Surface Quotients of Right-Angled Hyperbolic Buildings

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Exchange Seminar

November 26, 2024

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Definition of Hyperbolic Buildings

Definition 1.1 (Building)

A **building** is a simplicial complex Δ that can be expressed as the union of subcomplexes Σ (called **apartments**) satisfying the following axioms:

1. Each apartment Σ is a Coxeter complexes.
2. For any two simplices $A, B \in \Delta$ (called **chambers**), there is an apartment Σ containing both of them.
3. If Σ and Σ' are two apartments containing A and B, then there is an isomorphism $\Sigma \to \Sigma'$ fixing A and B pointwise.

By adding the condition that each chamber in the building Δ is a 2-dimensional polygon, this definition becomes that of a *2-dimensional building*.

Definition of Hyperbolic Buildings

Buildings can also be categorized based on the curvature of their apartments.

- positive curvature: spherical building
- zero curvature: Euclidean building
- negative curvature: hyperbolic building

Fuchsian buildings and **Bourdon's buildings** are special cases of hyperbolic buildings. These spaces possess many interesting properties, and their automorphism groups exhibit a rich variety of symmetries.

Definition of Hyperbolic Buildings

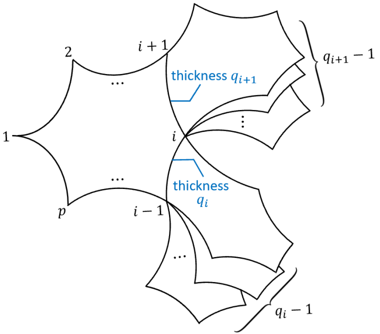


Figure: Fuchian building $I_{p,\mathbf{q}}$ for $\mathbf{q} = \{q_1, q_2, \cdots, q_p\}$

If $\mathbf{q} = \{v, v, \cdots, v\}$, we call this *Bourdon's building* and denote it by $I_{p,v}$.

Motivation and Main Question

Definition 1.2 (Surface Quotient Lattice)

Let Γ be a lattice of a hyperbolic building Δ. When the quotient space Δ/Γ is a genus g compact surface without boundary, such a lattice Γ is referred to as a **surface quotient lattice**. If Δ is a Fuchian building $I_{p,\mathbf{q}}$(resp. Bourdon's building ${}_{p,v}$), this Γ is denoted by $\Gamma_{p,\mathbf{q},g}$(resp. $\Gamma_{p,v,g}$).

Main Question. What are the conditions on p, $\mathbf{q}$, and g (resp. p, v, and g) for the surface quotient lattice $\Gamma_{p,\mathbf{q},g}$ (resp. $\Gamma_{p,v,g}$) to exist?

Motivation and Main Question

Theorem 1.2 (Futer-Thomas '12)

Let $p \geq 5$, $v \geq 2$, and $g \geq 2$ be integers, and let $I_{p,v}$ be Bourdon's building with a constant branching number v. Assume that $F = \frac{8(g-1)}{p-4}$ is a positive integer.

1. Existence of $\Gamma_{p,v,g}$.
   1. If $v \geq 2$ is even, then for all F, a lattice $\Gamma_{p,v,g}$ exists.
   2. If F is divisible by 4, then for all integers $v \geq 2$, a lattice $\Gamma_{p,v,g}$ exists.
   3. If F is composite, then for infinitely many odd integers $v \geq 3$, a lattice $\Gamma_{p,v,g}$ exists.
2. Non-existence of $\Gamma_{p,v,g}$.
   1. If F is odd, then for infinitely many odd integers $v \geq 3$, a lattice $\Gamma_{p,v,g}$ does not exist.

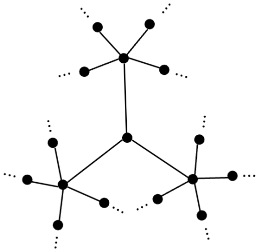
Our paper addresses the general case of Fuchsian buildings.

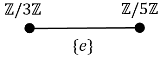
Complexes of Groups

Complexes of Groups can be considered as a high-dimensional analogue of graph of groups.

Definition 2.1 (Graph of Groups)

Let Y be a connected, nonempty graph. **Graph of groups** (G, Y) giving a group G_p for each P in $V(Y)$, group $G_y = G_{\bar{y}}$ with a monomorphism $\varphi_y : G_y \to G_{t(y)}$.





Complexes of Groups

Definition 2.2 (Complexes of Groups)

A complex of groups $G(X) = (G_\sigma, \psi_a, g_{a,b})$ over a polygonal complex X is given by:

1. a group G_σ for each $\sigma \in V(X')$ where X' is a barycentric subdivision of X, called the *local group* at σ;
2. a monomorphism $\psi_a : G_{i(a)} \to G_{t(a)}$ for each $a \in E(X')$; and
3. for each pair of composable edges a, b in X', an element $g_{a,b} \in G_{t(a)}$, such that

$$\mathrm{Ad}(g_{a,b}) \circ \psi_{ab} = \psi_a \circ \psi_b$$

where $\mathrm{Ad}(g_{a,b})$ is conjugation by $g_{a,b}$ in $G_{t(a)}$.

Universal Cover

Definition 2.3 (developability)

A complex of groups $G(Y)$ is **developable** if there exists a simply connected polygonal complex $\tilde{X}$ such that $\exists \Gamma \leq \mathrm{Aut}(\tilde{X})$ with $X = \tilde{X}/\Gamma$ and $G(X) \simeq G(Y)$, where $G(X)$ is a complex of groups generated by adding data about the stabilizers of $G(X)$.

Definition 2.4 (universal cover)

For a developable complex of groups $G(Y)$, $\tilde{X}$ is called the **universal cover** of X.

Remark. *Not* every complex of groups is globally developable.

Universal Cover

Definition 2.5 (Local Development)

Let $\widetilde{\mathrm{St}(\sigma)}$ be a complex of groups with an action of Γ_σ that is free of edge inversions, such that the quotient $\widetilde{\mathrm{St}(\sigma)}/\Gamma_\sigma$ is isomorphic to $\mathrm{St}(\sigma)$. We define $\widetilde{\mathrm{St}(\sigma)}$ as the **local development** of σ in X, and denote it by $\mathrm{St}(\tilde{\sigma})$.

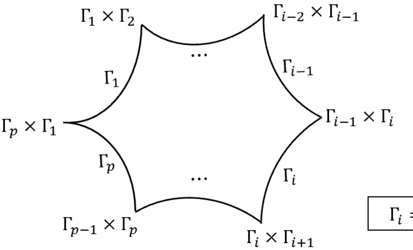
- Every complex of groups $G(X)$ has its local development at each vertex, even if $G(X)$ is not developable.
- A *nonpositively curved* complex of groups is developable.

Alternative Definition of a Fuchsian Building

The definition of the universal cover of complexes of groups and the developability of nonpositively curved complexes provide an alternative definition of Fuchsian buildings.

Definition 2.6 (Fuchsian Building)

For the complex of groups $G(P)$ defined below, the universal cover of $G(P)$ is called a **right-angled Fuchsian building**, denoted by $I_{p,\mathbf{q}}$.



$\boxed{\Gamma_i = \mathbb{Z}/i\mathbb{Z}}$

Construction of a Surface Tessellation

Theorem 2.7 (Edmond-Ewing-Kulkarina '82)

If a compact orientable surface of genus g with constant curvature -1 can be tessellated with F right-angled p-gons, the value of F is given by:

$$F = \frac{8(g-1)}{p-4}$$

Sketch of the proof.

The proof is based on computations using the Gauss-Bonnet theorem:

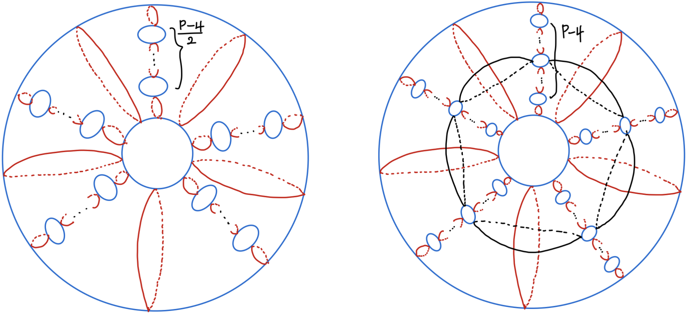
$$\int_M KdA + \int_{\partial M} k_g ds = 2\pi\chi(M)$$

$$-F(\text{area of a } p\text{-gon}) = -F\tfrac{\pi}{2}(p-4) = 2\pi(2-2g)$$

Construction of a Surface Tessellation

- Futer-Thomas's construction of hyperbolic surface tessellation

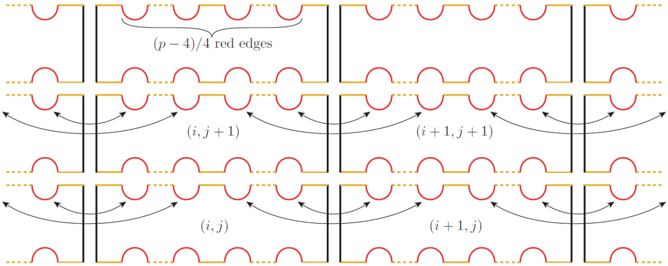
case 1. $4|F$



Construction of a Surface Tessellation

- Futer-Thomas's construction of hyperbolic surface tessellation

case 2. $4 \nmid F$



Main Theorem 1: Condition for the Existence

Definition 3.1 (alternating non-coprime sequence)

A sequence $\mathbf{q} = \{q_1, \cdots, q_p\}$ is called an **alternating non-coprime sequence** if it belongs to one of the following two cases:

1. If p is even, $q_1, q_3, \cdots, q_{p-1}$ are all divisible by $\exists d \in \mathbb{N}_{\geq 2}$, and $q_2, q_4, \cdots, q_p$ are all divisible by an integer $e > 1$.
2. If p is odd, $q_{i+1}, q_{i+3}, \cdots, q_{i+p-2}$ are all divisible by $\exists d \in \mathbb{N}_{\geq 2}$, and $q_{i+2}, q_{i+4}, \cdots, q_{i+p-1}$ are all divisible by an integer $e > 1$ for some integer i between 1 and p. Here, indices are $\bmod\ p$.

This condition on the p-tuple $\mathbf{q}$ provides a sufficient condition for the existence of a surface quotient, which we proved in this work.

Main Theorem 1: Condition for the Existence

Main Theorem 1 (Existence of $\Gamma_{p,\mathbf{q},g}$)

Assume that $\Gamma_{p,\mathbf{q},g} \leq \mathrm{Aut}(I_{p,\mathbf{q}})$ exists and $F = \frac{8(g-1)}{p-4}$ is an integer. Then, the following statements hold:

1. If F is divisible by 4, for all alternating non-coprime $\mathbf{q}$, a lattice $\Gamma_{p,\mathbf{q},g}$ exists.
2. If F is even and not divisible by 4, then for all alternating non-coprime and **2-symmetric** $\mathbf{q}$, a lattice $\Gamma_{p,\mathbf{q},g}$ exists.
3. If F is odd composite number, then for all alternating non-coprime and **4-symmetric** alternating non-coprime $\mathbf{q}$, a lattice $\Gamma_{p,\mathbf{q},g}$ exists if two alternating gcd d and e of $\mathbf{q}$ are both even.

Main Theorem 1: Condition for the Existence

- Homology group for tessellated surface

Theorem 3.2

Let X be a 2-dimensional compact orientable surface, and let $S_1, S_2, \cdots, S_k$ be homeomorphic images of S^1 on X such that $(X, S_1 \cup S_2 \cup \cdots \cup S_k)$ is a good pair. The intersection of S_i and $S_j (i \neq j)$ are either empty or consists of a single point, and the intersection of S_i, S_j, and S_k is empty for any distinct integers i, j, k. If $X/S_1 \cup S_2 \cup \cdots \cup S_k$ is orientable and has a trivial first homology group, then $[S_1], [S_2], \cdots, [S_k]$ generate the first homology group of X.

From this theorem, we can determine whether the set of **boundary geodesic loops** generates the basis of $H_1(X)$ for a given tessellation of X.

Main Theorem 1: Condition for the Existence

Sketch of the proof.

Let $A = S_1 \cup S_2 \cup \cdots \cup S_k$, and let U_i be a sufficiently small neighborhood of S_i.

From the generalized Mayer-Vietoris sequence,

$$\cdots \to \bigoplus_{i<j} \tilde{H}_1(U_i \cap U_j) \to \bigoplus_{i} \tilde{H}_1(U_i) \xrightarrow{\iota_*} \tilde{H}_1(A) \to \bigoplus_{i<j, U_i \cap U_j \neq \emptyset} \tilde{H}_0(U_i \cap U_j) \to \cdots 0$$

From the LES of a good pair (X, A),

$$\cdots \to \tilde{H}_2(X/A) \to \tilde{H}_1(A) \xrightarrow{i_*} \tilde{H}_1(X) \xrightarrow{j_*} \tilde{H}_1(X/A) \to \cdots$$

Main Theorem 1: Condition for the Existence

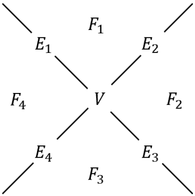
- A generalized lemma from Futer-Thomas about local groups:

Lemma 3.3

The local development $St(\tilde{\sigma})$ has a link L that is the complete bipartite graph $K_{q_i,q_{i+1}}$ if and only if

$$V = E_1E_2 = E_2E_3 = E_3E_4 = E_4E_1$$
$$E_1 \cap E_2 = F_1,\ E_2 \cap E_3 = F_2,\ E_3 \cap E_4 = F_3,\ E_4 \cap E_1 = F_4$$

$$\text{and } |V : E_1| + |V : E_3| = q_i,\ |V : E_2| + |V : E_4| = q_{i+1}.$$



Main Theorem 1: Condition for the Existence

Sketch of the proof of Main Theorem 1.

Step 1. Construct a complex of groups on the p-gon tessellation given by Futer-Thomas, using Lemma 3.3.

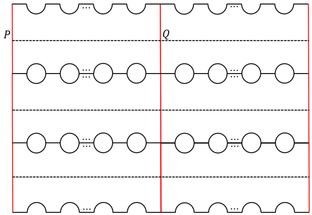
Remark. The most important part of the proof is to prove the well-definedness of this construction method.

Step 2. Prove that this construction is well-defined when the set of boundary geodesic loops generates $H_1(X)$.

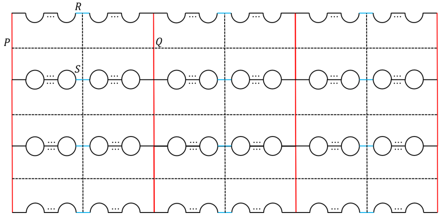
Step 3. Check that the tessellation by Futer and Thomas satisfies the condition in Step 2 as stated in Theorem 3.2.

Main Theorem 1: Condition for the Existence

Case 2: $2||F$



Case 3: $2 \nmid F$



Main Theorem 1: Condition for the Existence

- Corollary of Main Theorem 1

Corollary 3.4

For any even integer $p \geq 6$ and any alternating non-coprime sequence $\mathbf{q} = q_1, q_2, \cdots, q_p$, the automorphism group $\mathrm{Aut}(I_{p,\mathbf{q}})$ admits a lattice whose quotient is a compact, orientable hyperbolic surface.

$\Rightarrow$ Every Fuchsian building generated by an even-length alternating non-coprime sequence covers a compact surface.

Corollary 3.5

For any $g \geq 2$ and even $p \geq 6$ such that $\frac{8(g-1)}{p-4}$ is an integer, there exists a compact, orientable hyperbolic surface of genus g that is the quotient of a Fuchsian building with p-gons as chambers.

$\Rightarrow$ Every surface of genus is covered by a Fuchsian building.

Main Theorem 2: Condition for Non-Existence

Main Theorem 2 (Non-existence of $\Gamma_{p,\mathbf{q},g}$)

Let g be an integer such that $F = \frac{8(g-1)}{p-4}$ is an integer, and let $\mathbf{q}$ be a integer p-tuple. Then, the following statements hold:

1. If F is even and not divisible by 4, then $\mathbf{q}$ must be 2-symmetric for a lattice $\Gamma_{p,\mathbf{q},g}$ to exist.
2. If F is odd, then $\mathbf{q}$ must be 4-symmetric for a lattice $\Gamma_{p,\mathbf{q},g}$ to exist.

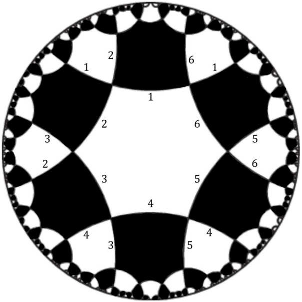
Main Theorem 2: Condition for Non-Existence

- Main idea of the proof

Observation 3.6

Let Γ be any subgroup of $\mathrm{Aut}(I_{p,\mathbf{q}})$. If $\Gamma \cdot B_i \supseteq B_j$, then $q_i = q_j$ in the given sequence $\mathbf{q}$.

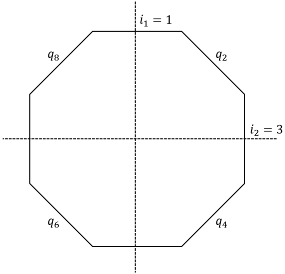
In Observation 3.6, B_i refers to the set of *type i edges* in $I_{p,\mathbf{q}}$.

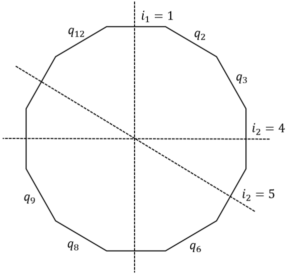


Main Theorem 2: Condition for Non-Existence

Sketch of the proof.

1. For the case where $2 \mid\mid F$, there exists $\gamma \in \Gamma_{p,\mathbf{q},g}$ such that $\gamma \cdot B_i = B_j$ for $i \neq j$. This γ induces a symmetry axis of $\mathbf{q}$.
2. For the case where $2 \nmid F$, there exist two or more symmetry axes in $\mathbf{q}$. These axes must include two orthogonal symmetry axes.





Review of the $I_{p,v}$ case

By restricting our Main Theorem 1 to the case of Bourdon's building $I_{p,v}$, we obtain an improved result for the second case of Futer-Thomas's existence theorem.

Theorem 1.2 (Futer-Thomas '12)

1. Existence of $\Gamma_{p,v,g}$.
   1. If $v \geq 2$ is even, then for all F, a lattice $\Gamma_{p,v,g}$ exists.
   2. If F is **divisible by 2**, then for all integers $v \geq 2$, a lattice $\Gamma_{p,v,g}$ exists.
   3. If F is composite, then for infinitely many odd integers $v \geq 3$, a lattice $\Gamma_{p,v,g}$ exists.
2. Non-existence of $\Gamma_{p,v,g}$.
   1. If F is odd, then for infinitely many odd integers $v \geq 3$, a lattice $\Gamma_{p,v,g}$ does not exist.

Further Directions

Remaining Questions:

- Is the alternating non-coprime condition on $\mathbf{q}$ necessary for the existence of $\Gamma_{p,\mathbf{q},g}$ when p is even?
- How about odd p?

Acknowledgment. I would like to express my gratitude to Prof. Seonhee Lim for supervising this research and for many stimulating conversations inspired by her broad insights into hyperbolic geometry.