\lambda_g \partial = \partial \lambda_g$

  i.e. can be regarded as the maps of cochain complexes
- $k = 2$: $\lambda_{g_1} \circ \lambda_{g_2} - \lambda_{g_1 g_2} = \partial \circ R_2(g_1, g_2) + R_2(g_1, g_2) \circ \partial$

  i.e. R_2 gives the **homotopy equivalence** between $\lambda_{g_1} \circ \lambda_{g_2}$ and $\lambda_{g_1 g_2}$

Structure Equation

Sketch of the proof.

1. There is a 1-1 correspondence between
   - elements $T \in C_G^k(\mathrm{End}^l(E))$
   - operators on $C(G; E)$ which rise the bigrading by (k, l) and which are $C(G)$-linear.

2. There is a 1-1 correspondence between quasi-actions λ of G on E and degree 1 operator $\hat{D}_\lambda$ on $C^\bullet(G; E)$ satisfying the Leibniz identity.

3. $\{R_k\}_{k \geq 0} \longleftrightarrow D = D_0 + D_1 + \cdots \longleftrightarrow \hat{D}_\lambda \longleftrightarrow \lambda$

Example: Adjoint Representation

Definition 3.3 (adjoint complex)

Given a Lie groupoid G over M with Lie algebroid A, the adjoint copmlex of G denoted $\mathrm{Ad}(G)$ is the complex of vector bundles

$$\mathrm{Ad}(G) := A \xrightarrow{\rho} TM,$$

where A has degree zero, TM has degree one and ρ is the anchor map.

Remark. In this vector bundle, we don't need to define R_k for $|k| \geq 2$.

Example: Adjoint Representation

Define $\{R_k\}$ as follows:

- $R_0 := \rho$ (anchor map)
- $R_1 := \lambda$ (quasi-action given by G)
- $R_2 := K_\sigma^{\text{bas}}$ (basic curvature for Ehresman connection)

Then we get a corresponding representation up to homotopy $\text{Ad}_\sigma(G)$.

Remark. For two different connections σ, σ', resulting representation $\text{Ad}_\sigma(G)$ and $\text{Ad}_{\sigma'}(G)$ are canonically isomorphic.

$\Rightarrow$ Ad_G is well-defined in $\text{Rep}^\infty(G)$.

Bott's Formula for Lie Groupoids

- Bott's spectral sequence for Lie group G

$$E_1^{p,q} = H^{p-q}(G; \mathcal{S}^q(\mathfrak{g}^*)) \Rightarrow H^{p+q}(BG)$$

- For a Lie groupoid G, this paper provides the generalized formula:

Theorem 4.1 (Generalized Bott's formula)

Let G be a Lie groupoid, there is a spectral sequence converging to the cohomology of BG:

$$E_1^{p,q} = H_{\text{diff}}^{p+q}(G; \mathcal{S}^q(\mathrm{Ad}^*)) \Rightarrow H^{p+q}(BG)$$

Remark. If we can compute the cohomology of classifying space $H^\bullet(BG)$, we can classify action of G.