Application towards Littlewood's Conjecture

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Contents

Introduction to Littlewood's Conjecture

Conjecture 1.1 (Littlewood's Conjecture)

For every $\alpha, \beta \in \mathbb{R}$,

$$\liminf_{n\to\infty} n\|n\alpha\|\|n\beta\| = 0$$

where $\|\omega\| = \min_{n\in\mathbb{Z}} |\omega - n|$.

It is possible to turn this conjecture into an equivalence statement using a **diagonal action**.

Introduction to Littlewood's Conjecture

Notations:

$$A^+ = \{a(s,t) : s,t \geq 0\},\ a(s,t) = \begin{pmatrix} e^{s+t} & & \\ & e^{-s} & \\ & & e^{-t} \end{pmatrix}$$

Theorem 1.2

The following are equivalent:

1) (α, β) satisfies $\liminf_{n\to\infty} n\|n\alpha\|\|n\beta\| = 0$

2) In $X_3 = SL(3,\mathbb{Z})\diagdown SL(3,\mathbb{R})$, the orbit of $x_{\alpha,\beta} = SL(3,\mathbb{Z}) \begin{pmatrix} 1 & \alpha & \beta \\ & 1 & 0 \\ & & 1 \end{pmatrix}$

under the semigroup A^+ is unbounded.

Mahler's Compactness Criterion

Theorem 1.3 (Mahler's Compactness Critetion)

A set $E \subset X_n = SL(n, \mathbb{Z}) \diagdown SL(n, \mathbb{R})$ is bounded if and only if $\exists \epsilon > 0$ s.t. for $\forall x = \pi_\Gamma(g) \in E$, there is no vector v in the lattice in $SL(n, \mathbb{R})$ spanned by the rows of g with $\|v\|_\infty < \epsilon$.

The above criterion implies,

$$B_\epsilon(0) \cap SL(3, \mathbb{Z})g = \phi \text{ for } \forall g \in SL(3, \mathbb{R}).$$

Proof of the Equivalence

- proof of Theorem 1.2

Let $g_{\alpha,\beta} = \begin{pmatrix} 1 & \alpha & \beta \\ & 1 & 0 \\ & & 1 \end{pmatrix}$.

Then the following equivalent holds:

$A^+ \cdot g_{\alpha,\beta}$ unbounded

$\Longleftrightarrow$ for $\forall \epsilon > 0$, $\exists a \in A^+$ such that there is a nonzero vector v with $\|v\|_\infty < \epsilon$ in the lattice generated by the rows of $g_{\alpha,\beta} a^{-1}$

Proof of the Equivalence

Since

$$g_{\alpha,\beta}a^{-1} = \begin{pmatrix} 1 & \alpha & \beta \\ & 1 & 0 \\ & & 1 \end{pmatrix} \begin{pmatrix} e^{-s-t} & & \\ & e^s & \\ & & e^t \end{pmatrix} = \begin{pmatrix} e^{-s-t} & \alpha e^s & \beta e^t \\ & e^s & 0 \\ & & e^t \end{pmatrix},$$

v is expressed as

$$v = (ne^{-s-t}, (n\alpha - m)e^s, (n\beta - k)e^t).$$

Let $\epsilon \in (0, \frac{1}{2})$ arbitrary, then $\|n\alpha\| = n\alpha - m$, $\|n\beta\| = n\beta - k$.

$$\Rightarrow n\|n\alpha\|\|n\beta\| \leq \|v\|_\infty^3 < \epsilon^3$$

$$\therefore \liminf_{n\to\infty} n\|n\alpha\|\|n\beta\| = 0$$

$\square$

Current Known Results for Littlewood's Conjecture

A recent result of Littlewood's conjecture is for the **dimensions** of the set that do not satisfy the conjecture.

Theorem 2.1

For any $\delta > 0$, the set $\Xi_\delta = \{(\alpha, \beta) \in [0,1]^2 : \liminf_n n\|n\alpha\|\|n\beta\| \geq \delta\}$ has zero upper box dimension.

- upper box dimension: $\dim(s) = \limsup_{\epsilon \to 0} \frac{\log N(\epsilon)}{log(1/\epsilon)}$

Current Known Results for Littlewood's Conjecture

Theorem 2.1 is proved using several lemmas.

Notation: $a_{\sigma,\tau}(t) = a(\sigma t, \tau t)$

Lemma 2.2

Suppose that $(\alpha, \beta) \in \mathbb{R}^2$ does not satisfy *Conjecture 1.1*.
(i.e. $A^+_{\alpha,\beta}$ is bounded)
Then for any $\sigma, \tau \geq 0$, the topological entropy of $a_{\sigma,\tau}$ acting on the compact set $\overline{\{a_{\sigma,\tau}(t) \cdot x_{\alpha,\beta} : t \in \mathbb{R}^+\}}$ vanishes.

This means that the topological entropy is zero for s by any 1-parameter subgroup of A^+.

Current Known Results for Littlewood's Conjecture

Definition 2.3

For fixed $\sigma, \tau \geq 0$, orbit closure of $x_{\alpha,\beta}$ by $a_{\sigma,\tau}$-action is defined as

$$X_{\alpha,\beta} = \overline{\{a_{\sigma,\tau}(t) \cdot x_{\alpha,\beta} : t \in \mathbb{R}^+\}}.$$

Question: What is the size of the set of $(\alpha, \beta) \in [0, 1]^2$ that satisfies $h_{top}(X_{\alpha,\beta}, a_{\sigma,\tau}) = 0$?

Current Known Results for Littlewood's Conjecture

For a general $\mathbb{R}$-action, the following property holds.

Observation 2.4

Let X' be a metric space equipped with a continuous $\mathbb{R}$-action $(t, x) \mapsto a_t \cdot x$. Let X'_0 be a compact a_t-invariant subset of X' such that for $\forall x \in X'_0$, $h_{top}(Y_x, a_t) = 0$ when $Y_x = \overline{\{a_t \cdot x : t \in \mathbb{R}^t\}}$.
Then $h_{top}(X'_0, a_t) = 0$.

That is, if the topological entropy vanishes in all orbit closures passing through the elements of the a_t-invariant compact set, then the entropy vanishes too for the entire compact set.

Current Known Results for Littlewood's Conjecture

- Proof of Observation 2.4

Assume that $h_{top}(X_0', a_t) > 0$.

$\Rightarrow$ a_t-invariant ergodic measure μ on X_0' such that $h_\mu(a_t) > 0$.
($\because$ vaiational principle)

$\Rightarrow$ μ is supported on Y_x for μ-almost everywhere $x \in X_0'$.
($\because$ Pointwise Ergodic Theorem)

$\Rightarrow 0 = h_{top}(Y_x, a_t) \geq h_\mu(a_t) > 0.$

This is contradiction. $\square$

Current Known Results for Littlewood's Conjecture

Notation: $X_C = \{x \in X_3 : A^+ \cdot x \subset C\}$

Lemma 2.5

For any compact $C \subset X_3$, $h_{top}(X_C, a_{\sigma,\tau}) = 0$ for $\forall \sigma, \tau \geq 0$.

- Proof of Lemma 2.5

Let $Y_x = \overline{\{a_{\sigma,\tau}(t) \cdot x : t \in \mathbb{R}^+\}}$.

Since $A^+ \cdot x$ is bounded, $h_{top}(Y_x, a_{\sigma,\tau}) = 0$ holds by *Lemma 2.2*.

Since X_C is $a_{\sigma,\tau}$-invariant, $h_{top}(X_C, a_{\sigma,\tau}) = 0$ holds by *Observation 2.4*. $\square$

Current Known Results for Littlewood's Conjecture

Using Lemma 2.5, Theorem 2.1 can be proved.

- Proof of Theorem 2.1

To show that Ξ_δ has upper box dimension zero, it is enough to show Ξ_δ can be covered by $\mathcal{O}_\epsilon r^{-\epsilon}$ boxes of side r for $\forall \epsilon > 0$, $r \in (0, 1)$.

($\Leftrightarrow$ Cardinality of maximal r-seperated subset of Ξ_δ is $\mathcal{O}_{\delta,\epsilon}(r^{-\epsilon})$)

Definition 2.6

C_δ is a compact subset of X_3 such that $A^+ \cdot x_{\alpha,\beta} \subset C_\delta$ for $\forall(\alpha, \beta)$ with $\liminf_n n\|n\alpha\|\|n\beta\| \geq \delta$.

Current Known Results for Littlewood's Conjecture

Let d is a left invariant Riemannian metric on $G = SL(3, \mathbb{R})$.
Then d induces a metric d on $X_3 = GL(3, \mathbb{Z}) \diagdown GL(3, \mathbb{R})$.
Let

$$g_{a,b} = \begin{pmatrix} 1 & a & b \\ & 1 & 0 \\ & & 1 \end{pmatrix} \; (a, b \in \mathbb{R})$$

$\forall x \in X_3$, there is an open ball $B_{r_x}(I_3)$ and constant c_x such that

$$d(x, g_{a,b} \cdot x) \geq c_x \max(|a|, |b|) \; \forall g_{a,b} \in B_{r_x}(I_3)$$

$\Rightarrow \exists c_0, r_0$ such that $d(x, g_{a,b} \cdot x) \geq c_0 \max(|a|, |b|)$ for any $x \in C_\delta$,
$|a|, |b| < r_0$ ($\because$ compactness of C_δ)

Current Known Results for Littlewood's Conjecture

From

$$\begin{pmatrix} 1 & \alpha & \beta \\ & 1 & 0 \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & \alpha' - \alpha & \beta' - \beta \\ & 1 & 0 \\ & & 1 \end{pmatrix} = \begin{pmatrix} 1 & \alpha' & \beta' \\ & 1 & 0 \\ & & 1 \end{pmatrix}$$

$x_{\alpha,\beta} = g_{\alpha'-\alpha,\beta'-\beta} \cdot x_{\alpha',\beta'}$ holds.

Generally, the following results can be obtained through simple calculations.

$$a_{1,1}^{n} \cdot x_{\alpha,\beta} = g_{e^{3n}(\alpha'-\alpha),e^{3n}(\beta'-\beta)} \cdot a_{1,1}^{n} \cdot x_{\alpha',\beta'} \text{ for any } n$$

Current Known Results for Littlewood's Conjecture

Let $r = e^{-3n} r_0 \in (0, r_0)$ and S be an r-separated subset of Ξ_δ.

Then a subset S' of X_3 is defined as

$$S' = \{x_{\alpha,\beta} : (\alpha, \beta) \in S\}$$

$\Rightarrow S'$ is $(n, c_0 r_0)$-separated subset of X_3 for $a_{1,1}$

Because of $< a_{1,1} > \subset A^+$ and definition of Ξ_δ, $S' \subset X_{C_\delta}$.

$h_{top}(X_{C_\delta}, a_{1,1}) = 0$ ($\because$ *Lemma 2.5*)

$\Rightarrow Card(S') \leq \mathcal{O}_{\delta,\epsilon}(\exp(\epsilon n))$

Since $r < r_0$,

$$Card(S) = Card(S') \leq \mathcal{O}_{\delta,\epsilon}\left(\left(\frac{r}{r_0}\right)^{-\frac{\epsilon}{3}}\right) \leq \mathcal{O}_{\delta,\epsilon}\left(\left(\frac{r}{r_0}\right)^{-\epsilon}\right) = \mathcal{O}_{\delta,\epsilon}(r^{-\epsilon})$$

$\square$

Another Results: Construction of Nontrivial (α, β)

Before the Hausdorff dimension of Ξ_δ was known, the Lebesgue measure of sets satisfying the Littlewood's conjecture was first proved.

This fact is easily shown by using **continued fraction expansion**.

Definition 3.1

The set of **badly approximable numbers** is denoted as follows.

$$\mathbf{Bad} := \{\zeta \in \mathbb{R} \mid \liminf_n n\|n\zeta\| > 0\}$$

It is known that **Bad** coincides with the set of real numbers with *bounded continued fractions*.

Another Results: Construction of Nontrivial (α, β)

Theorem 3.2 (Borel-Bernstein Theorem)

Let $(u_n)_{n\geq 1}$ be a sequence of positive real numbers.
If the sum $\Sigma_{n\geq 1} u_n^{-1}$ diverges, then for almost every $\xi = [0; a_1, a_2, \cdots] \in [0, 1]$, there exist infinitely many integers n such that $a_n \geq u_n$.
If this sum converges, there exists a finite number of integer n such that $a_n \geq u_n$ for almost every $\xi = [0; a_1, a_2, \cdots] \in [0, 1]$.

From *Borel-Bernstein Theorem*, it can be seen that almost every real numbers have unbounded partial quotients.

Observation 3.3

Bad is a set with *Lebesgue measure zero*.

Another Results: Construction of Nontrivial (α, β)

Notation:
$\mathbf{L} = \{$The set of ordered pairs (α, β) that satisfy Littlewood's conjecture $\}$

Theorem 3.4

$\mathbf{L}$ has full Lebesgue measure.

- Proof of Theorem 3.4

If α or β has an unbounded partial fraction, then the ordered pair (α, β) satisfiees Littlewood's Conjecture.

Because of *Observation 3.3*, $\mathbf{L}$ has full measure.

$\square$

Another Results: Construction of Nontrivial (α, β)

An ordered pair (α, β) that satisfies the following two conditions is called a *non-trivial pair*.

$$\text{(i) } \alpha, \beta \in \mathbf{Bad}$$
$$\text{(ii) } 1,\ \alpha \text{ and } \beta \text{ are linearly independent over } \mathbb{Q}$$

An easily recognizable nontrivial pair is $(\sqrt{2}, \sqrt{3})$.

$$\sqrt{2} = 1 + \cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\ddots}}}},\ \sqrt{3} = 1 + \cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\ddots}}}}$$

Another Results: Construction of Nontrivial (α, β)

But even whether $(\sqrt{2}, \sqrt{3})$ satisfies Littlewood's conjecture is *an open problem!*

Question: For a fixed elements α of **Bad**, is there any *explicit form* of a real number β such that (α, β) is a nontrivial pair satisfying the Littlewood's Conjecture?

Another Results: Construction of Nontrivial (α, β)

- Idea of construction

Let $\alpha := [0; a_1, a_2, a_3, \cdots] \in \mathbf{Bad}$ is a real number which partial quotients are bounded.

Take any increasing sequence of positive sequences $n = (n_i)_{i \geq 1}$, any $M \in \mathbb{Z}$, and $(t_j)_{j \geq 1} \in \{M+1, M+2\}^{\mathbb{N}}$.

Before constructing (α, β), let's look at some lemmas about partial fractions.

Another Results: Construction of Nontrivial (α, β)

Notations:
$\frac{p_n}{q_n} = [a_0; a_1, \cdots, a_n]$ is called **n-th convergent** of $[a_0; a_1, a_2, \cdots]$.
$\tilde{u} = a_j a_{j-1} \cdots a_1$ is called *mirror* of $u = a_1 a_2 \cdots a_j$.

Lemma 3.5 (The mirror formula)

Let $\zeta = [a_0; a_1, a_2, \cdots] \in \mathbb{R}$ and $\left(\frac{p_n}{q_n}\right)_{n \geq 0}$ is the sequence of convergents to ζ. Then,

$$\frac{q_n}{q_{n-1}} = [a_n; a_{n-1}, \cdots, a_1]$$

for every positive integer n.

Another Results: Construction of Nontrivial (α, β)

Lemma 3.6 (Speed of Convergence)

Let $\zeta = [a_0; a_1, a_2, \cdots]$, $\eta = [b_0; b_1, b_2, \cdots]$ and assume $a_0 = b_0, a_1 = b_1, \cdots, a_n = b_n$.
Then, $|\zeta - \eta| \leq \frac{1}{{q_n}^2}$ where q_n denotes the n-th convergent of ζ.
If partial quotients of ζ and η are bounded by M and $a_{n+1} \neq b_{n+1}$,

$$\frac{1}{(M+2)^3 {q_n}^2} \leq |\zeta - \eta| \leq \frac{1}{{q_n}^2}$$

holds.

Another Results: Construction of Nontrivial (α, β)

For a given sequence $(n_i)_{i\geq 1}$ and $(t_i)_{i\geq 1}$, construct a $\beta_{n,t}$ as

$$\beta_{n,t} := [0; \widetilde{u_{n_1}}, t_1, \widetilde{u_{n_2}}, t_2, \widetilde{u_{n_3}}, t_3,]$$
$$(\widetilde{u_{n_j}} = a_{n_j} a_{n_{j-1}} \cdots a_{n_1} \text{ where } \alpha = [0; a_1, a_2, \cdots])$$

If $(n_i)_{i\geq 1}$ is a **rapidly increasing sequence**, $(\alpha, \beta_{n,t})$ satisfies Littlewood's conjecture.

Theorem 3.7

Let ϵ is a positiver real number with $\epsilon < 1$.
Suppose n increasing rapidly like $\liminf_{i\to\infty} \frac{n_{i+1}}{n_i} > \frac{4\log(M+3)}{\epsilon \log 2}$,
Then the pair $(\alpha, \beta_{n,t})$ is a nontrivial pair satisfying

$$q\|q\alpha\|\|q\beta_{n,t}\| \leq \frac{1}{q^{1-\epsilon}}$$

for infinitely many positive integer q.

In particular, *Theorem 3.7* means $(\alpha, \beta_{n,t}) \in \mathbf{L}$.

Another Results: Construction of Nontrivial (α, β)

- A sketch of the proof of Theorem 3.7

Let $(p_j/q_j)_{j\geq 1}$ denote the sequence of convergents to α,
and $(r_j/s_j)_{j\geq 1}$ denote the sequence of convergents to $\beta_{n,t}$.
Let $m_j = n_1 + n_2 + \cdots + n_j + (j-1)$.

By *Lemma 3.5*,

$$\frac{s_{m_j-1}}{s_{m_j}} = [0; a_1, \cdots, a_{n_j}, t_{j-1}, a_1, \cdots, a_{n_{j-1}}, \cdots, t_1, a_1, \cdots, a_{n_1}]$$

By *Lemma 3.6*,

$$s_{m_j}|\alpha - \frac{s_{m_{j-1}}}{s_{m_j}}| = \|s_{m_j}\alpha\| \leq s_{m_j} {q_{n_j}}^{-2}$$

Another Results: Construction of Nontrivial (α, β)

From $\frac{1}{q_n(q_n+q_{n+1})} < |\zeta - \frac{p_n}{q_n}| < \frac{1}{q_n q_{n+1}}$,

$$s_{m_j} \cdot \|s_{m_j}\beta_{n,t}\| < 1$$

Through calculation, it can be obtained that for every positive integer j large enough

$$q_{n_j} > \left(s_{m_j}\right)^{1-\frac{\epsilon}{2}}$$

$$\Rightarrow s_{m_j}\|s_{m_j}\alpha\|\|s_{m_j}\beta_{n,t}\| \le s_{m_j}{q_{n_j}}^{-2} < \frac{1}{{s_{m_j}}^{1-\epsilon}}$$

It can also be proved that $(\alpha, \beta_{n,t})$ is a nontrivial pair.

□

Reference

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[2] Y.Bugeaud, *Approximations by Algebraic Numbers*(Cambridge University Press, 2004), 1-26, 410-449.