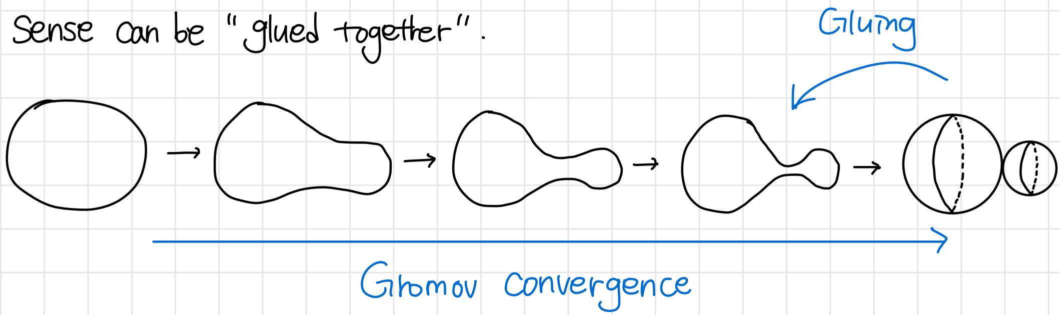
Mcduff – Salamon Ch 10. Gluing (10.1, 10.2, 10.5 intro)

* Motivations

Two J-holomorphic spheres that intersect transversally in the appropriate sense can be "glued together".



$\Rightarrow$ Gluing is a kind of converse to Gromov convergence.

* Settings

J: ω-tame almost complex structure on M

$A^0, A^\infty \in H_2(M;\mathbb{Z})$: two homology classes, $A := A^0 + A^\infty$

$D_u : \Omega^0(\Sigma, u^*TM) \to \Omega^{0,1}(\Sigma, u^*TM)$, vertical differential for $u : \Sigma \to M$

* Outlines

§ The moduli space of connected pairs (10.1)

§ Statement of gluing theorem (10.1)

§ Construction of pregluing (10.2)

§ Examples (10.2)

§ Construction of right inverse of D_u (10.5 + Appendix A.3)

§ The moduli space of connected pairs

$J^0 = \{J^0_z\}_{z\in S^2}$, $J^\infty = \{J^\infty_z\}_{z\in S^2}$: smooth families of ω-tame almost complex structure s.t. $\exists\, K>0$, $|z|<K \Rightarrow J^0_z = J^\infty_{1/z} =: J$ — ⊛

<u>Def</u> $(J^0, J^\infty) \in \mathcal{J}_\tau(S^2; M,\omega) \times \mathcal{J}_\tau(S^2; M,\omega)$ is called regular for (A^0, A^∞) if $J^0 \in \mathcal{J}_{reg}(S^2; A^0)$ and $J^\infty \in \mathcal{J}_{reg}(S^2; A^\infty)$

(1) Satisfy ⊛

(2) $ev: \mathcal{M}(A^0; J^0) \times \mathcal{M}(A^\infty; J^\infty) \longrightarrow M \times M$ is transverse to diagonal

$$(u^0, u^\infty) \longmapsto (u^0(0), u^\infty(\infty))$$

<u>RMK</u> meaning of (1) and (2):

(1) J^0 and J^∞ give the same equation near the intersection point $u^0(0) = u^\infty(\infty)$

(2) The condition to obtain a gluing w of u^0, u^∞ with surjective D_w.

Assume $u^0(0) = u^\infty(\infty)$, D_{u^0}, D_{u^∞} surjective : $2n + 2c_1(A^0), 2n + 2c_1(A^\infty) > 0$

If (2) holds, $\dim(\mathcal{M}(A^0; J^0)) + \dim(\mathcal{M}(A^\infty; J^\infty)) \geq 2n$

$\Rightarrow 2n + 2c_1(A^0) + 2c_1(A^\infty) = 2n + 2c_1(A^0 + A^\infty) \geq 0$

<u>RMK</u> From now on, fix J^0 and J^∞ to J.

Generally, we use new $J^R \in \mathcal{J}_{reg}(S^2; A^0 + A^\infty)$ for regular J^0, J^∞,

where $J^R := \begin{cases} J^0_z, & \text{if } |z| \geq 1/R \\ J^\infty_{R^2 z}, & \text{if } |z| \leq 1/R \end{cases}$ for gluing process.

Denote the moduli space $\mathcal{M}(A^{0,\infty}) := \{(u^0, u^\infty) \in \mathcal{M}(A^0) \times \mathcal{M}(A^\infty) \mid u^0(0) = u^\infty(\infty)\}$

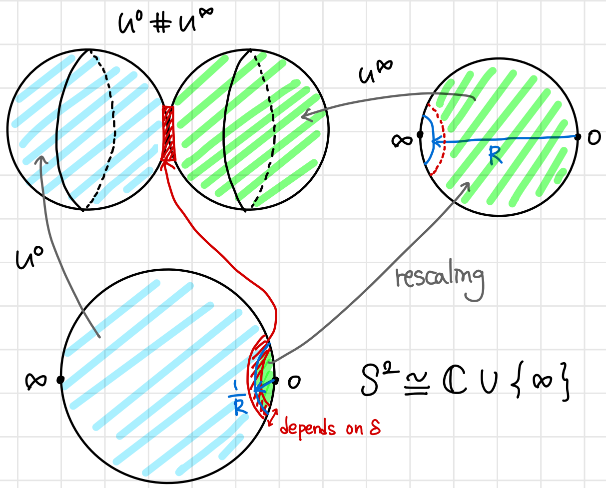
and $\mathcal{M}(c) := \mathcal{M}(A^{0,\infty}; c)$ the compact subset of $(u^0, u^\infty) \in \mathcal{M}(A^{0,\infty})$ that satisfies $\|du^0\|_{L^\infty} \leq c$, $\|du^\infty\|_{L^\infty} \leq c$.

§ Statement of gluing theorem

The gluing map $\mathcal{M}(c) \to \mathcal{M}(A)$ depends on two parameter: R and δ

R: rescaling parameter, δ: "length of the neck"

Brief description



Given a large constant $c > 0$ and a small constant $\delta_0(c) > 0$,
we need parameters δ, R s.t. $\delta < \delta_0(c)$ and $\delta R > 1/\delta_0(c)$ to maintain red region thin.

Def For each $\delta_0 > 0$, the set of annulus parameters $\mathcal{A}(\delta_0)$ by

$$\mathcal{A}(\delta_0) := \{ (\delta, R) \mid 0 < \delta < \delta_0 ,\ \delta R > \frac{1}{\delta_0} \}.$$

The next theorem states the main properties of the gluing map $\iota_c^{\delta, R} : \mathcal{M}(c) \to \mathcal{M}(A)$ for each pair $(\delta, R) \in \mathcal{A}(\delta_0)$.

Gluing Theorem [Thm 10.1.2]

For a given constant $c>0$, $\exists\ \delta_0>0$, $\varepsilon>0$, and a family of maps

$\iota_c^{\delta,R} = \iota_c^R : \mathcal{M}(c) \to \mathcal{M}(A)$ for each pair $(\delta, R) \in \mathcal{A}(\delta_0)$ satisfying followings:

$\boxed{1}$ Fix a constant $0<\delta<\delta_0$. Then the map ι_c^R varying smoothly on $R \in (1/\delta\delta_0, \infty)$ and ι_c^R is an embedding $\forall R > 1/\delta\delta_0$.

$\Rightarrow$ gluing map depends smoothly on R.

$\boxed{2}$ Let $(u_\nu^0, u_\nu^\infty) \to (u^0, u^\infty)$ in C^∞ topology, $R_\nu \to \infty$ and $\tilde{u}_\nu := \iota_c^{\delta, R_\nu}(u_\nu^0, u_\nu^\infty)$

Then $\tilde{u}_\nu^0 \rightrightarrows u^0$ on $S^2 \setminus \{0\}$, $\tilde{u}_\nu^\infty(z/R^2) \rightrightarrows u^\infty(z)$ on $S^2 \setminus \{\infty\}$.

$\Rightarrow$ C^0 – Gromov Convergence.

$\boxed{3}$ $ev \circ (\iota_c^{\delta,R} \times id) : \mathcal{M}(c) \times (S^2 \setminus \mathrm{int}\, B_{1/c}(0)) \longrightarrow M$ Converges to ev^0 s.t. $ev^0(u^0, u^\infty, z) = u^0(z)$ in C^1–topology as $R \to \infty$.

(Simirarily, $\mathcal{M}(c) \times B_c(0) \longrightarrow M : (u^0, u^\infty, z) \mapsto \iota_c^{\delta,R}(z/R^2)$ converges to ev^∞ s.t. $ev^\infty(u^0, u^\infty, z) = u^\infty(z)$ in C^1–top. as $R \to \infty$)

$\Rightarrow$ C^1 – Convergence of the evaluation maps ; to Control transversality

$\boxed{4}$ $\iota_c^{\delta,R}$: orientation – preserving embedding $\forall (\delta, R) \in \mathcal{A}(\delta_0)$

$\boxed{5}$ $(\delta, R) \in \mathcal{A}(\delta_0)$, $(u^0, u^\infty) \in \mathcal{M}(c-1)$, $v \in \mathcal{M}(A)$ Satisfy

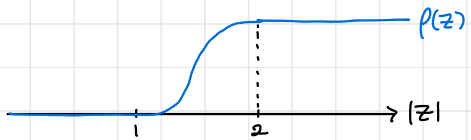
$$\sup_{|z| \geq 1} d(v(z), u^0(z)) < \varepsilon, \quad \sup_{|z| \leq 1} d(v(z/R^2), u^\infty(z)) < \varepsilon \Rightarrow v \in \mathrm{Im}(\iota_c^{\delta,R})$$

$\Rightarrow$ Surjectivity.

* $\boxed{1} \sim \boxed{3}$ will be proved in Ch 10.5.

§ Construction of a pregluing

Fix a smooth cutoff function $\rho : \mathbb{C} \to [0,1]$ s.t. $\rho(z) = \begin{cases} 0 & \text{if } |z| \le 1 \\ 1 & \text{if } |z| \ge 2 \end{cases}$.



For a small constant $\delta_0 > 0$ and $(\delta, \delta) \in \mathcal{A}(\delta_0)$, we can construct an approximate J-holomorphic curve (pregluing) $u^{\delta,R} := u^R : S^2 \to M$ s.t.

$$u^R(z) = \begin{cases} u^\infty(R^2 z) & ; \ |z| \le \frac{\delta}{2R} \\ \exp_x\left(\rho(\delta R z)\, \xi^0(z) + \rho\left(\frac{\delta}{Rz}\right) \xi^\infty(R^2 z)\right) & ; \ \frac{\delta}{2R} \le |z| \le \frac{2}{\delta R} \\ u^0(z) & ; \ |z| \ge \frac{2}{\delta R} \end{cases}$$

for $\xi^0(z), \xi^\infty(z) \in T_x M$ s.t. $\begin{cases} u^0(z) = \exp_x(\xi^0(z)) \text{ if } |z| \text{ is sufficiently small} \\ u^\infty(z) = \exp_x(\xi^\infty(z)) \text{ if } |z| \text{ is sufficiently large} \end{cases}$

(x denotes $u^0(0) = u^\infty(\infty)$ for $(u^0, u^\infty) \in \mathcal{M}(c)$)

$*\ \exists K > 0$ s.t. $\|du^0\|_{L^\infty}, \|du^\infty\|_{L^\infty} < C$ for $|z| < K$

$d_{FS}(z, 0) = \tan^{-1}(|z|)$

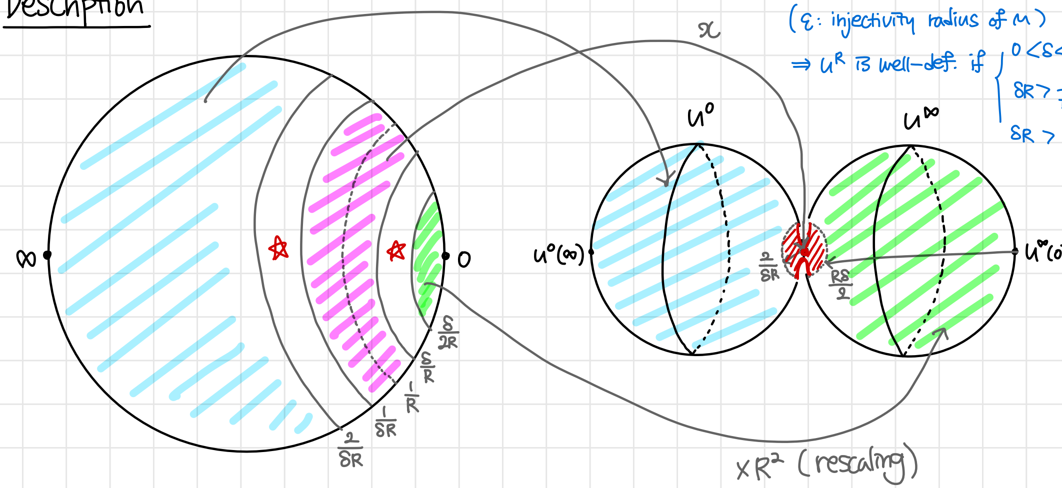
$\Rightarrow \begin{cases} |z| < \tan(\varepsilon/c),\ |z| < K \Rightarrow d_J(u^0(z), x) < \varepsilon \\ |z| > \cot(\varepsilon/c),\ |z| > 1/K \Rightarrow d_J(u^\infty(z), x) < \varepsilon \end{cases}$

(ε: injectivity radius of M)

$\Rightarrow u^R$ is well-def. if $\begin{cases} 0 < \delta < 1 \\ \delta R > \frac{2}{\tan(\varepsilon/c)} \\ \delta R > \frac{2}{K} \end{cases}$

RMK $u^R(z) = x \ \ \forall z$ s.t. $\frac{\delta}{R} \le |z| \le \frac{1}{\delta R}$

Description



RMK u^R is NOT a J-holomorphic curve ($\bar{\partial}_J(u) \neq 0$)

$\Rightarrow$ Goal) Construct a true J-holomorphic curve $\tilde{u}^R := \tilde{u}^{S.R}$ near u^R.

$\boxed{\text{How ?}}$ ① Construct an approximate solution u^R

$\downarrow$

② Construct an approximately (resp. real) right inverse T_{u^R} (resp. Q_{u^R})

$\downarrow$

③ Using Newton–Picard iteration : Find a real solution $\tilde{u}^R$

To construct T_{u^R}, we need mediate pairs of curves between $u^{0.\infty}$ and u^R.

For $r: \delta R$, $u^{0.r}, u^{\infty.r} : S^2 \to M$ are perturbed curves of u^0 and u^∞.

$$u^{0.r}(z) := \begin{cases} u^R(z) \ ; & |z| \geq 1/r \\ u^0(0) \ ; & |z| \leq 1/r \end{cases}, \quad u^{\infty.r}(z) := \begin{cases} u^R(z/R^2) \ ; & |z| \leq r \\ u^\infty(\infty) \ ; & |z| \geq r \end{cases}$$

$\Rightarrow$ as $r \to \infty$, $\boxed{1}$ $u^{0.r} \to u^0$, $u^{\infty.r} \to u^\infty$ in the $W^{1.p}$– norm

$\boxed{2}$ $D_{u^{0.r}} \to D_{u^0}$, $D_{u^{\infty.r}} \to D_{u^\infty}$ in the operator norm

§ Examples

<u>Ex 1</u> [Example 10.2.3] Let $M = S^2 \cong \mathbb{C} \cup \{\infty\}$.

$u^0(z) = 1 + z$, $u^\infty(z) = 1 + 1/z$: intersecting holomorphic spheres

$(u^0(0) = u^\infty(\infty) = 1)$

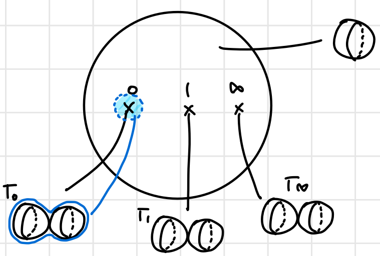
$$\Rightarrow u^R(z) = \begin{cases} 1 + 1/R^2 z & \text{if } |z| \leq \delta/2R \\ 1 & \text{if } \delta/R \leq |z| \leq 1/\delta R \\ 1 + z & \text{if } 2/\delta R \leq |z| \end{cases} \quad : \text{pregluing}$$

nearby J-holomorphic curve : $\tilde{u}^R(z) = \dfrac{z^2 + z + 1/R^2}{z}$

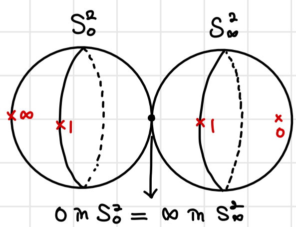
<u>Ex 2</u> [Remark 10.2.4] moduli space $\overline{\mathcal{M}}_{0,4}$

open stratum : $S^2 \setminus \{0, 1, \infty\}$ Consists of the element $T_z = [0, 1, \infty, z]$

Then compactifying by T_0, T_1, T_∞

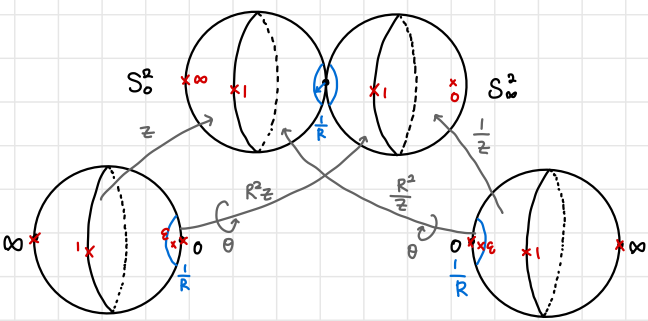


T_0 :



Q) How to build the neighborhood of T_0 ?

A) By gluing of the stable curve T_0, and let $|\varepsilon| = \dfrac{1}{R^2}$, $\varepsilon = |\varepsilon| e^{i\theta}$



§ Construction of right inverse of D_u

We need the right inverse of D_{u^R} to find the exact solution of $\bar{\partial}_J(\tilde{u}_R) = 0$

near the pregluing u_R.

Thm [Appendix A.3, Proposition A.3.4]

X, Y : Banach sp. $U \subset X$: open set, $f : U \to Y$ be a C^1 map

$x_0 \in U$: $D := df(x_0) : X \to Y$ is surjective & has a right inverse $Q : Y \to X$

Let $\delta, c > 0$ be constants s.t. $\|Q\| < c$, $B_\delta(x_0, X) \subseteq U$,

$\|x - x_0\| < \delta \Rightarrow \|df(x) - D\| < \frac{1}{2c}$

Assumptions

Then, for $x_1 \in X$,

$\|f(x_1)\| < \frac{\delta}{4c}$, $\|x_1 - x_0\| < \frac{\delta}{8} \Rightarrow \exists!\, x \in X$ s.t. $f(x) = 0$, $x - x_1 \in \operatorname{im} Q$, $\|x - x_0\| < \delta$

RMK At the above theorem, f : equation, x_1 : trial solution, x : real solution

RMK Correspondence to find real J-holomorphic curve $\tilde{u}^R$:

$X : W^{1,p}(S^2, (u^R)^* TM)$, $Y : L^p(S^2, \Lambda^{0,1} \otimes_J (u^R)^* TM)$, $U = X$

$f : \mathcal{F}_u \left(\mathcal{F}_u(\xi) := \Phi_u(\xi)^{-1} (\bar{\partial}_J (\exp_u(\xi))) \right)$

$x_0 = 0$, $df(x_0) = d\mathcal{F}_u(0) = D_u$

If $\exists\, \xi \in X$ s.t. $\mathcal{F}_u(\xi) = 0$, $\bar{\partial}_J(\exp_u(\xi)) = 0$

$$\Rightarrow \exp_u(\xi) : \begin{array}{ccc} S^2 & \longrightarrow & M \\ z & \longmapsto & \exp_u(\xi(z)) \end{array} \quad \text{be a real gluing}$$

Thus, we have to find a right inverse Q_{u^R} of D_{u^R} and prove that this Q_{u^R} satisfies the above assumptions.

Construction process

$\ast$ Notations

For smooth map $u : S^2 \to M$,

$W^{1,p}_u := W^{1,p}(S^2, u^*TM)$, $L^p_{u,J} := L^p(S^2, \Lambda^{0,1} \otimes_J u^*TM)$,

$W^{1,p}_{u^{0,\infty}} := \{ (\xi^0, \xi^\infty) \in W^{1,p}_{u^0} \times W^{1,p}_{u^\infty} \mid \xi^0(0) = \xi^\infty(\infty) \}$

$$D_{0,\infty} := D_{u^0,u^\infty} : W^{1,p}_{u^{0,\infty}} \longrightarrow L^p_{u^0} \times L^p_{u^\infty}$$
$$(\xi^0, \xi^\infty) \mapsto (D_{u^0}\xi^0, D_{u^\infty}\xi^\infty)$$

$\ast$ $D_{0,\infty,r} := D_{u^{0,r},u^{\infty,r}}$, $Q_{0,\infty,r} := Q_{u^{0,r},u^{\infty,r}}$ defined similarily

For $\underbrace{(\ker(D_{0,\infty}))^{\perp}}_{L^2\text{-orthogonal Complement}} =: W_{u^{0,\infty}}$, $Q_{0,\infty} := Q_{u^0,u^\infty} := (D_{0,\infty}|_{W_{u^{0,\infty}}})^{-1}$

$\Rightarrow Q_{0,\infty}$ is a right inverse of $D_{0,\infty}$

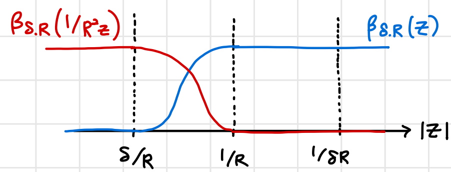
Idea) Define an approximate right inverse from $Q_{0,\infty,r}$

$$\begin{tikzcd}
Q_{0,\infty,r}(\eta^0,\eta^\infty) \quad W^{1,p}_{u^{0,\infty,r}} \arrow[d, "\text{Cutoff function}"'] \arrow[r, dashed, shift right, "D_{0,\infty,r}"'] \arrow[dr, phantom, "\circlearrowright"] & L^p_{u^{0,r}} \times L^p_{u^{\infty,r}} \quad (\eta^0,\eta^\infty) \arrow[l, shift right, "Q_{0,\infty,r}"'] \arrow[d] \\
\xi^R \quad W^{1,p}_{u^R} & L^p_{u^R} \quad \eta \arrow[l, "T_{u^R}"']
\end{tikzcd}$$

$$\eta^0(z) := \begin{cases} \eta(z) & ; |z| \geq 1/R \\ 0 & ; |z| \leq 1/R \end{cases}, \quad \eta^\infty(z) := \begin{cases} R^{-2}\eta(z/R^2) & ; |z| \leq R \\ 0 & ; |z| \geq R \end{cases}$$

Let $\beta_{\delta,R}$ be a cutoff function defined as

$$\beta_{\delta,R}(z) := \begin{cases} 0 & ; |z| \leq \delta/R \\ \log(R|z|/\delta)/\log(1/\delta) & ; \frac{\delta}{R} \leq |z| \leq \frac{1}{R} \\ 1 & ; |z| \geq \frac{1}{R} \end{cases}$$



Define $(\xi^0, \xi^\infty) := Q_{0,\infty,r}(\eta^0, \eta^\infty)$. Then $T_{uR}\eta := \xi^R$ can be obtained by connecting $\xi^0(z)$ and $\xi^\infty(R^2 z)$ by cutoff function β:

$$\xi^R(z) := \begin{cases} \xi^0(z) & ; |z| \geq \frac{1}{\delta R} \\ \xi^0(z) + \beta_{\delta R}(1/R^2 z)\left(\xi^\infty(R^2 z) - \xi_0\right) & ; \frac{1}{R} \leq |z| \leq \frac{1}{\delta R} \\ \xi^0(z) + \xi^\infty(R^2 z) - \xi_0 & ; |z| = \frac{1}{R} \\ \xi^\infty(R^2 z) + \beta_{\delta, R}(z)\left(\xi^0(z) - \xi_0\right) & ; \frac{\delta}{R} \leq |z| \leq \frac{1}{R} \\ \xi^\infty(R^2 z) & ; |z| \leq \frac{\delta}{R} \end{cases}$$

This $T_{uR} : L^p_{uR} \to W^{1,p}_{uR}$ be an approximate right inverse in the following sense:

<u>Thm</u> [Proposition 10.5.1]

T_{uR} is a bounded linear operator for each $(\delta, R) \in \mathcal{A}(\delta_0)$ s.t.

$\| D_{uR} T_{uR} \eta - \eta \|_{0,p,R} \leq \frac{1}{2} \|\eta\|_{0,p,R}$, $\| T_{uR} \eta \|_{1,p,R} \leq \frac{C_0}{2} \|\eta\|_{0,p,R}$ $\forall \eta \in L^p_{uR}$

($\|\cdot\|_{0,p,R}$, $\|\cdot\|_{1,p,R}$ are weighted norms in §10.3)

For the above T_{uR}, we get a real right inverse $Q_{uR} := T_{uR}(D_{uR} T_{uR})^{-1}$ and this Q_{uR} satisfies the assumptions in Newton–Picard iteration.